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An Efficient Spectral Method for Nonlinear Volterra Integro-Differential Equations with Weakly Singular Kernels

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Abstract. For Volterra integro-differential equations (VIDEs) with weakly singular kernels, their solutions are singular at the initial time. This property brings a great challenge to traditional numerical methods. Here, we investigate the numerical approximation for the solution of the nonlinear model with weakly singular kernels. Due to its characteristic, we split the interval and focus on the first one to save operation. We employ the corresponding singular functions as basis functions in the first interval to simulate its singular behavior, and take the Legendre polynomials as basis functions in the other one. Then the corresponding *hp*-version spectral method is proposed, the existence and uniqueness of solution to the numerical scheme are proved, the *hp*-version optimal convergence is derived. Numerical experiments verify the effectiveness of the proposed method.

Keywords: spectral element method, Volterra integro-differential equation, weak singularity, exponential convergence.

AMS Subject Classification: 65N35; 65M70; 41A25; 42B20; 45D05.

1 Introduction

In this paper, we consider the numerical approximation of the nonlinear Volterra integro-differential equations (VIDEs) with weakly singular kernels:

$$\begin{cases} y'(t) + y(t) = f(t) + \int_0^t (t-s)^{-\mu} K(t,s) G(s, y(s)) ds, \quad t \in (0,T], \\ y(0) = y_0. \end{cases} \tag{1.1}$$

Here, the parameter $\mu \in (0,1)$, y_0 is the initial data, f and G are given continuous functions. In addition, $K \in C(D)$ where $D := \{(t,s) : 0 \le s < t \le T\}$.

It is proved that the solution of (1.1) at $t = 0$ appears singularly [3], which poses a great challenge to traditional numerical methods. To overcome this difficulty, various numerical methods, such as collocation methods, Runge-Kutta methods, spectral methods, have been proposed during the past few decades, see [1, 2, 3, 4, 6, 7, 10, 11, 14, 15, 16, 17, 18, 20] and the references therein.

Due to its characteristic, if we divide the interval $[0, T]$, the singularity only appears in the first one. Therefore, it is not advisable to use only one family of functions on the whole interval to approximate the solution. We can take the basis functions on the first interval which are different from the others. Fortunately, the *hp*-version method is suitable to our thought. Combined with the basis of spectral method, the *hp*-version spectral method can approximate smooth solutions with possible local singularities at high algebraic or even exponential rates of convergence. Wang et al. [12, 13] used three types of polynomial interpolation techniques to express the numerical solution, and proposed an *hp*-version spectral collocation method for weakly singular VIDEs, where the *hp*-version optimal convergence is obtained. Since Müntz-Jacobi functions [5, 9] can capture the singularity of the solution exactly, we employ Müntz-Jacobi functions as the basis in the first interval, and Legendre polynomials as the basis in the other intervals. An *hp*-version spectral method is designed and developed to approximate weakly singular VIDEs. We prove the existence and uniqueness of solution to the numerical scheme and derive *hp*-version error estimates for the singular solution.

The remainder of the paper is arranged as follows. The next section is for preliminaries. We introduce the shifted Müntz-Jacobi functions and Legendre polynomials, and provide some approximation results which are significant in the convergence analysis. In Section 3, we propose an *hp*-version spectral method to approximate nonlinear weakly singular VIDEs, and prove the existence and uniqueness of solution to the numerical scheme. Meanwhile, optimal error estimates in the *hp*-version are derived under the H^1-norm. Numerical results are performed to demonstrate the effectiveness of the new method in Section 4. The final section is for conclusion remarks.

2 Preliminaries

We divide the interval $I := [0, T]$ as $I_h = \{t_n : 0 = t_0 < t_1 < \ldots < t_N = T\}$ and let $h_n = t_n - t_{n-1}, h_{\max} = \max_{1 \le n \le N} h_n$, $I_n = (t_{n-1}, t_n]$ and $y^n(t)$ the solution of (1.1) on the n-th element, namely

$$y^n(t) := y(t), \quad \forall t \in I_n, \quad 1 \le n \le N.$$

The model (1.1) can be rewritten as

$$\begin{cases} \dfrac{d}{dt} y^n(t) + y^n(t) = f(t) + \displaystyle\sum_{k=1}^{n-1} \int_{I_k} (t-\xi)^{-\mu} K(t,\xi) G(\xi, y^k(\xi)) d\xi \\ \qquad\qquad + \displaystyle\int_{t_{n-1}}^{t} (t-s)^{-\mu} K(t,s) G(s, y^n(s)) ds, \quad t \in I_n, \\ y(0) = y_0. \end{cases} \tag{2.1}$$

Let

$$s = s(t,\tau) = t_{n-1} + (\tau - t_{n-1})(t - t_{n-1})/h_n, \quad \tau \in I_n.$$

Then, the problem (2.1) becomes

$$\begin{cases} \dfrac{d}{dt} y^n(t) + y^n(t) := f(t) + \mathcal{V}_1^n y(t) + \mathcal{V}_2^n y^n(t), \\ y(0) = y_0, \end{cases} \tag{2.2}$$

where

$$\mathcal{V}_1^n y(t) = \sum_{k=1}^{n-1} \int_{I_k} (t-\xi)^{-\mu} K(t,\xi) G(\xi, y^k(\xi)) d\xi,$$
$$\mathcal{V}_2^n y^n(t) = \Big(\frac{t - t_{n-1}}{h_n}\Big)^{1-\mu} \int_{I_n} (t_n - \tau)^{-\mu} K(t, s(t,\tau)) G(s(t,\tau), y^n(s(t,\tau))) d\tau.$$

2.1 The shifted Müntz-Jacobi functions on I_1

For $\alpha, \beta > -1$, the shifted Müntz-Jacobi function of degree p on I_1 is defined by

$$J_{1,p}^{\alpha,\beta,\lambda}(t) = J_p^{\alpha,\beta,\lambda}\Big(\frac{t}{t_1}\Big) = J_p^{\alpha,\beta}\Big(2\big(\frac{t}{t_1}\big)^{\lambda} - 1\Big), \ t \in I_1, \ 0 < \lambda \le 1,$$

where $J_p^{\alpha,\beta}(x)$ is the standard Jacobi polynomial of degree p defined on $(-1,1)$.

Let

$$\tilde{\omega}^{\alpha,\beta,\lambda}(t) = \frac{\lambda}{t_1}\Big(1 - \Big(\frac{t}{t_1}\Big)^{\lambda}\Big)^{\alpha} \Big(\frac{t}{t_1}\Big)^{(\beta+1)\lambda - 1}.$$

The set of $J_{1,p}^{\alpha,\beta,\lambda}(t)$ forms a complete $L^2_{\tilde{\omega}^{\alpha,\beta,\lambda}}(I_1)$-orthogonal system, and satisfy that

$$\int_{I_1} J_{1,p}^{\alpha,\beta,\lambda}(t) J_{1,q}^{\alpha,\beta,\lambda}(t) \tilde{\omega}^{\alpha,\beta,\lambda}(t) dt = \widehat{\gamma}_p^{\alpha,\beta} \delta_{p,q}$$

with

$$\widehat{\gamma}_p^{\alpha,\beta} = \begin{cases} \dfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}, & p = 0, \\ \dfrac{\Gamma(p+\alpha+1)\Gamma(p+\beta+1)}{(2p+\alpha+\beta+1)p!\Gamma(p+\alpha+\beta+1)}, & p \ge 1. \end{cases}$$

For any given integer $M_1 \geq 0$, let $t_{1,j}^{\alpha,\beta,\lambda}$ be the shifted Müntz-Jacobi function quadrature nodes on the interval I_1,

$$t_{1,j}^{\alpha,\beta,\lambda} = t_1 \left(\frac{x_j^{\alpha,\beta} + 1}{2} \right)^{\frac{1}{\lambda}}, \ 0 \leq j \leq M_1,$$

where $\{x_j^{\alpha,\beta}\}_{j=0}^{M_1}$ are the nodes of the standard Jacobi-Gauss interpolation on the interval $(-1, 1)$.

We introduce the finite-dimensional approximation space as follows:

$$P_{M_1}^{\lambda}(I_1) = \text{span}\{J_{1,p}^{\alpha,\beta,\lambda}, 0 \leq p \leq M_1\}.$$

Due to the property of the standard Jacobi-Gauss quadrature, there holds that for any $\phi \in P_{2M_1+1}^{\lambda}(I_1)$,

$$\int_{I_1} \phi(t)\tilde{\omega}^{\alpha,\beta,\lambda}(t)dt = \frac{1}{2^{\alpha+\beta+1}} \sum_{j=0}^{M_1} \phi(t_{1,j}^{\alpha,\beta,\lambda})\omega_j^{\alpha,\beta}, \tag{2.3}$$

where $\{\omega_j^{\alpha,\beta}\}_{j=0}^{M_1}$ are the corresponding Christoffel numbers of the standard Jacobi-Gauss interpolation on the interval $(-1, 1)$.

Let $(\cdot, \cdot)_{\tilde{\omega}^{\alpha,\beta,\lambda}}$, $\|\cdot\|_{\tilde{\omega}^{\alpha,\beta,\lambda}}$ be the inner product and norm of space $L^2_{\tilde{\omega}^{\alpha,\beta,\lambda}}(I_1)$, respectively. We introduce the following discrete inner product and norm:

$$\langle u, v\rangle_{\tilde{\omega}^{\alpha,\beta,\lambda}} = \frac{1}{2^{\alpha+\beta+1}} \sum_{j=0}^{M_1} u(t_{1,j}^{\alpha,\beta,\lambda})v(t_{1,j}^{\alpha,\beta,\lambda})\omega_j^{\alpha,\beta}, \quad \|v\|_{M_1,\tilde{\omega}^{\alpha,\beta,\lambda}} = \langle v, v\rangle^{\frac{1}{2}}_{\tilde{\omega}^{\alpha,\beta,\lambda}}.$$

Thanks to (2.3), for any $\phi\psi \in P_{2M_1+1}^{\lambda}(I_1)$, it holds that

$$(\phi, \psi)_{\tilde{\omega}^{\alpha,\beta,\lambda}} = \langle \phi, \psi\rangle_{\tilde{\omega}^{\alpha,\beta,\lambda}}.$$

We also introduce the non-uniformly Jacobi-weighted Sobolev space:

$$B_{\alpha,\beta}^{m,1}(I_1) := \{v : \partial_t^k v \in L^2_{\tilde{\omega}^{\alpha+k,\beta+k,1}}(I_1), 0 \leq k \leq m\}, \ m \in \mathbb{N}.$$

Denote the projection by $\pi_{I_1,M_1}^{\alpha,\beta,\lambda} : L^2_{\tilde{\omega}^{\alpha,\beta,\lambda}}(I_1) \to P_{M_1}^{\lambda}(I_1)$ as

$$(\pi_{I_1,M_1}^{\alpha,\beta,\lambda} v - v, \psi)_{\tilde{\omega}^{\alpha,\beta,\lambda}} = 0, \quad \forall \psi \in P_{M_1}^{\lambda}(I_1).$$

Lemma 1. ([19]) *For any* $v(t^{\frac{1}{\lambda}}) \in B_{\alpha,\beta}^{m_1,1}(I_1)$, *and* $0 \leq m_1 \leq M_1 + 1$,

$$\left\|\pi_{I_1,M_1}^{\alpha,\beta,\lambda} v - v\right\|_{\tilde{\omega}^{\alpha,\beta,\lambda}} \leq ch_1^{m_1} M_1^{-m_1} \left\|\partial_t^{m_1}\{v(t^{\frac{1}{\lambda}})\}\right\|_{\tilde{\omega}^{\alpha+m_1,\beta+m_1,1}}.$$

2.2 The shifted Legendre polynomials on I_n

The shifted Legendre polynomial $L_{n,p}(t)$ on I_n is defined by

$$L_{n,p}(t) = L_p\left(\frac{2t - t_{n-1} - t_n}{h_n}\right), \quad t \in I_n, \quad p \geq 0,$$

where $L_p(x)$ be the standard Legendre polynomial of degree p defined on $(-1,1)$. They form a complete $L^2(I_n)$-orthogonal system, i.e.,

$$\int_{I_n} L_{n,p}(t)L_{n,q}(t)dt = \frac{h_n}{2p+1}\delta_{p,q}.$$

For any given integer $M_n \geq 0$, let $\mathcal{P}_{M_n}(I_n) = \{L_{n,0}, L_{n,1}, \ldots, L_{n,M_n}\}$, $\{x_j, \omega_j\}_{j=0}^{M_n}$ be the Legendre-Gauss interpolation nodes and the corresponding Christoffel numbers on $(-1,1)$.

Define the shifted Legendre-Gauss quadrature nodes on I_n by

$$t_{n,j} = \frac{h_n x_j + t_{n-1} + t_n}{2},\ 1 \leq n \leq N,\ 0 \leq j \leq M_n.$$

According to the standard Legendre-Gauss quadrature, it follows that

$$\int_{I_n} \phi(t)dt = \frac{h_n}{2}\sum_{j=0}^{M_n} \phi(t_{n,j})\omega_j,\ \forall \phi \in \mathcal{P}_{2M_n+1}(I_n). \tag{2.4}$$

Let $(\cdot,\cdot)_{I_n}$ and $\|\cdot\|_{I_n}$ be the inner product and norm of space $L^2(I_n)$, respectively. We further introduce the following discrete inner product and norm:

$$\langle u, v\rangle_{I_n} = \frac{h_n}{2}\sum_{j=0}^{M_n} u(t_{n,j})v(t_{n,j})\omega_j, \quad \|v\|_{M_n,I_n} = \langle v, v\rangle_{I_n}^{\frac{1}{2}}.$$

Thanks to (2.4), we have that

$$(\phi,\psi)_{I_n} = \langle \phi,\psi\rangle_{I_n}, \quad \forall \phi\psi \in \mathcal{P}_{2M_n+1}(I_n).$$

We introduce the $L^2(I_n)$-orthogonal projection $\pi_{I_n,M_n} : L^2(I_n) \to \mathcal{P}_{M_n}(I_n)$, satisfying that

$$(\pi_{I_n,M_n} v - v, \psi)_{I_n} = 0, \ \ \forall \psi \in \mathcal{P}_{M_n}(I_n).$$

Lemma 2. ([19]) *For any $v \in H^{m_n}(I_n)$, and $1 \leq m_n \leq M_n + 1$,*

$$\|\pi_{I_n,M_n} v - v\|_{I_n} \leq c h_n^{m_n} M_n^{-m_n} \|\partial_t^{m_n} v\|_{I_n}.$$

3 An *hp*-version spectral method for VIDEs

3.1 Spectral scheme of problem (2.2)

Let $P_{M_1}^{\lambda,0}(I_1) = P_{M_1}^{\lambda}(I_1) \cap \{y(0) = y_0\}$. The *hp*-version spectral scheme for (2.2) is to look for $Y^1(t) (\in P_{M_1}^{\lambda,0}(I_1))$ and $Y^n(t) (\in \mathcal{P}_{M_n-1}(I_n))$, such that

$$\begin{cases} \left(\frac{d}{dt}Y^1, \varphi\right)_{\tilde{\omega}^{\alpha,\beta,\lambda}} + (Y^1,\varphi)_{\tilde{\omega}^{\alpha,\beta,\lambda}} \\ \qquad\qquad = (f,\varphi)_{\tilde{\omega}^{\alpha,\beta,\lambda}} + (\mathcal{V}_2^1 Y^1, \varphi)_{\tilde{\omega}^{\alpha,\beta,\lambda}}, \forall \varphi \in P_{M_1}^{\lambda,0}(I_1); \\ \left(\frac{d}{dt}Y^n, \psi\right)_{I_n} + (Y^n,\psi)_{I_n} = (f,\psi)_{I_n} + (\mathcal{V}_1^n Y, \psi)_{I_n} \\ \qquad\qquad + (\mathcal{V}_2^n Y^n, \psi)_{I_n}, \quad \forall \psi \in \mathcal{P}_{M_n-1}(I_n),\ n \geq 2; \\ Y^n(t_{n-1}) = Y^{n-1}(t_{n-1}). \end{cases} \tag{3.1}$$

We may rewrite (3.1) as

$$\begin{cases} \left(\frac{d}{dt}Y^1,\varphi\right)_{\tilde\omega^{\alpha,\beta,\lambda}} + (Y^1,\varphi)_{\tilde\omega^{\alpha,\beta,\lambda}} = (f,\varphi)_{\tilde\omega^{\alpha,\beta,\lambda}} + (\mathcal{V}_2^1 Y^1,\varphi)_{\tilde\omega^{\alpha,\beta,\lambda}}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall\varphi\in P_{M_1}^{\lambda,0}(I_1); \\ Y^n(t_n)\psi(t_n) - Y^{n-1}(t_{n-1})\psi(t_{n-1}) - (Y^n,\frac{d}{dt}\psi)_{I_n} + (Y^n,\psi)_{I_n} \\ = (f,\psi)_{I_n} + (\mathcal{V}_1^n Y,\psi)_{I_n} + (\mathcal{V}_2^n Y^n,\psi)_{I_n}, \forall\psi\in\mathcal{P}_{M_n-1}(I_n),\ n\ge 2. \end{cases} \tag{3.2}$$

Using the shifted Müntz-Jacobi functions on I_1 and the shifted Legendre functions on I_n, we expand the numerical solutions as

$$\begin{cases} Y^1(t) &= \sum_{p=0}^{M_1-1} y_p^1 \mathcal{J}_{1,p}^{\alpha,\beta,\lambda}(t), \quad t\in I_1, \\ Y^n(t) &= \sum_{p=0}^{M_n-1} y_p^n L_{n,p}(t), \qquad t\in I_n, \quad n\ge 2, \end{cases} \tag{3.3}$$

where

$$\mathcal{J}_{1,p}^{\alpha,\beta,\lambda}(t) := J_{1,p}^{\alpha,\beta,\lambda}(t) + s_p J_{1,p+1}^{\alpha,\beta,\lambda}(t). \tag{3.4}$$

Remark 1. According to [8], the Jacobi polynomials have the following property:

$$J_p^{\alpha,\beta}(-1) = (-1)^p \frac{\Gamma(p+\beta+1)}{p!\Gamma(\beta+1)},$$

then the coefficient s_p in (3.4) can be uniquely determined as

$$s_p = \frac{(p+1)!\Gamma(\beta+1)y_0}{(-1)^{p+1}\Gamma(p+\beta+2)} + \frac{p+1}{p+\beta+1}.$$

We substitute the expression (3.3) into (3.2), take $\varphi = \mathcal{J}_{1,q}^{\alpha,\beta,\lambda}(t)$, $\psi = L_{n,q}(t)$, and obtain that

$$\begin{aligned} &\sum_{p=0}^{M_1-1} y_p^1\Big((\frac{d}{dt}\mathcal{J}_{1,p}^{\alpha,\beta,\lambda}, \mathcal{J}_{1,q}^{\alpha,\beta,\lambda})_{\tilde\omega^{\alpha,\beta,\lambda}} + (\mathcal{J}_{1,p}^{\alpha,\beta,\lambda}, \mathcal{J}_{1,q}^{\alpha,\beta,\lambda})_{\tilde\omega^{\alpha,\beta,\lambda}} \Big) \\ &\qquad - (\mathcal{V}_2^1 Y^1, \mathcal{J}_{1,q}^{\alpha,\beta,\lambda})_{\tilde\omega^{\alpha,\beta,\lambda}} = (f, \mathcal{J}_{1,q}^{\alpha,\beta,\lambda})_{\tilde\omega^{\alpha,\beta,\lambda}}, \quad 0\le q\le M_1-1, \\ &\sum_{p=0}^{M_n-1} y_p^n\Big(-(L_{n,p}, \frac{d}{dt}L_{n,q})_{I_n} + (L_{n,p},L_{n,q})_{I_n} + L_{n,p}(t_n)\cdot L_{n,q}(t_n) \Big) \\ &\qquad - (\mathcal{V}_2^n Y^n, L_{n,q})_{I_n} = (f,L_{n,q})_{I_n} + (\mathcal{V}_1^n Y, L_{n,q})_{I_n} \\ &\qquad + Y^{n-1}(t_{n-1})\, L_{n,q}(t_{n-1}), \quad 0\le q\le M_n-1,\ n\ge 2. \end{aligned}$$

Introducing the entries as

$$\begin{aligned} &s_{qp}^1 = (\frac{d}{dt}\mathcal{J}_{1,p}^{\alpha,\beta,\lambda}, \mathcal{J}_{1,q}^{\alpha,\beta,\lambda})_{\tilde\omega^{\alpha,\beta,\lambda}}, \quad s_{qp}^n = (L_{n,p}, \frac{d}{dt}L_{n,q})_{I_n}, \quad n\ge 2, \\ &a_{qp}^1 = (\mathcal{J}_{1,p}^{\alpha,\beta,\lambda}, \mathcal{J}_{1,q}^{\alpha,\beta,\lambda})_{\tilde\omega^{\alpha,\beta,\lambda}}, \qquad a_{qp}^n = (L_{n,p},L_{n,q})_{I_n}, \qquad n\ge 2, \\ &w_q^1 = (\mathcal{V}_2^1 Y^1, \mathcal{J}_{1,q}^{\alpha,\beta,\lambda})_{\tilde\omega^{\alpha,\beta,\lambda}}, \quad w_q^n = (\mathcal{V}_2^n Y^n, L_{n,q})_{I_n}, \qquad n\ge 2, \\ &f_q^1 = (f, \mathcal{J}_{1,q}^{\alpha,\beta,\lambda})_{\tilde\omega^{\alpha,\beta,\lambda}}, \quad f_q^n = (f,L_{n,q})_{I_n}, \quad n\ge 2, \end{aligned}$$

$$v_q^n = (\mathcal{V}_1^n Y, L_{n,q})_{I_n}, \quad d_q^n = Y^{n-1}(t_{n-1}) \cdot L_{n,q}(t_{n-1}), \quad n \geq 2,$$
$$\mathbf{S}^n = (s_{qp}^n)_{0 \leq p,q \leq M_n-1}, \quad \mathbf{A}^n = (a_{qp}^n)_{0 \leq p,q \leq M_n-1}, \quad \boldsymbol{w}^n = (w_0^n, \ldots, w_{M_n-1}^n)^T,$$
$$\boldsymbol{v}^n = (v_0^n, \ldots, v_{M_n-1}^n)^T, \quad \boldsymbol{d}^n = (d_0^n, \ldots, d_{M_n-1}^n)^T,$$
$$\boldsymbol{f}^n = (f_0^n, \ldots, f_{M_n-1}^n)^T, \quad \boldsymbol{y}^n = (y_0^n, \ldots, y_{M_n-1}^n)^T,$$

then we can obtain the following compact system as

$$\begin{cases} (\mathbf{S}^1 + \mathbf{A}^1)\boldsymbol{y}^1 - \boldsymbol{w}^1 = \boldsymbol{f}^1, \\ (-\mathbf{S}^n + \mathbf{A}^n + \mathbf{E}^n)\boldsymbol{y}^n - \boldsymbol{w}^n = \boldsymbol{f}^n + \boldsymbol{v}^n + \boldsymbol{d}^n, \quad n \geq 2, \end{cases} \tag{3.5}$$

where $\mathbf{E}^n$ is $M_n \times M_n$ order matrix whose elements are all one and $\boldsymbol{w}^m = \boldsymbol{w}^m(\boldsymbol{y}^m),\ m = 1, 2, \ldots, N$ are implicit terms.

In the actual computation, we employ an iterative algorithm to evaluate the expansion coefficients $\{y_p^n\}_{p=0}^{M_n-1}$. Briefly, we obtain the successive values of Y^n in terms of previously computed quantities $\{y_p^k\}_{p=0}^{M_k-1}, 1 \leq k \leq n-1$. The following algorithm gives the concrete calculation steps.

Algorithm A simple iterative process

for $1 \leq n \leq N$
give initial value $\boldsymbol{y}^{n,(0)} \equiv (1, \cdots, 1)^T$, perform the following iteration procedure
$(\mathbf{S}^1 + \mathbf{A}^1)\boldsymbol{y}^{1,(k)} = \boldsymbol{f}^1 + \boldsymbol{w}^1(\boldsymbol{y}^{1,(k-1)}),\ k = 1, 2, \ldots$
$(-\mathbf{S}^n + \mathbf{A}^n + \mathbf{E}^n)\boldsymbol{y}^{n,(k)} = \boldsymbol{f}^n + \boldsymbol{v}^n + \boldsymbol{d}^n + \boldsymbol{w}^n(\boldsymbol{y}^{n,(k-1)}),\ k = 1, 2, \ldots$

3.2 Theoretical analysis

We first prove the existence and uniqueness of solution to the scheme (3.1) under the reasonable assumption on the nonlinearity.

Theorem 1 [existence and uniqueness]. *Assume that $K(t,s) \in C(D)$, and G fulfills the following Lipschitz condition:*

$$|G(s, y_1) - G(s, y_2)| \leq \gamma |y_1 - y_2|, \quad \gamma \geq 0, \tag{3.6}$$

then for any $1 \leq n \leq N$ and sufficiently small $h_{\max}^{2-2\mu}$, the scheme (3.1) *has a unique solution.*

To be undistracted from the main results, we postpone the derivation of the formulas to Appendix A.

Let $y^n(t)$ be the solution of (2.2) and $Y^n(t)$ be the solution of (3.1), respectively. Denote $e_n(t) = y^n(t) - Y^n(t),\ 1 \leq n \leq N$.

Theorem 2 [local error]. *Assume that $K(t,s) \in C(D), y(t^{\frac{1}{\lambda}})|_{t \in I_1} \in B_{\alpha,\beta}^{m_1,1}(I_1)$, $y_t(t^{\frac{1}{\lambda}})|_{t \in I_1} \in B_{\alpha,\beta}^{m_1-1,1}(I_1)$, $y(t)|_{t \in I_n} \in H^{m_n}(I_n)$ with $2 \leq n \leq N$ and integers $1 \leq m_n \leq M_n + 1$, G fulfills the Lipschitz condition* (3.6), *and*

$$0 < \mu < \frac{1}{2}, 0 < \lambda \leq 1, -1 < \alpha < 0, -1 < \beta < \frac{1}{\lambda} - 1,$$

then for sufficiently small $h_{\max}^{2-2\mu}$,

$$
\begin{aligned}
\left\|y^1-Y^1\right\|^2_{H^1(I_1)} \le\, & ch_1^{2m_1-1}M_1^{-2m_1+2}\left\|\partial_t^{m_1-1}\{y_t^1(t^{\frac{1}{\lambda}})\}\right\|^2_{\tilde{\omega}^{\alpha+m_1-1,\beta+m_1-1,1}} \\
& + ch_1^{2m_1+1}M_1^{-2m_1}\left\|\partial_t^{m_1}\{y^1(t^{\frac{1}{\lambda}})\}\right\|^2_{\tilde{\omega}^{\alpha+m_1,\beta+m_1,1}}, \\
\|y^n-Y^n\|^2_{H^1(I_n)} \le\, & ce^T(1+h_n)\Big(\sum_{k=2}^{n-1} h_k^{2m_k-2}M_k^{-2m_k+2}\left\|\partial_t^{m_k}y^k\right\|^2_{I_k} \\
& + h_1^{2m_1-1}M_1^{-2m_1+2}\left\|\partial_t^{m_1-1}\{y_t^1(t^{\frac{1}{\lambda}})\}\right\|^2_{\tilde{\omega}^{\alpha+m_1-1,\beta+m_1-1,1}} + h_1^{2m_1+1}M_1^{-2m_1} \\
& \times\left\|\partial_t^{m_1}\{y^1(t^{\frac{1}{\lambda}})\}\right\|^2_{\tilde{\omega}^{\alpha+m_1,\beta+m_1,1}}\Big) + ch_n^{2m_n-2}M_n^{-2m_n+2}\left\|\partial_t^{m_n}y^n\right\|^2_{I_n}. \qquad (3.7)
\end{aligned}
$$

We provide the proof in Appendix B.

According to Theorem 2, summing n from 1 to N in (3.7), then we can obtain the global errors as follows.

Theorem 3 [global error]. *Let y be the solution of* (1.1) *and Y be its numerical solution, respectively. Under the assumption conditions in Theorem* 2, *there holds that*

$$
\begin{aligned}
& \|y-Y\|^2_{H^1(I)} \le c(1+T)e^T\sum_{n=2}^{N} h_n^{2m_n-2}M_n^{-2m_n+2}\left\|\partial_t^{m_n}y^n\right\|^2_{I_n} \\
& + c(1+T)e^T\Big(h_1^{2m_1-1}M_1^{-2m_1+2}\left\|\partial_t^{m_1-1}\{y_t^1(t^{\frac{1}{\lambda}})\}\right\|^2_{\tilde{\omega}^{\alpha+m_1-1,\beta+m_1-1,1}} \\
& + h_1^{2m_1+1}M_1^{-2m_1}\left\|\partial_t^{m_1}\{y^1(t^{\frac{1}{\lambda}})\}\right\|^2_{\tilde{\omega}^{\alpha+m_1,\beta+m_1,1}}\Big). \qquad (3.8)
\end{aligned}
$$

4 Numerical results

In this section, some numerical examples will be presented to show the convergence and accuracy of the proposed scheme. For this, we first introduce some quantities to measure the errors

$$
\begin{aligned}
E_1(T) &= \max_{\substack{1\le k\le N\\ 0\le j\le M_k}} |y(t_{k,j})-Y(t_{k,j})|, \\
E_2(T) &= \Big(\sum_{k=1}^{N}\frac{h_k}{2}\sum_{j=0}^{M_k}(y^k(t_{k,j})-Y^k(t_{k,j}))^2\omega_j\Big)^{\frac{1}{2}}, \\
E_3(T) &= \Big(\sum_{k=1}^{N}\frac{h_k}{2}\sum_{j=0}^{M_k}(\frac{d}{dt}y^k(t_{k,j})-\frac{d}{dt}Y^k(t_{k,j}))^2\omega_j\Big)^{\frac{1}{2}}.
\end{aligned}
$$

In the forthcoming numerical tests, we choose the uniform mode $M_k\equiv M$ and the uniform step size $h_k\equiv h$.

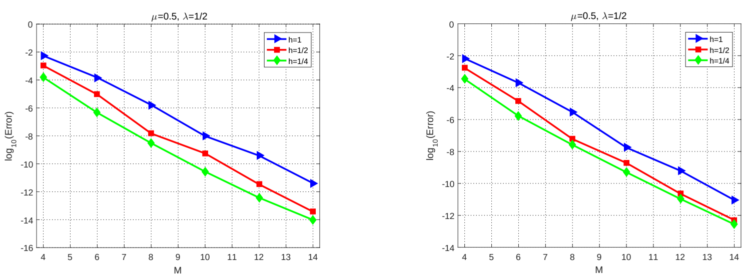
Table 1. A comparison of maximum errors for (4.1).

DOF	method of [12]	DOF	new method
47	1.59E-08	16	1.48E-08
90	2.91E-11	22	9.42E-12
191	1.29E-14	28	3.74E-14

Example 1. Consider the following linear VIDE with singular kernel (cf. Example 4 of [12]):

$$\begin{cases} y'(t) = -\frac{1}{\Gamma(1-\mu)} \int_0^t (t-s)^{-\mu} y(s) ds + (2-\mu) t^{1-\mu}, \quad t \in (0,1], \\ y(0) = 0, \end{cases} \tag{4.1}$$

with the exact solution $y(t) = \Gamma(3-\mu)(1 - E_{2-\mu}(-t^{2-\mu}))$, where the Mittag-Leffler function is $E_\sigma(x) = \sum_{p=0}^{\infty} x^p / \Gamma(1+p\sigma)$. Thus, $y(t)$ behaves like $t^{2-\mu}$ as $t \to 0^+$, which has a weak singularity at $t = 0$ for $\mu \in (0,1)$. In Figure 1, we plot the H^1-errors and maximum errors in semi-log scale with $\mu = 0.5$ and $\lambda = \frac{1}{2}$, which show exponential decay with respect to M. This result is in a good agreement with the theoretical prediction given in Theorem 3, stating that the convergence of numerical solution is exponential if $y(t^{\frac{1}{\lambda}})$ is smooth.



(a) (b)

Figure 1. (a)The H^1-errors of (4.1); (b)The maximum point-wise errors of (4.1).

In order to show the effectiveness of the new method, the maximum errors with $\mu = 0.5, \lambda = \frac{1}{2}, h = \frac{1}{2}$ obtained by our method and the Legendre-Jacobi spectral collocation method (geometric mesh, the parameter $\rho = 0.2$ and $\theta = 1.5$ in [12]) are compared in Table 1. It is clear that our spectral element method is capable of providing more accurate numerical results with relatively less degree of freedom.

Example 2. Consider the following linear VIDE with singular kernel:

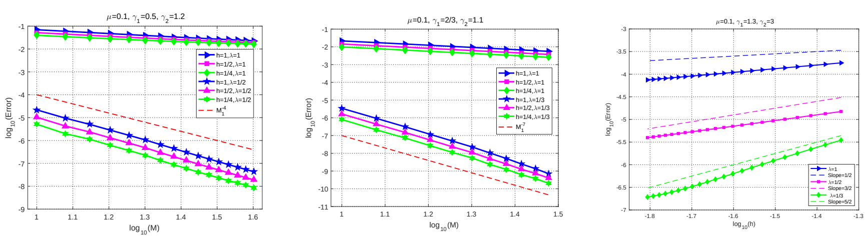
$$\begin{cases} y'(t) + y(t) = -\int_0^t (t-s)^{-\mu} y(s) ds + f(t), \quad t \in (0,1], \\ y(0) = 0, \end{cases} \tag{4.2}$$

with the exact solution $y(t) = t^{\gamma_1} + t^{\gamma_2}$. The source function is

$$f(t)=t^{\gamma_1}+t^{\gamma_2}+\gamma_1 t^{\gamma_1-1}+\gamma_2 t^{\gamma_2-1}+t^{1+\gamma_1-\mu}B(1-\mu,1+\gamma_1)$$
$$+\,t^{1+\gamma_2-\mu}B(1-\mu,1+\gamma_2),$$

where $B(\cdot,\cdot)$ is the Beta function.

The exact solution consists of two fractional power functions, which can further illustrate the benefits of using our new method to solve general one-point singular problems. In Figure 2 (a) and (b), we plot the H^1-errors of (4.2) with different parameters $\lambda, \gamma_1, \gamma_2$ in log-log scale. We observe that, comparing with the classical Legendre polynomial (i.e., $\lambda = 1$) in the first interval I_1, the Müntz case (i.e., $\lambda < 1$) enhances the convergence rates. A reasonable explanation for this excellent result is that the regularity of the solution $y(t^{\frac{1}{\lambda}})$ is improved as shown in the theoretical estimate of the Müntz approximation. Specifically, the main error depends on the first interval I_1 because of the singularity of the solution at the initial time. Hence, as shown in Theorem 3 that the convergence rate is determined by the regularity m_1, i.e., the maximum m_1 such that $\partial_t^{m_1-1}\{y_t^1(t^{\frac{1}{\lambda}})\} \in L^2_{\tilde{\omega}^{\alpha+m_1-1,\beta+m_1-1,1}}(I_1)$. The theoretical convergence curves drawn with dash lines in Figure 2 show that the convergence rates verify the theoretical prediction given in (3.8). In Figure 2 (c), we plot the H^1-errors of (4.2) with $\mu = 0.1, \gamma_1 = 1.3, \gamma_2 = 3$, fixed $M = 6$ and different λ in log-log scale. The lines of slopes $h^{\frac{1}{2}}, h^{\frac{3}{2}}$ and $h^{\frac{5}{2}}$ are also plotted in Figure 2 (c), which clearly indicate that the convergence rate is close to $h^{\frac{1}{2}}$ for $\lambda = 1$, $h^{\frac{3}{2}}$ for $\lambda = \frac{1}{2}$, and $h^{\frac{5}{2}}$ for $\lambda = \frac{1}{3}$. This is in a good agreement with the theoretical estimate in (3.8).



(a) (b) (c)

Figure 2. The H^1-errors of (4.2).

Note that the main advantage of new spectral element method lies in the capability of dealing with more complicated nonlinear weakly singular VIDEs with singular solutions, the following challenging cases are designed to validate the high-efficiency of the new method.

Example 3. Consider the following nonlinear VIDE with weakly singular kernel:

$$\begin{cases} y'(t) + y(t) = \int_0^t (t-s)^{-\mu} e^{2s} y^2(s)ds + f(t), & t \in (0,1], \\ y(0) = 0, \end{cases} \tag{4.3}$$

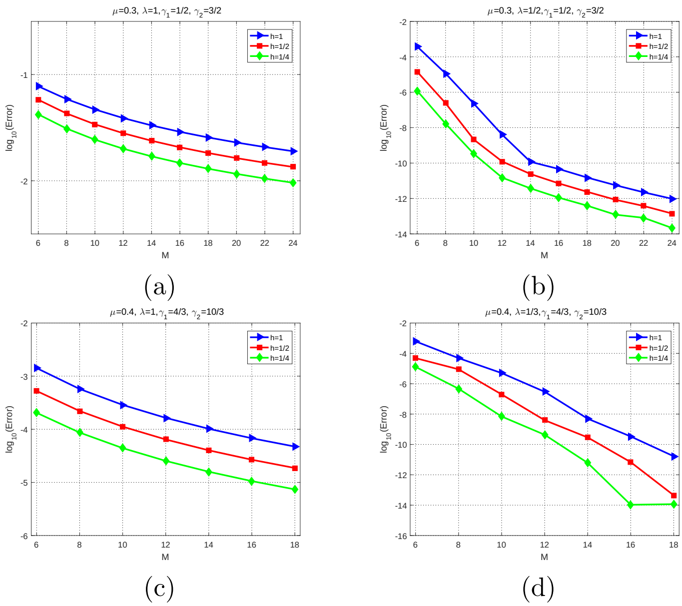


Figure 3. The maximum point-wise errors of (4.3).

with the exact solution $y(t) = (t^{\gamma_1} + t^{\gamma_2})e^{-t}$. The source function is

$$\begin{aligned} f(t) =&(\gamma_1 t^{\gamma_1 - 1} + \gamma_2 t^{\gamma_2 - 1})e^{-t} - 2t^{\gamma_1+\gamma_2+1-\mu}B(1-\mu, \gamma_1+\gamma_2+1) \\ &- t^{2\gamma_1+1-\mu}B(1-\mu, 2\gamma_1+1) - t^{2\gamma_2+1-\mu}B(1-\mu, 2\gamma_2+1), \end{aligned}$$

where $B(\cdot,\cdot)$ is the Beta function.

We apply the numerical scheme (3.5) and corresponding iterative algorithm to solve the above model. The related numerical results with different parameters $\mu, \lambda, \gamma_1, \gamma_2$ are presented in Figure 3. Convergence rates are quite low in (a) and (c) for the reason that classical Legendre polynomial ((i.e., $\lambda = 1$)) cannot approximate the solution very well due to the singularity at the initial time. Therefore, we choose suitable small parameter $\lambda < 1$ so that the regularity of $y(t^{\frac{1}{\lambda}})$ can be improved. We can observe that convergence rates are greatly enhanced in (b) and (d), which show the efficiency of Müntz-Jacobi functions for singular solutions.

Example 4. Consider the following nonlinear VIDE with weakly singular kernel:

$$\begin{cases} y'(t) + y(t) = \int_0^t (t-s)^{-0.5}\sin^2(y(s))ds + f(t), & t \in (0, T], \\ y(0) = 0, \end{cases} \tag{4.4}$$

with the exact solution $y(t) = t^{\frac{3}{2}}e^{-t}$.

In Figure 4 (a) and (b), we plot the H^1-errors of (4.4) with $T = 4, \lambda = \frac{1}{2}$. The numerical results again confirm that new spectral element method (i.e., $\lambda < 1$) performs much better than classical spectral element method (i.e., $\lambda = 1$).

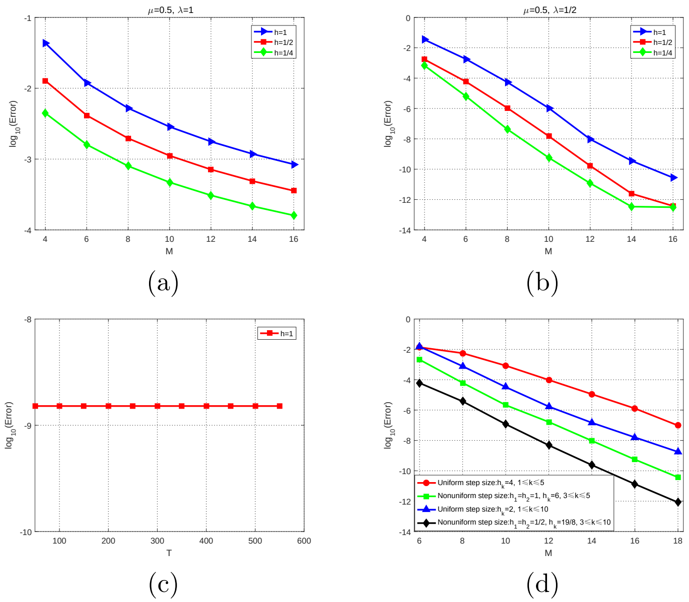


Figure 4. The H^1-errors and maximum point-wise errors of (4.4).

Next, we use new method to test (4.4) with large time interval $[0,T]$. In Figure 4 (c), we plot the maximum point-wise errors with $h=1, M=12, \lambda=\frac{1}{2}$ and various T. The error curve shows that the new method is accurate and stable for long time numerical simulation.

Finally, we provide the numerical results with variable steps to show the consistency between theoretical analysis and numerical results. In Figure 4 (d), we plot the maximum point-wise errors of (4.4) with $T=20, \lambda=\frac{1}{2}$, the uniform mode $M_k=M$, and the nonuniform step size distribution. More precisely, we take

(1) $N=5$, the uniform step size $h_k\equiv 4$;

(2) $N=5$, the nonuniform step size $h_1=h_2=1, h_k=6, 3\leq k\leq 5$;

(3) $N=10$, the uniform step size $h_k\equiv 2$;

(4) $N=10$, the nonuniform step size $h_1=h_2=\frac{1}{2}, h_k=\frac{19}{8}, 3\leq k\leq 10$.

Numerical results show that such a delicate mesh can obtain better approximation results than the simple uniform mesh.

5 Conclusions

In this paper, we constructed Müntz-Jacobi functions according to the singularity expansion of the solution. An *hp*-version spectral method combining Müntz-Jacobi functions and Legendre polynomials was proposed. The innovation of the approach is that Müntz-Jacobi functions are capable of capturing the singularity of the solution exactly. We applied the new method to nonlinear Volterra integro-differential equations with weakly singular kernels. Then we proved the existence and uniqueness of solution to the numerical scheme and derived the *hp*-version optimal convergence under some reasonable assumptions. Finally, we conducted numerical simulation on various models. Numerical results show that the *hp*-version spectral method is more effective than the traditional spectral method.

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Appendix A: Proof of Theorem 1

Proof. Consider the following iteration process $(m = 1, 2, \ldots)$:

$$\begin{cases} \left(\frac{d}{dt}Y^{1,(m)}, \varphi\right)_{\tilde{\omega}^{\alpha,\beta,\lambda}} + (Y^{1,(m)}, \varphi)_{\tilde{\omega}^{\alpha,\beta,\lambda}} = (f, \varphi)_{\tilde{\omega}^{\alpha,\beta,\lambda}} \\ \qquad + (\mathcal{V}_2^1 Y^{1,(m-1)}, \varphi)_{\tilde{\omega}^{\alpha,\beta,\lambda}}, \forall \varphi \in P_{M_1}^{\lambda,0}(I_1), \end{cases}$$

$$\begin{cases} \left(\frac{d}{dt}Y^{n,(m)},\psi\right)_{I_n} + (Y^{n,(m)},\psi)_{I_n} = (f,\psi)_{I_n} + (\mathcal{V}_1^n Y,\psi)_{I_n} \\ \qquad +(\mathcal{V}_2^n Y^{n,(m-1)},\psi)_{I_n}, \forall\psi\in\mathcal{P}_{M_n-1}(I_n),\ n\geq 2, \\ Y^{n,(m)}(t_{n-1}) = Y^{n-1,(m)}(t_{n-1}). \end{cases} \tag{5.1}$$

According to the definition of $\pi_{I_1,M_1}^{\alpha,\beta,\lambda}$ and π_{I_n,M_n}, we know from (5.1) that

$$\begin{cases} \frac{d}{dt}Y^{1,(m)} + Y^{1,(m)} = \pi_{I_1,M_1}^{\alpha,\beta,\lambda}(f+\mathcal{V}_2^1 Y^{1,(m-1)}), \\ \frac{d}{dt}Y^{n,(m)} + Y^{n,(m)} = \pi_{I_n,M_n}(f+\mathcal{V}_1^n Y + \mathcal{V}_2^n Y^{n,(m-1)}),\ n\geq 2. \end{cases} \tag{5.2}$$

Let $\widetilde{Y}^{n,(m)}(t) = Y^{n,(m)}(t) - Y^{n,(m-1)}(t)(n\geq 1)$. Obviously, $\widetilde{Y}^{n,(m)}(t_{n-1}) = 0$. We further get from (5.2) that

$$\begin{cases} \frac{d}{dt}\widetilde{Y}^{1,(m)} + \widetilde{Y}^{1,(m)} = \pi_{I_1,M_1}^{\alpha,\beta,\lambda}(\mathcal{V}_2^1 Y^{1,(m-1)} - \mathcal{V}_2^1 Y^{1,(m-2)}), \\ \frac{d}{dt}\widetilde{Y}^{n,(m)} + \widetilde{Y}^{n,(m)} = \pi_{I_n,M_n}(\mathcal{V}_2^n Y^{n,(m-1)} - \mathcal{V}_2^n Y^{n,(m-2)}),\ n\geq 2. \end{cases} \tag{5.3}$$

Set

$$U^{k,(m)}(t) = G(t,Y^{k,(m)}(t)) - G(t,Y^{k,(m-1)}(t)),\ t\in I_k,\ 1\leq k\leq n. \tag{5.4}$$

Using (5.3), (2.2), (5.4), Cauchy-Schwarz (C-S) inequality and (3.6) successively, we obtain that

$$\begin{aligned} &\left\|\frac{d}{dt}\widetilde{Y}^{1,(m)} + \widetilde{Y}^{1,(m)}\right\|_{I_1}^2 \\ &\overset{(5.3)}{=} \left\|\pi_{I_1,M_1}^{\alpha,\beta,\lambda}(\mathcal{V}_2^1 Y^{1,(m-1)} - \mathcal{V}_2^1 Y^{1,(m-2)})\right\|_{I_1}^2 \leq \left\|\mathcal{V}_2^1 Y^{1,(m-1)} - \mathcal{V}_2^1 Y^{1,(m-2)}\right\|_{I_1}^2 \\ &\overset{(2.2)}{=} \int_{I_1}\Big(\int_{t_0}^t (t-s)^{-\mu}K(t,s)(G(s,Y^{1,(m-1)}(s)) - G(s,Y^{1,(m-2)}(s)))ds\Big)^2 dt \\ &\overset{(5.4)}{\leq} c\int_{I_1}\Big(\int_{t_0}^t (t-s)^{-\mu}U^{1,(m-1)}(s)ds\Big)^2 dt \\ &\overset{\mathrm{C-S}}{\leq} c\int_{I_1}\Big(\int_{t_0}^t (t-s)^{-\mu}ds\int_{t_0}^t (t-s)^{-\mu}(U^{1,(m-1)}(s))^2 ds\Big)dt \\ &\leq ch_1^{1-\mu}\int_{I_1}\int_{t_0}^t (t-s)^{-\mu}(U^{1,(m-1)}(s))^2 dsdt \\ &\leq ch_1^{1-\mu}\int_{I_1}(U^{1,(m-1)}(s))^2\Big(\int_s^{t_1}(t-s)^{-\mu}dt\Big)ds \overset{(3.6)}{\leq} ch_1^{2-2\mu}\left\|\widetilde{Y}^{1,(m-1)}\right\|_{I_1}^2. \end{aligned} \tag{5.5}$$

Meanwhile, we know that

$$\begin{aligned} &\left\|\frac{d}{dt}\widetilde{Y}^{1,(m)} + \widetilde{Y}^{1,(m)}\right\|_{I_1}^2 \\ &= \left\|\frac{d}{dt}\widetilde{Y}^{1,(m)}\right\|_{I_1}^2 + \left\|\widetilde{Y}^{1,(m)}\right\|_{I_1}^2 + 2\int_{I_1}\frac{d}{dt}\widetilde{Y}^{1,(m)}(t)\cdot\widetilde{Y}^{1,(m)}(t)dt \\ &= \left\|\frac{d}{dt}\widetilde{Y}^{1,(m)}\right\|_{I_1}^2 + \left\|\widetilde{Y}^{1,(m)}\right\|_{I_1}^2 + (\widetilde{Y}^{1,(m)}(t_1))^2 \geq \left\|\widetilde{Y}^{1,(m)}\right\|_{I_1}^2. \end{aligned} \tag{5.6}$$

Thus, the combination of (5.5) and (5.6) gives that

$$\left\|\widetilde{Y}^{1,(m)}\right\|_{I_1}^2 \le ch_1^{2-2\mu}\left\|\widetilde{Y}^{1,(m-1)}\right\|_{I_1}^2.$$

Similarly, for $n \ge 2$, it holds that

$$\left\|\widetilde{Y}^{n,(m)}\right\|_{I_n}^2 \le ch_n^{2-2\mu}\left\|\widetilde{Y}^{n,(m-1)}\right\|_{I_n}^2.$$

Hence, for sufficiently small $h_{\max}^{2-2\mu}$, the existence and uniqueness are verified due to $\|\widetilde{Y}^{1,(m)}\|_{I_1} \to 0$ and $\|\widetilde{Y}^{n,(m)}\|_{I_n} \to 0$ as $m \to \infty$. □

Appendix B: Proof of Theorem 2

Proof. First, we know from (3.1) that

$$\begin{cases} Y_t^1 + Y^1 = \pi_{I_1,M_1}^{\alpha,\beta,\lambda} f + \pi_{I_1,M_1}^{\alpha,\beta,\lambda}\mathcal{V}_2^1 Y^1, \\ Y_t^n + Y^n = \pi_{I_n,M_n} f + \pi_{I_n,M_n}\mathcal{V}_1^n Y + \pi_{I_n,M_n}\mathcal{V}_2^n Y^n. \end{cases} \tag{5.7}$$

By subtracting (2.2) from (5.7), we obtain that

$$\begin{cases} y_t^1 - Y_t^1 + y^1 - Y^1 = f - \pi_{I_1,M_1}^{\alpha,\beta,\lambda} f + \mathcal{V}_2^1 y^1 - \pi_{I_1,M_1}^{\alpha,\beta,\lambda}\mathcal{V}_2^1 Y^1, \\ y_t^n - Y_t^n + y^n - Y^n = f - \pi_{I_n,M_n} f + \mathcal{V}_1^n y - \pi_{I_n,M_n}\mathcal{V}_1^n Y \\ \qquad\qquad +\mathcal{V}_2^n y^n - \pi_{I_n,M_n}\mathcal{V}_2^n Y^n, \qquad n \ge 2. \end{cases} \tag{5.8}$$

We also know from (2.2) that

$$\begin{cases} f - \pi_{I_1,M_1}^{\alpha,\beta,\lambda} f &= y_t^1 - \pi_{I_1,M_1}^{\alpha,\beta,\lambda} y_t^1 + y^1 - \pi_{I_1,M_1}^{\alpha,\beta,\lambda} y^1 + (\pi_{I_1,M_1}^{\alpha,\beta,\lambda} - \mathcal{I})\mathcal{V}_2^1 y^1, \\ f - \pi_{I_n,M_n} f &= y_t^n - \pi_{I_n,M_n} y_t^n + y^n - \pi_{I_n,M_n} y^n + (\pi_{I_n,M_n} - \mathcal{I})\mathcal{V}_1^n y \\ & \quad +(\pi_{I_n,M_n} - \mathcal{I})\mathcal{V}_2^n y^n, \qquad n \ge 2, \end{cases} \tag{5.9}$$

where $\mathcal{I}$ is the identity operator. The combination of (5.8) and (5.9) leads to

$$\begin{cases} e_1' + e_1 &= y_t^1 - \pi_{I_1,M_1}^{\alpha,\beta,\lambda} y_t^1 + y^1 - \pi_{I_1,M_1}^{\alpha,\beta,\lambda} y^1 + \pi_{I_1,M_1}^{\alpha,\beta,\lambda}(\mathcal{V}_2^1 y^1 - \mathcal{V}_2^1 Y^1), \\ e_n' + e_n &= y_t^n - \pi_{I_n,M_n} y_t^n + y^n - \pi_{I_n,M_n} y^n + \pi_{I_n,M_n}(\mathcal{V}_1^n y - \mathcal{V}_1^n Y) \\ & \quad +\pi_{I_n,M_n}(\mathcal{V}_2^n y^n - \mathcal{V}_2^n Y^n), \qquad n \ge 2. \end{cases}$$

Clearly, we derive from the Cauchy-Schwarz inequality that

$$\begin{cases} \|e_1' + e_1\|_{I_1}^2 \le 2\left\|y_t^1 - \pi_{I_1,M_1}^{\alpha,\beta,\lambda} y_t^1 + y^1 - \pi_{I_1,M_1}^{\alpha,\beta,\lambda} y^1\right\|_{I_1}^2 \\ \qquad +2\left\|\pi_{I_1,M_1}^{\alpha,\beta,\lambda}(\mathcal{V}_2^1 y^1 - \mathcal{V}_2^1 Y^1)\right\|_{I_1}^2 \le 2(D_1 + D_2), \\ \|e_n' + e_n\|_{I_n}^2 \le 2\left\|y_t^n - \pi_{I_n,M_n} y_t^n + y^n - \pi_{I_n,M_n} y^n\right\|_{I_n}^2 \\ +2\left\|\pi_{I_n,M_n}(\mathcal{V}_1^n y - \mathcal{V}_1^n Y) + \pi_{I_n,M_n}(\mathcal{V}_2^n y^n - \mathcal{V}_2^n Y^n)\right\|_{I_n}^2 \le 2(D_3 + D_4), \end{cases} \tag{5.10}$$

where

$$D_1 = \left\| y_t^1 - \pi_{I_1,M_1}^{\alpha,\beta,\lambda} y_t^1 + y^1 - \pi_{I_1,M_1}^{\alpha,\beta,\lambda} y^1 \right\|_{I_1}^2, \quad D_2 = \left\| \mathcal{V}_2^1 y^1 - \mathcal{V}_2^1 Y^1 \right\|_{I_1}^2,$$
$$D_3 = \| y_t^n - \pi_{I_n,M_n} y_t^n + y^n - \pi_{I_n,M_n} y^n \|_{I_n}^2,$$
$$D_4 = \| (\mathcal{V}_1^n y - \mathcal{V}_1^n Y) + (\mathcal{V}_2^n y^n - \mathcal{V}_2^n Y^n) \|_{I_n}^2. \tag{5.11}$$

Therefore, it is sufficient to estimate $D_j, 1 \le j \le 4$. First, for $t \in I_1$, if $-1 < \alpha < 0$, $-1 < \beta < \frac{1}{\lambda} - 1$, then we have that

$$(\tilde{\omega}^{\alpha,\beta,\lambda}(t))^{-1} = \frac{t_1}{\lambda}\Big(1 - \Big(\frac{t}{t_1}\Big)^{\lambda}\Big)^{-\alpha} \Big(\frac{t}{t_1}\Big)^{-(\beta+1)\lambda+1} \le ch_1. \tag{5.12}$$

According to (5.11), (5.12) and Lemma 1, we obtain that

$$\begin{aligned} D_1 &\overset{(5.11)}{\le} 2\left\| y_t^1 - \pi_{I_1,M_1}^{\alpha,\beta,\lambda} y_t^1 \right\|_{I_1}^2 + 2\left\| y^1 - \pi_{I_1,M_1}^{\alpha,\beta,\lambda} y^1 \right\|_{I_1}^2 \\ &\overset{(5.12)}{\le} ch_1 \int_{I_1} (y_t^1 - \pi_{I_1,M_1}^{\alpha,\beta,\lambda} y_t^1)^2 \tilde{\omega}^{\alpha,\beta,\lambda}(t)dt + ch_1 \int_{I_1} (y^1 - \pi_{I_1,M_1}^{\alpha,\beta,\lambda} y^1)^2 \tilde{\omega}^{\alpha,\beta,\lambda}(t)dt \\ &\le ch_1^{2m_1-1} M_1^{-2m_1+2} \left\| \partial_t^{m_1-1} \{ y_t^1(t^{\frac{1}{\lambda}}) \} \right\|_{\tilde{\omega}^{\alpha+m_1-1,\beta+m_1-1,1}}^2 \\ &\quad + ch_1^{2m_1+1} M_1^{-2m_1} \left\| \partial_t^{m_1} \{ y^1(t^{\frac{1}{\lambda}}) \} \right\|_{\tilde{\omega}^{\alpha+m_1,\beta+m_1,1}}^2. \end{aligned} \tag{5.13}$$

We set

$$\widetilde{U}^k(t) = G(t, y^k(t)) - G(t, Y^k(t)), \; t \in I_k, \; 1 \le k \le n. \tag{5.14}$$

By (5.11), (5.14), (3.6) and the Cauchy-Schwarz inequality, we derive that

$$\begin{aligned} D_2 &\overset{(5.11)}{=} \left\| \mathcal{V}_2^1 y^1 - \mathcal{V}_2^1 Y^1 \right\|_{I_1}^2 \overset{(5.14)}{=} \int_{I_1} \Big(\int_{t_0}^t (t-s)^{-\mu} K(t,s) \widetilde{U}^1(s) ds \Big)^2 dt \\ &\overset{\text{C-S}}{\le} c \int_{I_1} \Big(\int_{t_0}^t (t-s)^{-\mu} ds \Big) \Big(\int_{t_0}^t (t-s)^{-\mu} (\widetilde{U}^1(s))^2 ds \Big) dt \\ &\le ch_1^{1-\mu} \int_{I_1} (\widetilde{U}^1(s))^2 \Big(\int_s^{t_1} (t-s)^{-\mu} dt \Big) ds \overset{(3.6)}{\le} ch_1^{2-2\mu} \| e_1 \|_{I_1}^2. \end{aligned} \tag{5.15}$$

It is clear that each e_k satisfies that

$$\begin{cases} e_k^2(t_k) - e_k^2(t_{k-1}) = 2\int_{I_k} e_k'(t) e_k(t) dt \le \| e_k \|_{H^1(I_k)}^2, \\ e_k(t_{k-1}) = e_{k-1}(t_{k-1}), \quad e_1(t_0) = 0. \end{cases} \tag{5.16}$$

Summing up all these inequalities, we obtain

$$e_{n-1}^2(t_{n-1}) \le \sum_{k=1}^{n-1} \| e_k \|_{H^1(I_k)}^2. \tag{5.17}$$

By virtue of (5.10), (5.13), (5.15) and (5.16) with $k=1$, we can get that

$$\begin{aligned}&\|e_1'\|_{I_1}^2+(1-ch_1^{2-2\mu})\|e_1\|_{I_1}^2\\&\le ch_1^{2m_1-1}M_1^{-2m_1+2}\left\|\partial_t^{m_1-1}\{y_t^1(t^{\frac{1}{\lambda}})\}\right\|^2_{\tilde{\omega}^{\alpha+m_1-1,\beta+m_1-1,1}}\\&\quad+ch_1^{2m_1+1}M_1^{-2m_1}\left\|\partial_t^{m_1}\{y^1(t^{\frac{1}{\lambda}})\}\right\|^2_{\tilde{\omega}^{\alpha+m_1,\beta+m_1,1}}.\end{aligned}\tag{5.18}$$

We can find constants η_1 and η_2 such that

$$0<\eta_1\le 1-ch_n^{2-2\mu}\le\eta_2<1,\quad 1\le n\le N,\tag{5.19}$$

then we may rewrite (5.18) as

$$\begin{aligned}\|e_1\|^2_{H^1(I_1)}&\le ch_1^{2m_1-1}M_1^{-2m_1+2}\left\|\partial_t^{m_1-1}\{y_t^1(t^{\frac{1}{\lambda}})\}\right\|^2_{\tilde{\omega}^{\alpha+m_1-1,\beta+m_1-1,1}}\\&\quad+ch_1^{2m_1+1}M_1^{-2m_1}\left\|\partial_t^{m_1}\{y^1(t^{\frac{1}{\lambda}})\}\right\|^2_{\tilde{\omega}^{\alpha+m_1,\beta+m_1,1}}.\end{aligned}$$

On the other hand, according to Lemma 2, for any integer $1\le m_n\le M_n+1$,

$$\begin{aligned}D_3&\le 2\|y_t^n-\pi_{I_n,M_n}y_t^n\|_{I_n}^2+2\|y^n-\pi_{I_n,M_n}y^n\|_{I_n}^2\\&\le ch_n^{2m_n-2}M_n^{-2m_n+2}\|\partial_t^{m_n}y^n\|_{I_n}^2+ch_n^{2m_n}M_n^{-2m_n}\|\partial_t^{m_n}y^n\|_{I_n}^2\\&\le ch_n^{2m_n-2}M_n^{-2m_n+2}\|\partial_t^{m_n}y^n\|_{I_n}^2.\end{aligned}\tag{5.20}$$

By (5.11) and (2.2) we obtain that

$$D_4=\|\mathcal{V}_1^ny-\mathcal{V}_1^nY+\mathcal{V}_2^ny^n-\mathcal{V}_2^nY^n\|_{I_n}^2\le 2\|D_5\|_{I_n}^2+2\|D_6\|_{I_n}^2,$$

where

$$\begin{aligned}D_5&=\int_0^{t_{n-1}}(t-s)^{-\mu}K(t,s)(G(s,y(s))-G(s,Y(s)))ds,\\D_6&=\int_{t_{n-1}}^t(t-s)^{-\mu}K(t,s)(G(s,y^n(s))-G(s,Y^n(s)))ds.\end{aligned}$$

Thus, by (3.6) and the Cauchy-Schwarz inequality, for $0<\mu<\frac{1}{2}$, we obtain that

$$\begin{aligned}\|D_5\|_{I_n}^2&=\int_{I_n}\Big(\int_0^{t_{n-1}}(t-s)^{-\mu}K(t,s)(G(s,y(s))-G(s,Y(s)))ds\Big)^2dt\\&\overset{\text{C}-\text{S}}{\le} c\int_{I_n}\Big(\int_0^{t_{n-1}}(t-s)^{-2\mu}ds\Big)\Big(\int_0^{t_{n-1}}(G(s,y(s))-G(s,Y(s)))^2ds\Big)dt\\&\le c\int_{I_n}t^{1-2\mu}\Big(\int_0^{t_{n-1}}(G(s,y(s))-G(s,Y(s)))^2ds\Big)dt\\&\overset{(3.6)}{\le} c\int_{I_n}\int_0^{t_{n-1}}(y(s)-Y(s))^2dsdt\\&\le ch_n\int_0^{t_{n-1}}(y(s)-Y(s))^2ds\le ch_n\sum_{k=1}^{n-1}\|e_k\|_{I_k}^2.\end{aligned}$$

Similar to the estimate of D_2, we have that

$$\|D_6\|^2_{I_n} \le c h_n^{2-2\mu} \|e_n\|^2_{I_n} . \tag{5.21}$$

By virtue of (5.10), (5.17), (5.19) and (5.20)–(5.21), we can deduce that

$$\begin{aligned}\|e_n\|^2_{H^1(I_n)} \le & c\sum_{k=1}^{n-1} h_n \|e_k\|^2_{I_k} + c\sum_{k=1}^{n-1} \|e_k\|^2_{H^1(I_k)} + c h_n^{2m_n-2} M_n^{-2m_n+2} \|\partial_t^{m_n} y^n\|^2_{I_n} \\ \le & c\sum_{k=1}^{n-1} (1+h_n) \|e_k\|^2_{H^1(I_k)} + c h_n^{2m_n-2} M_n^{-2m_n+2} \|\partial_t^{m_n} y^n\|^2_{I_n} .\end{aligned}$$

Consequently, we obtain from Gronwall inequality that

$$\begin{aligned}\|e_n\|^2_{H^1(I_n)} \le & c(1+h_n)e^T \Big(h_1^{2m_1-1} M_1^{-2m_1+2} \left\|\partial_t^{m_1-1}\{y_t^1(t^{\frac{1}{\lambda}})\}\right\|^2_{\tilde{\omega}^{\alpha+m_1-1,\beta+m_1-1,1}} \\ & + h_1^{2m_1+1} M_1^{-2m_1} \left\|\partial_t^{m_1}\{y^1(t^{\frac{1}{\lambda}})\}\right\|^2_{\tilde{\omega}^{\alpha+m_1,\beta+m_1,1}} \Big) \\ & + c(1+h_n)e^T \sum_{k=2}^{n-1} h_k^{2m_k-2} M_k^{-2m_k+2} \left\|\partial_t^{m_k} y^k\right\|^2_{I_k} \\ & + c h_n^{2m_n-2} M_n^{-2m_n+2} \|\partial_t^{m_n} y^n\|^2_{I_n} .\end{aligned}$$

The proof is ended. □