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Simultaneous Identification of the Right-Hand Side and Time-Dependent Coefficients in a Two-Dimensional Parabolic Equation

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Abstract. This paper investigates the simultaneous identification of time-dependent lowest and source terms in a two-dimensional (2D) parabolic equation from the additional measurements. To investigate the solvability of the inverse problem, we first examine an auxiliary inverse boundary value problem and prove its equivalence to the original problem in a certain sense. Then, applying the contraction mappings principle existence and uniqueness of the solution of an equivalent problem is proved. Furthermore, using the equivalency, the existence and uniqueness theorem for the classical solution of the original problem is obtained and some discussions on the numerical solutions for this inverse problem are presented including numerical examples.

Keywords: inverse identification problem, 2D parabolic equation, Fourier method, classical solution, nonlinear optimization, Tikhonov regularization.

AMS Subject Classification: 35R30; 35K10; 35A09; 35A01; 35A02.

1 Introduction

In the past few decades a great deal of interest has been directed towards to the study of the problem of determining the coefficients and right-hand sides simultaneously in partial differential equations from some additional data. Such problems constitute a prominent branch of the theory of differential equations and are called inverse problems. Inverse problems arise in various fields of human activity, such as seismology, mineral exploration, biology, medical visualization, computed tomography, Earth remote sensing, spectral analysis, nondestructive control, etc. From historical background can be seen, that the foundations of the theory and practice of studying inverse problems were established and developed in the pioneering works of Tikhonov [34], Lavrentiev [25], Ivanov [20], and their followers (see, for example, [18, 26, 30, 32] and references therein).

One-dimensional nonlocal inverse problems for the various partial differential equations have been extensively studied, notably in [6, 21, 24, 28, 33]. Let us now browse the content of some works devoted to inverse coefficient problems for parabolic equations: in [2] an inverse problem for a 2D parabolic equation with an integral overdetermination condition is investigated. First, the considered problem was reduced to some auxiliary problem with trivial boundary conditions, and the existence and uniqueness of a solution to an equivalent problem applying the contraction mapping principle is shown. Then, using the equivalence, the existence and uniqueness theorem for the classical solution of the original problem is proved. The authors Baglan and Kanca [3] studied a coefficient problem for a quasi-linear 2D parabolic inverse problem with periodic boundary and integral overdetermination conditions. In the work, the existence, uniqueness, and continuously dependence upon the data of the solution is proved by iteration method. In the paper by Huntul and Lesnic [15] the retrieval of the timewise-dependent intensity of a free boundary and the temperature in a 2D parabolic problem is solved numerically for the first time. A stability theorem is proved based on the Green function theory and Volterra's integral equations of the second kind. The resulting nonlinear minimization is numerically solved using the MATLAB optimization routine. The determination of a time-dependent perfusion coefficient of 2D heat equation with a classical and total energy integral overdetermination condition was studied by Ismailov et al. [17]. In [19], Ivanchov and Kinash the conditions for the existence and uniqueness of a smooth solution to the inverse problem for the 2D heat equation with unknown leading coefficient depending on time and the space variable is studied. In the article published by Kinash [23] an inverse problem of identifying the time-dependent coefficient in a 2D parabolic equation is considered and the existence and uniqueness conditions for the classical solution to this problem are established. The work [31] discusses the existence of a positive radial solution to the generalized $p(x)$-Laplacian problem using the variational methods. In the paper [35] on the basis of the formula for solving the first initial-boundary problem for the inhomogeneous heat equation, inverse problems of finding the initial condition and the right-hand side are studied. For this problem, theorems of uniqueness, existence, and stability of solutions are proved.

Numerical aspects of inverse problems for 2D parabolic equations with various boundary conditions were studied in [8, 9], and the references therein. Huntul et al. [11, 12, 13, 14, 16], numerically investigated the inverse problems of reconstructing the timewise coefficients in a 2D parabolic equation.

A distinctive feature of the presented paper is the study of the inverse problem for a 2D parabolic equation with nonlocal boundary conditions in both space and time variables. It should be noted that in the literature, the term "nonlocal boundary value problems" refers to problems that contain conditions relating the values of the solution and/or its derivatives either at different points of the boundary or at boundary points and some interior points. The term "nonlocal conditions" and their classification were introduced by Dezin [10]. The applied significance of problems with nonlocal conditions is associated with the study of processes occurring in the turbulent plasma, the processes of diffusion, heat conductivity, moisture transfer in a capillary-porous medium, problems of mathematical biology, as well as some inverse problems of mathematical physics.

This article is outlined as: the 2D inverse problems have been stated in Section 2. The unique solvability of the inverse problems is proved in Section 3. The numerical discretization of the forward problem based on ADE scheme is briefly presented in Section 4. The numerical procedure for solving the inverse problems is given in Section 5. The results outcomes for benchmark test example are discussed in Section 6. Finally, the conclusions remarks are highlighted in Section 7.

2 Mathematical formulation of the problem

Let $T > 0$ be a fixed time moment and let $D_T = \bar{Q}_{xy} \times [0, T]$ denote a closed bounded region in space, where Q_{xy} is defined by the inequalities $0 < x < 1$ and $0 < y < 1$. We consider the problem of determining the unknown functions $u(x, y, t) \in C^{2,2,1}(D_T)$ and $a(t), b(t) \in C[0, T]$ such that the triple $\{u(x, y, t), a(t), b(t)\}$ satisfies in D_T, the following two-dimensional parabolic equation

$$\begin{aligned} &u_t(x, y, t) - c(t)\left(u_{xx}(x, y, t) + u_{yy}(x, y, t)\right) \\ &\quad = a(t)u(x, y, t) + b(t)g(x, y, t) + f(x, y, t),\ (x, y, t) \in D_T, \end{aligned} \tag{2.1}$$

with the nonlocal condition

$$u(x, y, 0) + \delta u(x, y, T) + \int_0^T p(t)u(x, y, t)dt = \varphi(x, y),\ (x, y) \in \bar{Q}_{xy}, \tag{2.2}$$

the boundary conditions

$$u(0, y, t) = u_x(1, y, t) = 0, \quad (y, t) \in [0, 1] \times [0, T], \tag{2.3}$$

$$u_y(x, 0, t) = u(x, 1, t) = 0, \quad (x, t) \in [0, 1] \times [0, T], \tag{2.4}$$

and over-specification conditions

$$u(1,0,t) = h_1(t), \quad t \in [0,T], \tag{2.5}$$

$$u(x_0,y_0,t) = h_2(t), \quad (x_0,y_0) \in Q_{xy}, \quad t \in [0,T], \tag{2.6}$$

where $\delta \geq 0$ is a given number and $0 < c(t)$, $0 \leq p(t)$, $f(x,y,t)$, $g(x,y,t)$, $\varphi(x,y)$, $h_1(t)$, and $h_2(t)$ are known functions.

The method used in Theorem 2.3 from [1], is easily adapted to prove the following theorem.

Theorem 1. *Assume that* $\delta \geq 0$, $0 < c(t) \in C[0,T]$, $0 \leq p(t) \in C[0,T]$, $f(x,y,t)$, $g(x,y,t) \in C(D_T)$, $\varphi(x,y) \in C(\bar{Q}_{xy})$, $h_1(t), h_2(t) \in C^1[0,T]$, $h(t) \equiv h_1(t)g(x_0,y_0,t) - h_2(t)g(1,0,t) \neq 0$, $0 \leq t \leq T$, $h(t) \in C[0,T]$, *and the compatibility conditions*

$$h_1(0) + \delta h_1(T) + \int_0^T p(t)h_1(t)dt = \varphi(1,0),$$

$$h_2(0) + \delta h_2(T) + \int_0^T p(t)h_2(t)dt = \varphi(x_0,y_0), \tag{2.7}$$

hold. Then the problem of finding a classical solution of (2.1)–(2.6) *is equivalent to the problem of determining the functions* $u(x,y,t) \in C^{2,2,1}(D_T)$, $a(t) \in C[0,T]$ *and* $b(t) \in C[0,T]$ *satisfying* (2.1)–(2.4), *and the conditions*

$$\begin{aligned} &h_1'(t) - c(t)(u_{xx}(1,0,t) + u_{yy}(1,0,t)) \\ &\quad = a(t)h_1(t) + b(t)g(1,0,t) + f(1,0,t),\ 0 \leq t \leq T, \end{aligned} \tag{2.8}$$

$$\begin{aligned} &h_2'(t) - c(t)(u_{xx}(x_0,y_0,t) + u_{yy}(x_0,y_0,t)) \\ &\quad = a(t)h_2(t) + b(t)g(x_0,y_0,t) + f(x_0,y_0,t),\ 0 \leq t \leq T. \end{aligned} \tag{2.9}$$

3 Classical solvability of IBV problem

We seek the first component of classical solution $\{u(x,y,t), a(t), b(t)\}$ of the problem (2.1)–(2.4), (2.8) and (2.9) in the form

$$u(x,y,t) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} u_{k,n}(t) \sin\lambda_k x \cos\gamma_n y, \tag{3.1}$$

where

$$u_{k,n}(t) = 4\int_0^1\int_0^1 u(x,y,t)\sin\lambda_k x\cos\gamma_n y dxdy,$$

$$\lambda_k = \frac{\pi}{2}(2k-1),\ \gamma_n = \frac{\pi}{2}(2n-1),\ k,n = 1,2,\ldots.$$

Applying the formal scheme of the Fourier method, from (2.1) and (2.2), we have

$$u_{k,n}'(t) + (\lambda_k^2 + \gamma_n^2)c(t)u_{k,n}(t) = F_{k,n}(t;u,a,b),\ 0 \leq t \leq T, \tag{3.2}$$

$$u_{k,n}(0) + \delta u_{k,n}(T) + \int_0^T p(t)u_{k,n}(t)dt = \varphi_{k,n},\ k,n = 1,2,\ldots, \tag{3.3}$$

where

$$F_{k,n}(t;u,a,b) = f_{k,n}(t) + a(t)u_{k,n}(t) + b(t)g_{k,n}(t),$$
$$f_{k,n}(t) = 4\int_0^1\int_0^1 f(x,y,t)\sin\lambda_k x\cos\gamma_n y dxdy,$$
$$g_{k,n}(t) = 4\int_0^1\int_0^1 g(x,y,t)\sin\lambda_k x\cos\gamma_n y dxdy,$$
$$\varphi_{k,n} = 4\int_0^1\int_0^1 \varphi(x,y)\sin\lambda_k x\cos\gamma_n y dxdy.$$

Solving problem (3.2), (3.3), we find

$$u_{k,n}(t) = \frac{\exp\left(-\int_0^t \mu_{k,n}^2 c(s)ds\right)}{1+\delta\exp\left(-\int_0^T \mu_{k,n}^2 c(s)ds\right)}\left(\varphi_{k,n} - \int_0^T p(t)u_{k,n}(t)dt\right)$$
$$+\int_0^t F_{k,n}(\tau;u,a,b)\exp\Big(-\int_\tau^t \mu_{k,n}^2 c(s)ds\Big)d\tau$$
$$-\frac{\delta\exp\big(-\int_0^T \mu_{k,n}^2 c(s)ds\big)}{1+\delta\exp\big(-\int_0^T \mu_{k,n}^2 c(s)ds\big)}\int_0^T F_{k,n}(\tau;u,a,b)e^{-\int\limits_\tau^t \mu_{k,n}^2 c(s)ds}d\tau, \quad (3.4)$$

where $\mu_{k,n}^2 = \lambda_k^2 + \gamma_n^2$. Substituting the expressions $u_{k,n}(t)$ $(k,n = 1,2,\ldots)$ described by (3.4) into (3.1), gives

$$u(x,y,t) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left\{\frac{\exp\big(-\int_0^t \mu_{k,n}^2 c(s)ds\big)}{1+\delta\exp\big(-\int_0^T \mu_{k,n}^2 c(s)ds\big)}\left(\varphi_{k,n} - \int_0^T p(t)u_{k,n}(t)dt\right)\right.$$
$$+\int_0^t F_{k,n}(\tau;u,a,b)\exp\left(-\int_\tau^t \mu_{k,n}^2 c(s)ds\right)d\tau - \frac{\delta\exp\big(-\int_0^T \mu_{k,n}^2 c(s)ds\big)}{1+\delta\exp\big(-\int_0^T \mu_{k,n}^2 c(s)ds\big)}$$
$$\left.\times\int_0^T F_{k,n}(\tau;u,a,b)\exp\left(-\int_\tau^t \mu_{k,n}^2 c(s)ds\right)d\tau\right\}\sin\lambda_k x\cos\gamma_n y. \quad (3.5)$$

Furthermore, taking into account $h(t)\neq 0$ and substituting the expressions represented by (3.1) into (2.8) and (2.9), and using and (3.4), we get

$$a(t) = [h(t)]^{-1}\Big\{(h_1'(t) - f(1,0,t))g(x_0,y_0,t) - (h_2'(t) - f(x_0,y_0,t))g(1,0,t)$$
$$+c(t)\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}((-1)^{k+1}g(x_0,y_0,t) - g(1,0,t)\sin\lambda_k x_0\cos\gamma_n y_0)$$
$$\times\Big[\frac{\exp\big(-\int_0^t \mu_{k,n}^2 c(s)ds\big)}{1+\delta\exp\big(-\int_0^T \mu_{k,n}^2 c(s)ds\big)}\Big(\varphi_{k,n} - \int_0^T p(t)u_{k,n}(t)dt\Big)$$
$$+\int_0^t F_{k,n}(\tau;u,a,b)e^{-\int\limits_\tau^t \mu_{k,n}^2 c(s)ds}d\tau$$

$$-\frac{\delta\exp\big(-\int_0^T\mu_{k,n}^2c(s)ds\big)}{1+\delta\exp\big(-\int_0^T\mu_{k,n}^2c(s)ds\big)}\int_0^T F_{k,n}(\tau;u,a,b)e^{-\int\limits_\tau^t\mu_{k,n}^2c(s)ds}d\tau\Big]\Big\}, \quad (3.6)$$

$$b(t)=[h(t)]^{-1}\Big\{(h_2'(t)-f(x_0,y_0,t))h_1(t)-(h_1'(t)-f(1,0,t))h_2(t)$$

$$+c(t)\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\big(h_1(t)\sin\lambda_kx_0\cos\gamma_ny_0-(-1)^{k+1}h_2(t)\big)$$

$$\times\Big[\frac{\exp\big(-\int_0^t\mu_{k,n}^2c(s)ds\big)}{1+\delta\exp\big(-\int_0^T\mu_{k,n}^2c(s)ds\big)}\Big(\varphi_{k,n}-\int_0^T p(t)u_{k,n}(t)dt\Big)$$

$$+\int_0^t F_{k,n}(\tau;u,a,b)e^{-\int\limits_\tau^t\mu_{k,n}^2c(s)ds}d\tau$$

$$-\frac{\delta\exp\big(-\int_0^T\mu_{k,n}^2c(s)ds\big)}{1+\delta\exp\big(-\int_0^T\mu_{k,n}^2c(s)ds\big)}\int_0^T F_{k,n}(\tau;u,a,b)e^{-\int\limits_\tau^t\mu_{k,n}^2c(s)ds}d\tau\Big]\Big\}. \quad (3.7)$$

Thus, the solution of problem (2.1)–(2.4), (2.8), (2.9) was reduced to the solution by systems (3.5)–(3.7) with respect to unknown functions $u(x,y,t)$, $a(t)$, and $b(t)$.

The following lemma plays an important role in studying the uniqueness of the solution to problem (2.1)–(2.4), (2.8), (2.9). But we omit the proof of the following lemma to avoid a lengthy digression.

Lemma 1. *If* $\{u(x,y,t),a(t),b(t)\}$ *is any solution of* (2.1)–(2.4), (2.8), (2.9), *then the functions*

$$u_{k,n}(t)=4\int_0^1\int_0^1 u(x,y,t)\sin\lambda_kx\cos\gamma_nydxdy, \quad k,n=1,2,\ldots,$$

satisfy the system (3.4) *on the interval* $[0,T]$.

It is clear that, if

$$u_{k,n}(t)=4\int_0^1\int_0^1 u(x,y,t)\sin\lambda_kx\cos\gamma_nydxdy,\ k,n=1,2,\ldots$$

is a solution to system (3.4), then the functions

$$u(x,y,t)=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}u_{k,n}(t)\sin\lambda_kx\cos\gamma_ny,$$

$a(t)$ and $b(t)$ will also be solution of system (3.5)–(3.7).

From Lemma 1 it follows:

Corollary 1. Suppose that system (3.5)–(3.7) has a unique solution. Then the problem (2.1)–(2.4), (2.8), (2.9) couldn't have more than one solution, in other words, if problem (2.1)–(2.4), (2.8), (2.9) has a solution, then it is unique.

In order to study the problem (2.1)–(2.4), (2.8), (2.9), we consider the following spaces. Let us consider the functional space $B_{2,T}^3$, introduced in the study by Khudaverdiyev and Veliyev [22], such that $B_{2,T}^3$ denotes a set of all functions of the form

$$u(x,y,t)=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}u_{k,n}(t)\sin\lambda_k x\cos\gamma_n y,$$
$$\lambda_k=\frac{\pi}{2}(2k-1),\ \gamma_n=\frac{\pi}{2}(2n-1),\quad k,n=1,2,\ldots,$$

considered in domain D_T. Moreover, the functions $u_{k,n}(t)$ $(k,n=1,2,\ldots)$ contained in last double sum are continuously differentiable on $[0,T]$, and the norm in the space $B_{2,T}^3$ is defined as follows

$$\|u(x,y,t)\|_{B_{2,T}^3}=\left\{\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(\mu_{k,n}^3\|u_{k,n}(t)\|_{C[0,T]})^2\right\}^{\frac{1}{2}}<+\infty.$$

Let E_T^3 denote the space consisting of the topological product $B_{2,T}^3\times C[0,T]\times C[0,T]$, which is the norm of the element $z=\{u,a,b\}$ described by the formula

$$\|z\|_{E_T^3}=\|u(x,y,t)\|_{B_{2,T}^3}+\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}.$$

It is known that [22,27] the spaces $B_{2,T}^3$ and E_T^3 are Banach spaces.

Let us now consider the operator

$$\Phi(u,a,b)=\{\Phi_1(u,a,b),\Phi_2(u,a,b),\Phi_3(u,a,b)\}$$

in the space E_T^3, where

$$\Phi_1(u,a,b)=\tilde{u}(x,y,t)\equiv\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\tilde{u}_{k,n}(t)\cos\lambda_k x\sin\gamma_n y,$$
$$\Phi_2(u,a,b)=\tilde{a}(t),\quad \Phi_3(u,a,b)=\tilde{b}(t),$$

and the functions $\tilde{u}_{k,n}(t)$ $(k,n=1,2,...)$, $\tilde{a}(t)$, and $\tilde{b}(t)$ are equal to the right-hand sides of (3.5), (3.6) and (3.7), respectively.

If the data for the problem (2.1)–(2.4), (2.8) and (2.9) meets the following four conditions

(C_1) $\varphi(x,y),\varphi_x(x,y),\varphi_{xx}(x,y),\varphi_y(x,y),\varphi_{xy}(x,y),\varphi_{yy}(x,y)\in C(\bar{Q}_{xy})$,
$\varphi_{xxy}(x,y),\varphi_{xyy}(x,y),\varphi_{xxx}(x,y),\varphi_{yyy}(x,y)\in L_2(Q_{xy})$,
$\varphi(0,y)=\varphi_x(1,y)=\varphi_{xx}(0,y)=0,\ 0\le y\le 1$,
$\varphi_y(x,0)=\varphi(x,1)=\varphi_{yy}(x,1)=0,\ 0\le x\le 1$;

(C_2) $f(x,y,t),f_x(x,y,t),f_{xx}(x,y,t),f_y(x,y,t),f_{xy}(x,y,t),f_{yy}(x,y,t)\in C(D_T)$,
$f_{xxy}(x,y,t),f_{xyy}(x,y,t),f_{xxx}(x,y,t),f_{yyy}(x,y,t)\in L_2(D_T)$,
$f(0,y,t)=f_x(1,y,t)=f_{xx}(0,y,t)=0,\ 0\le y\le 1,\ 0\le t\le T$,
$f_y(x,0,t)=f(x,1,t)=f_{yy}(x,1,t)=0,\ 0\le x\le 1,\ 0\le t\le T$;

(C_3) $g(x,y,t), g_x(x,y,t), g_{xx}(x,y,t), g_y(x,y,t), g_{xy}(x,y,t), g_{yy}(x,y,t) \in C(D_T)$,
$g_{xxy}(x,y,t), g_{xyy}(x,y,t), g_{xxx}(x,y,t), g_{yyy}(x,y,t) \in L_2(D_T)$,
$g(0,y,t) = g_x(1,y,t) = g_{xx}(0,y,t) = 0,\ 0 \le y \le 1,\ 0 \le t \le T$,
$g_y(x,0,t) = g(x,1,t) = g_{yy}(x,1,t) = 0,\ 0 \le x \le 1,\ 0 \le t \le T$;

(C_4) $\delta \ge 0, 0 \le p(t) \in C[0,T], 0 < c(t) \in C[0,T], h_i(t) \in C^1[0,T]\ (i=1,2)$,
$h(t) \equiv h_1(t)g(x_0,y_0,t) - h_2(t)g(1,0,t) \ne 0,\ 0 \le t \le T,\ h(t) \in C[0,T]$,

then we obtain

$$\begin{aligned}\|\tilde{u}(x,y,t)\|_{B^3_{2,T}} &\le A_1(T) + B_1(T)\|a(t)\|_{C[0,T]}\|u(x,y,t)\|_{B^3_{2,T}} \\ &\quad + C_1(T)\|b(t)\|_{C[0,T]} + D_1(T)\|u(x,y,t)\|_{B^3_{2,T}}, \qquad (3.8)\end{aligned}$$

$$\begin{aligned}\|\tilde{a}(t)\|_{C[0,T]} &\le A_2(T) + B_2(T)\|a(t)\|_{C[0,T]}\|u(x,y,t)\|_{B^3_{2,T}} \\ &\quad + C_2(T)\|b(t)\|_{C[0,T]} + D_2(T)\|u(x,y,t)\|_{B^3_{2,T}}, \qquad (3.9)\end{aligned}$$

$$\begin{aligned}\|\tilde{b}(t)\|_{C[0,T]} &\le A_3(T) + B_3(T)\|a(t)\|_{C[0,T]}\|u(x,y,t)\|_{B^3_{2,T}} \\ &\quad + C_3(T)\|b(t)\|_{C[0,T]} + D_3(T)\|u(x,y,t)\|_{B^3_{2,T}}, \qquad (3.10)\end{aligned}$$

where

$$\begin{aligned}
A_1(T) &= \sqrt{11}\|\varphi_{xxx}(x,y)\|_{L_2(Q_{xy})} + \sqrt{11}\|\varphi_{xyy}(x,y)\|_{L_2(Q_{xy})} \\
&\quad + \sqrt{11}\|\varphi_{xxy}(x,y)\|_{L_2(Q_{xy})} + \sqrt{11}\|\varphi_{yyy}(x,y)\|_{L_2(Q_{xy})} \\
&\quad + (1+\delta)\sqrt{11T}(\|f_{xxx}(x,y,t)\|_{L_2(D_T)} + \|f_{xyy}(x,y,t)\|_{L_2(D_T)} \\
&\quad + \|f_{xxy}(x,y,t)\|_{L_2(D_T)} + \|f_{xxx}(x,y,t)\|_{L_2(D_T)} + \|f_{yyy}(x,y,t)\|_{L_2(D_T)}), \\
B_1(T) &= \sqrt{11}(1+\delta)T, \\
C_1(T) &= (1+\delta)\sqrt{11T}(\|g_{xxx}(x,y,t)\|_{L_2(D_T)} + \|g_{xyy}(x,y,t)\|_{L_2(D_T)} \\
&\quad + \|g_{xxy}(x,y,t)\|_{L_2(D_T)} + \|g_{xxx}(x,y,t)\|_{L_2(D_T)} + \|g_{yyy}(x,y,t)\|_{L_2(D_T)}), \\
D_1(T) &= \sqrt{11}T\|p(t)\|_{C[0,T]}, \\
A_2(T) &= \|[h(t)]^{-1}\|_{C[0,T]} \\
&\quad \times \{\|(h_1'(t) - f(1,0,t))g(x_0,y_0,t) - (h_2'(t) - f(x_0,y_0,t))g(1,0,t)\|_{C[0,T]} \\
&\quad \times \{\|(h_1'(t) - f(1,0,t))g(x_0,y_0,t) - (h_2'(t) - f(x_0,y_0,t))g(1,0,t)\|_{C[0,T]} \\
&\quad + \|c(t)(|g(x_0,y_0,t)| + |g(1,0,t)|)\|_{C[0,T]} \\
&\quad \times \left(\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\mu_{k,n}^{-2}\right)^{\frac{1}{2}} \left[\|\varphi_{xxx}(x,y)\|_{L_2(Q_{xy})} + \|\varphi_{xyy}(x,y)\|_{L_2(Q_{xy})}\right. \\
&\quad + \|\varphi_{xxy}(x,y)\|_{L_2(Q_{xy})} + \|\varphi_{yyy}(x,y)\|_{L_2(Q_{xy})} \\
&\quad + (1+\delta)\sqrt{T}(\|f_{xxx}(x,y,t)\|_{L_2(D_T)} + \|f_{xyy}(x,y,t)\|_{L_2(D_T)} \\
&\quad + \|f_{xyy}(x,y,t)\|_{L_2(D_T)} + \|f_{xxy}(x,y,t)\|_{L_2(D_T)} \\
&\quad \left. + \|f_{xxx}(x,y,t)\|_{L_2(D_T)} + \|f_{yyy}(x,y,t)\|_{L_2(D_T)})\right]\}, \\
B_2(T) &= \|[h(t)]^{-1}\|_{C[0,T]}\|c(t)(|g(x_0,y_0,t)| + |g(1,0,t)|)\|_{C[0,T]}
\end{aligned}$$

$$\times \left(\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\mu_{k,n}^{-2}\right)^{\frac{1}{2}} (1+\delta)T,$$

$$C_2(T) = \|[h(t)]^{-1}\|_{C[0,T]}\|c(t)(|g(x_0,y_0,t)| + |g(1,0,t)|)\|_{C[0,T]}$$

$$\times \left(\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\mu_{k,n}^{-2}\right)^{\frac{1}{2}} (1+\delta)\sqrt{T}(\|g_{xxx}(x,y,t)\|_{L_2(D_T)} + \|g_{xyy}(x,y,t)\|_{L_2(D_T)}$$

$$+ \|g_{xxy}(x,y,t)\|_{L_2(D_T)} + \|g_{xxx}(x,y,t)\|_{L_2(D_T)} + \|g_{yyy}(x,y,t)\|_{L_2(D_T)})]\},$$

$$D_2(T) = \|[h(t)]^{-1}\|_{C[0,T]}$$

$$\times \|c(t)(|g(x_0,y_0,t)| + |g(1,0,t)|)\|_{C[0,T]} \left(\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\mu_{k,n}^{-2}\right)^{\frac{1}{2}} T\|p(t)\|_{C[0,T]},$$

$$A_3(T) = \|[h(t)]^{-1}\|_{C[0,T]}$$

$$\times \{\|(h_2'(t) - f(x_0,y_0,t))h_1(t) - (h_1'(t) - f(1,0,t))h_2(t)\|_{C[0,T]}$$

$$+ \|c(t)(|h_1(t)| + |h_2(t)|)\|_{C[0,T]} \left(\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\mu_{k,n}^{-2}\right)^{\frac{1}{2}}$$

$$\times \left[\|\varphi_{xxx}(x,y)\|_{L_2(Q_{xy})} + \|\varphi_{xyy}(x,y)\|_{L_2(Q_{xy})} + \|\varphi_{xxy}(x,y)\|_{L_2(Q_{xy})}\right.$$

$$+ \|\varphi_{yyy}(x,y)\|_{L_2(Q_{xy})} + (1+\delta)\sqrt{T}(\|f_{xxx}(x,y,t)\|_{L_2(D_T)}$$

$$+ \|f_{xyy}(x,y,t)\|_{L_2(D_T)} + \|f_{xxy}(x,y,t)\|_{L_2(D_T)}$$

$$+ \|f_{xxx}(x,y,t)\|_{L_2(D_T)} + \|f_{yyy}(x,y,t)\|_{L_2(D_T)})]\},$$

$$B_3(T) = \|[h(t)]^{-1}\|_{C[0,T]}\|c(t)(|h_1(t)| + |h_2(t)|)\|_{C[0,T]}$$

$$\times \left(\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\mu_{k,n}^{-2}\right)^{\frac{1}{2}} (1+\delta)T,$$

$$C_3(T) = \|[h(t)]^{-1}\|_{C[0,T]}\|c(t)(|h_1(t)| + |h_2(t)|)\|_{C[0,T]} \left(\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\mu_{k,n}^{-2}\right)^{\frac{1}{2}}$$

$$\times (1+\delta)\sqrt{T}(\|g_{xxx}(x,y,t)\|_{L_2(D_T)} + \|g_{xyy}(x,y,t)\|_{L_2(D_T)}$$

$$+ \|g_{xxy}(x,y,t)\|_{L_2(D_T)} + \|g_{xxx}(x,y,t)\|_{L_2(D_T)} + \|g_{yyy}(x,y,t)\|_{L_2(D_T)})]\},$$

$$D_3(T) = \|[h(t)]^{-1}\|_{C[0,T]}\|c(t)(|h_1(t)| + |h_2(t)|)\|_{C[0,T]}$$

$$\times \left(\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\mu_{k,n}^{-2}\right)^{\frac{1}{2}} T\|p(t)\|_{C[0,T]}.$$

From inequalities (3.8)–(3.10), we conclude

$$\|\tilde{u}(x,y,t)\|_{B^3_{2,T}} + \|\tilde{a}(t)\|_{C[0,T]} + \|\tilde{b}(t)\|_{C[0,T]} \leq A(T) + B(T)$$
$$\times \|a(t)\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}} + C(T)\|b(t)\|_{C[0,T]} + D(T)\|u(x,t)\|_{B^3_{2,T}},$$

where

$$A(T) = A_1(T) + A_2(T) + A_3(T), \quad B(T) = B_1(T) + B_2(T) + B_3(T),$$

$$C(T) = C_1(T) + C_2(T) + C_3(T), \quad D(T) = D_1(T) + D_2(T) + D_3(T).$$

Thus, the following theorem is valid.

Theorem 2. *Let the conditions* (C_1)–(C_4) *and*

$$\big(B(T)(A(T)+2) + C(T) + D(T)\big)(A(T)+2) < 1 \tag{3.11}$$

be fulfilled. Then, problem (2.1)–(2.4), (2.8), (2.9) *has a unique solution in the ball* $K = K_R(\|z\|_{E_T^3} \leq R = A(T)+2)$ *of the space* E_T^3.

Remark 1. Inequality (3.11) is satisfied for sufficiently small values of T.

Proof. Consider the following operator equation

$$z = \Phi z, \tag{3.12}$$

in the space E_T^3, for which $z = \{u, a, b\}$ and the components $\Phi_i(u, a, b)$ $(i = 1, 2, 3)$ of operator $\Phi(u, a, b)$ defined by the right side of Equations (3.5), (3.6), (3.7), respectively. Then we obtain that for any $z, z_1, z_2 \in K_R$ the following inequalities hold

$$\begin{aligned} \|\Phi z\|_{E_T^3} \leq\ & A(T) + B(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B_{2,T}^3} \\ & + C(T)\|b(t)\|_{C[0,T]} + D(T)\|u(x,t)\|_{B_{2,T}^3}, \end{aligned} \tag{3.13}$$

$$\begin{aligned} \|\Phi z_1 - \Phi z_2\|_{E_T^3} \leq\ & B(T)R(\|u_1(x,y,t) - u_2(x,y,t)\|_{B_{2,T}^3} + \|a_1(t) - a_2(t)\|_{C[0,T]}) \\ & + C(T)\|b_1(t) - b_2(t)\|_{C[0,T]} + D(T)\|u_1(x,y,t) - u_2(x,y,t)\|_{B_{2,T}^3}. \end{aligned} \tag{3.14}$$

So, by (3.11), from estimates (3.13) and (3.14) it is clear that the operator Φ acts in the ball $K = K_R$, and is contractive. Therefore, the operator Φ has a unique fixed point in the ball $K = K_R$, which is a unique solution of Equation (3.12), i.e., $\{u, a, b\}$ is a unique solution of the systems (3.5)–(3.7) in the ball $K = K_R$. Thus, we obtain that the function $u(x, y, t)$, as an element of the space $B_{2,T}^3$ is continuous and has continuous derivatives $u_x(x,y,t), u_{xx}(x,y,t), u_y(x,y,t), u_{xy}(x,y,t), u_{yy}(x,y,t)$ in D_T.

Hence, it is easy to show that $u_t(x, y, t)$ is also continuous in the region D_T. Furthermore, it is not hard to verify that Equation (2.1) and conditions (2.2)–(2.4), (2.8), (2.9) are satisfied in the usual sense. Consequently, $\{u(x,y,t), a(t), b(t)\}$ is a solution of problem (2.1)–(2.4), (2.8), (2.9), and by Lemma 1 this solution is unique in the ball $K = K_R$. □

In summary, from Theorem 1 and Theorem 2 we conclude the following main result.

Theorem 3. *Assume that the hypotheses of Theorem 2 and compatibility conditions* (2.7) *hold. Then problem* (2.1)–(2.6) *has a unique classical solution in the ball* $K = K_R$ *of space* E_T^3.

4 Numerical solution of the direct problem

The discretization of the forward problem (2.1)–(2.4) is considered in this section when $a(t)$, $b(t)$, $c(t)$, $g(x,y,t)$, $\varphi(x,y)$ and $f(x,y,t)$ are given functions, and the solution $u(x,y,t)$ is to be identified. We divide the domain Q_T into a mesh of the equal length by M_1, M_2, N, and $\Delta x = 1/M_1$, $\Delta y = 1/M_2$, $\Delta t = T/N$. To represent the forward problem, we denote $u^n_{i,j} := u(x_i, y_j, t_n)$, where $x_i = i\Delta x$, $y_j = j\Delta y$, $t_n = n\Delta t$, $a^n := a(t_n)$, $b^n := b(t_n)$, $p^n := p(t_n)$, $c^n := c(t_n)$, $g^n_{i,j} := g(x_i, y_j, t_n)$, $f^n_{i,j} := f(x_i, y_j, t_n)$ for $i = \overline{0, M_1}$, $j = \overline{0, M_2}$, $n = \overline{0, N}$. The ADE scheme [4,5,29] is used to discretize Equation (2.1), which is an unconditionally stable method, as follows.

Let $z^n_{i,j}$ and $v^n_{i,j}$ be the solutions of the following equations:

$$\begin{aligned} z^{n+1}_{i,j} &= A^n z^n_{i,j} + B^n(z^n_{i+1,j} + z^{n+1}_{i-1,j}) + C^n(z^n_{i,j+1} + z^{n+1}_{i,j-1}) + D^n(b^n g^n_{i,j} + f^n_{i,j}),\\ &\quad i = \overline{1, M_1 - 1}, \quad j = \overline{1, M_2 - 1}, \quad n = \overline{0, N},\\ v^{n+1}_{i,j} &= A^n v^n_{i,j} + B^n(v^{n+1}_{i+1,j} + v^n_{i-1,j}) + C^n(v^{n+1}_{i,j+1} + v^n_{i,j-1}) + D^n(b^n g^n_{i,j} + f^n_{i,j}),\\ &\quad i = \overline{M_1 - 1, 1}, \quad j = \overline{M_2 - 1, 1}, \quad n = \overline{0, N}, \end{aligned}$$

where

$$\begin{aligned} A^n &= \frac{1-\lambda^n}{1+\lambda^n}, \quad B^n = \frac{(\Delta t)c^n}{(\Delta x)^2(1+\lambda^n)}, \quad C^n = \frac{(\Delta t)c^n}{(\Delta y)^2(1+\lambda^n)},\\ D^n &= \frac{\Delta t}{1+\lambda^n}, \quad \lambda^n = \Delta t\left(\frac{c^n}{(\Delta x)^2} + \frac{c^n}{(\Delta y)^2} - \frac{a^n}{2}\right). \end{aligned}$$

The conditions (2.2)–(2.4) are given as:

$$\begin{aligned} &z^0_{i,j} + \delta z^T_{i,j} + \int_0^T p^n z(x,y,t)dt = v^0_{i,j} + \delta v^T_{i,j} + \int_0^T p^n v(x,y,t)dt = \varphi(x_i, y_j),\\ &\quad i = \overline{0, M_1}, \quad j = \overline{0, M_2}. \end{aligned} \tag{4.1}$$

In Equation (4.1), the integral is calculated using trapezoidal rule as follows:

$$\begin{aligned} \int_0^T p^n z(x_i, y_j, t)dt \approx \frac{1}{2N}\left(p^0 z^0_{i,j} + 2\sum_{n=0}^{N-1} p^n z^n_{i,j} + p^T z^T_{i,j}\right),\\ i = \overline{0, M_1},\ j = \overline{0, M_2}, \end{aligned}$$

and similar expression is used for $\int_0^T p^n v(x,y,t)dt$,

$$z^n_{0,j} = z^n_{x_{M_1,j}} = v^n_{0,j} = v^n_{x_{M_1,j}} = 0, \quad j = \overline{0, M_2}, \quad n = \overline{1, N}, \tag{4.2}$$

$$z^n_{y_{i,0}} = z^n_{i,M_2} = v^n_{y_{i,0}} = v^n_{i,M_2} = 0, \quad i = \overline{0, M_1}, \quad n = \overline{1, N}. \tag{4.3}$$

In expressions (4.2) and (4.3), the derivative of $z^n_{x_{M_1,j}}$, $z^n_{y_{i,0}}$ is approximated, for simplicity, using forward finite differences as

$$z^n_{x_{M_1,j}} = \frac{z^{n+1}_{M_1+1,j} - z^{n+1}_{M_1,j}}{\Delta x}, \quad z^n_{y_{i,0}} = \frac{z^{n+1}_{i,1} - z^{n+1}_{i,0}}{\Delta y},$$

and similar expression is used for $v^n_{x_{M_1},j}$ and $v^n_{y_{i,0}}$. These values are then replaced into the simple arithmetic mean approximation

$$u^{n+1}_{i,j} = \frac{z^{n+1}_{i,j} + v^{n+1}_{i,j}}{2}.$$

5 Numerical solution of the inverse problem

The numerical solution of (2.1)–(2.6) is obtained by minimizing

$$F(\underline{a},\underline{b}) = \sum_{n=1}^{N} \Big[u(1,0,t_n) - h_1(t_n)\Big]^2 + \sum_{n=1}^{N} \Big[u(x_0,y_0,t_n) - h_2(t_n)\Big]^2, \quad (5.1)$$

where $t_n = n\Delta t$, $\Delta t = T/N$, N is the number of time steps and $u(x,y,t)$ solves numerically using the ADE scheme [4, 5, 29] the direct problem (2.1)–(2.4) for given $a(t)$ and $b(t)$. The minimization of (5.1) is accomplished using the *lsqnonlin* subroutine in MATLAB, which does not require supplying by the user the gradient of the objective function. This routine attempts to find the minimum of a sum of squares by starting from the initial guesses and marching to the next iterate according to a trust region reflective search method [7].

6 Numerical results

The approximation solutions for $a(t)$, $b(t)$ and $u(x,y,t)$ are constructed in this section. We use

$$\mathrm{RMSE}(a) = \left[\frac{T}{N}\sum_{n=1}^{N}\Big(a^{Numerical}(t_n) - a^{Exact}(t_n)\Big)^2\right]^{1/2}, \quad (6.1)$$

$$\mathrm{RMSE}(b) = \left[\frac{T}{N}\sum_{n=1}^{N}\Big(b^{Numerical}(t_n) - b^{Exact}(t_n)\Big)^2\right]^{1/2}, \quad (6.2)$$

for measuring the accuracy. Now, we choose $T = 1$, for simplicity. The lower bound for $a(t)$, $b(t)$ is selected as -10^2 while 10^2 for the upper bound.

To measure the errors in this data, the $h_1(t_n)$, $h_2(t_n)$, in (5.1) is substituted by perturbed data $h_1^{\epsilon 1}(t_n)$, $h_2^{\epsilon 1}(t_n)$, as follows:

$$h_1^{\epsilon 1}(t_n) = h_1(t_n) + \epsilon 1_n, \quad h_2^{\epsilon 2}(t_n) = h_2(t_n) + \epsilon 2_n, \quad n = \overline{1,N}, \quad (6.3)$$

where $\epsilon 1_n$ and $\epsilon 2_n$ are are random variables with mean zero and with S.D. The standard deviations σ_1 and σ_2 are taken as:

$$\sigma_1 = \max_{t\in[0,T]} |h_1(t)| \times p, \quad \sigma_2 = \max_{t\in[0,T]} |h_2(t)| \times p, \quad (6.4)$$

where p represents the noise.

Let us investigate the problem proposed in Equations (2.1)–(2.6) with unknown functions $a(t)$ and $b(t)$, with:

$$\begin{aligned}
&\delta = 0,\ \ p(t) = 0,\ \ \varphi(x,y) = -x^5(x-2)^5(y^2-1)^5, u(0,y,t) = u_x(1,y,t) = 0,\\
&c(t)=\frac{1+t}{300},\ u_y(x,0,t) = u(x,1,t) = 0,\quad g(x,y,t) = x^3y^3(x-1)^3(y-1)^3e^t,\\
&f(x,y,t) = \frac{1}{30}e^tx^3\Big[-30(1+t)(x-1)^3(y-1)^3y^3 - 30(x-2)^5x^2(y^2-1)^5\\
&\qquad + 30t(x-2)^5x^2(y^2-1)^5 + (1+t)(x-2)^3(-1+y^2)^3(x^3(4-36y^2)\\
&\qquad + 8(y^2-1)^2 - 18x(y^2-1)^2 + x^4(9y^2-1) + x^2(5+18y^2+9y^4))\Big],\\
&h_1(t) = u(1,0,t) = -e^t, \qquad\qquad (6.5)\\
&h_2(t) = u(x_0,y_0,t) = -59049e^t/1048576,\ \ x_0 = 0.5,\ \ y_0 = 0.5,
\end{aligned}$$

where

$$h(t) = h_1(t)g(x_0,y_0,t) - h_2(t)g(1,0,t) = \frac{59049e^{2t}}{4294967296} \neq 0,\ \forall t \in [0,1].$$

It is shown that the criteria of Theorem 1 is fulfilled, indicating that a unique solution is observed. The analytical solution is considered as

$$u(x,y,t) = -x^5(x-2)^5(y^2-1)^5e^t,\quad (x,y,t) \in D_T, \qquad (6.6)$$

$$a(t) = t,\quad b(t) = 1+t,\quad t \in [0,1]. \qquad (6.7)$$

First, when $a(t)$, $b(t)$ is supplied by (6.7) the accuracy of (2.1)–(2.4) is tested using the data (6.5). Figure 1 shows the analytical (6.6) and estimated $u(x,y,t)$, as well as absolute errors, for various grid sizes.

Also, the RMSE defined by

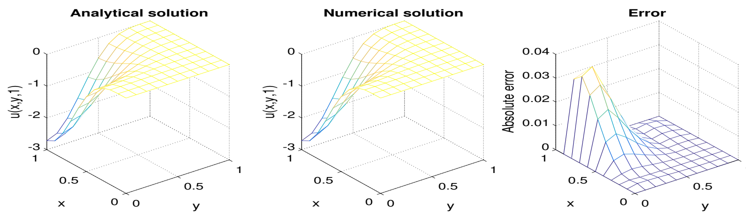
$$\text{RMSE}(h_1) = \left[\frac{1}{N}\sum_{n=1}^{N}\left(h_1^{numerical}(t_n) - h_1^{exact}(t_n)\right)^2\right]^{1/2}, \qquad (6.8)$$

$$\text{RMSE}(h_2) = \left[\frac{1}{N}\sum_{n=1}^{N}\left(h_2^{numerical}(t_n) - h_2^{exact}(t_n)\right)^2\right]^{1/2}, \qquad (6.9)$$

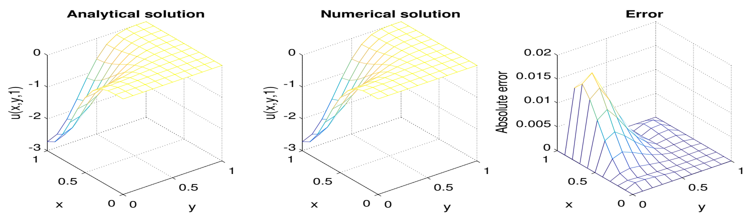
indicated in Table 1 for estimated $h_1(t)$, $h_2(t)$, shows more clearly their decreases as the grid size becomes smaller.

Table 1. The RMSE values ((6.8) and (6.9)) for $h_1(t)$ and $h_2(t)$, with $M_1 = M_2 = 10$ and with various $N \in \{10, 20, 40, 80\}$ for forward problem.

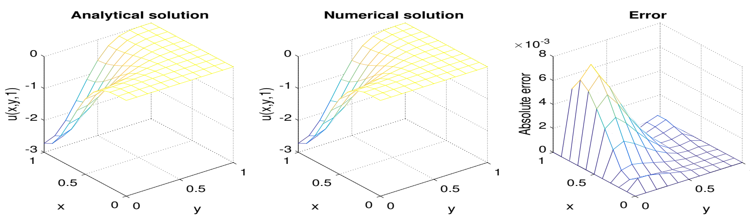
$M_1 = M_2 = 10$	RMSE(h_1)	RMSE(h_2)
$N = 10$	0.001002	0.001953
$N = 20$	3.0359E-3	0.001116
$N = 40$	7.4728E-4	2.7236E-3
$N = 80$	1.0135E-4	5.7539E-4



(a)



(b)



(c)

Figure 1. The analytical (6.6) and approximate $u(x, y, 1)$, with absolute errors for $\Delta x = \Delta y = \frac{1}{10}$ and (a) $\Delta t = \frac{1}{10}$, (b) $\Delta t = \frac{1}{20}$ and (c) $\Delta t = \frac{1}{40}$, for forward problem.

Now, we investigate the IP. We fix $M_1 = M_2 = 10$ and $N = 40$ and start the investigation for reconstructing the functions $a(t)$, $b(t)$ and $u(x, y, t)$ in absence of noise in the measured data (6.3). The initial guesses for $\underline{a}$ and $\underline{b}$ are chosen as follows:

$$a^0(t_n) = a(0) = 0, \quad b^0(t_n) = b(0) = 1, \quad n = \overline{1, N}.$$

The cost function (5.1) is depicted in Figure 2(a), where a monotonically decreasing convergence is achieved in 5 for a prescribed tolerance of $O(10^{-32})$. The exact (6.7) and estimated functions $a(t)$ and $b(t)$ are portrayed in Figures 2(b) and 2(c). It is observed that the numerical outcomes are accurate with RMSE(a) =1.3012E-4 and RMSE(b) =0.009159. It should be noted that in case of exact data, i.e., $p = 0$, no regularization was needed to penalise the cost function (5.1). Nevertheless, for higher noise the instability in recovering the functions $a(t)$ and $b(t)$ will become obvious and regularization would need to be applied.

Next, we associate 0.1%, 1% noise with the simulated data (2.5) and (2.6), as in Equation (6.4). It is significant to note that the IP is not well posed

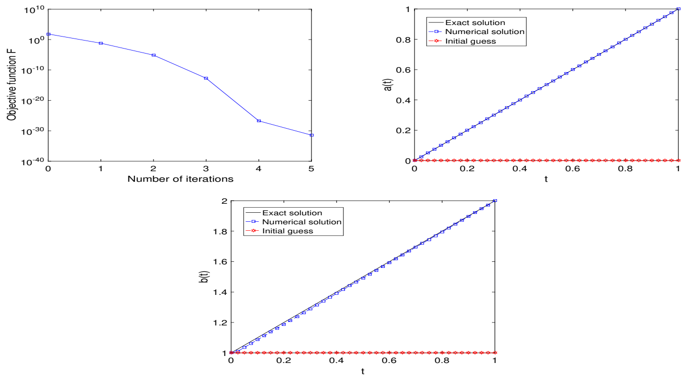


Figure 2. The cost function (5.1), and the analytical exact curves (6.7) and approximate $a(t)$, $b(t)$, with $p = 0$.

therefore, we anticipate that the cost function needs to be regularized for the sake of stability and accuracy in results.

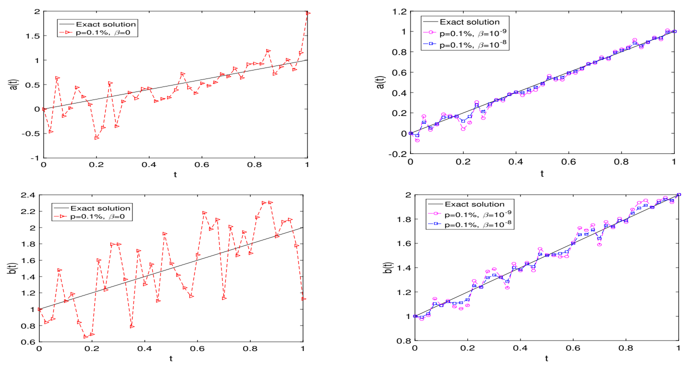


Figure 3. The analytical exact curves (6.7) and approximate $a(t)$, $b(t)$, for $p = 0.1\%$, with $\beta \in \{0, 10^{-9}, 10^{-8}\}$.

Figures 3 and 4 show visuals of the reconstructed terms $a(t)$, $b(t)$. From Figures 3(a), 3(c) and 4(a), 4(c) it is clear that, as expected, we obtain inaccurate and unstable solutions with RMSE$(a) = 0.2686$ and RMSE$(b) = 0.3231$ for $p = 0.1\%$, and RMSE$(a) = 1.3429$ and RMSE$(b) = 1.8279$ for $p = 1\%$, respectively, as the problem is noise sensitive and ill-posed. Hence, regularization process is crucial for stable solutions. For this, we penalise the cost function F (5.1) by adding penalty term β to it, where $\beta > 0$ is the Tikhonov's regularization parameter to be selected. Then, the Tikhonov functional recasts as

$$F_\beta(\underline{a}, \underline{b}) = F(\underline{a}, \underline{b}) + \beta \left(\sum_{n=1}^{N} \left(\frac{a^n - a^{n-1}}{\Delta t} \right)^2 + \sum_{n=1}^{N} \left(\frac{b^n - b^{n-1}}{\Delta t} \right)^2 \right).$$

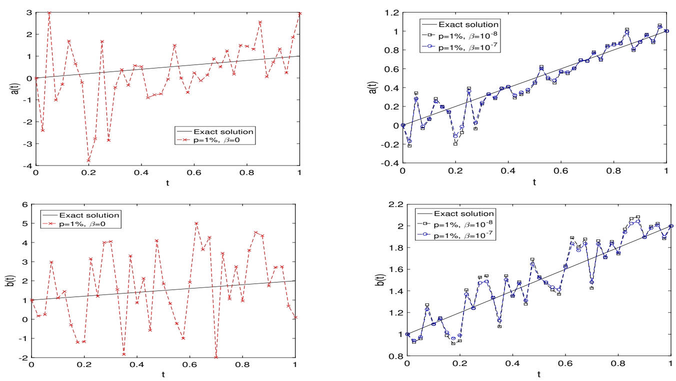


Figure 4. The analytical exact curves (6.7) and approximate $a(t)$, $b(t)$, for $p = 1\%$, with $\beta \in \{0, 10^{-8}, 10^{-7}\}$.

The regularization parameter β is chosen to be 10^{-9}, 10^{-8} for $p = 0.1\%$ noise (see Figures 3(b) and 3(d) obtaining $\mathrm{RMSE}(a) \in \{0.0537, 0.0269\}$ and $\mathrm{RMSE}(b) \in \{0.0629, 0.0320\}$, and $\beta \in \{10^{-8}, 10^{-7}\}$ for $p = 1\%$ noise (see Figures 4(b) and 4(d) obtaining $\mathrm{RMSE}(a) \in \{0.1343, 0.1074\}$ and $\mathrm{RMSE}(b) \in \{0.1587, 0.1265\}$, which provide stable and comparitively accurate approximations for the time dependent functions $a(t)$ and $b(t)$. Although not presented, it is illustrated that the regularized cost function F_β versus no. of iterations monotonically decreasing convergence is observed. Other details about the RMSE ((6.1) and 6.2)), and the iterations, with and without regularization are listed in Table 2. Eventually, from Figures 2–4 and Table 2, it is observed that the MATLAB simulation results are fairly stable and accurate.

Table 2. RMSE values and iterations with $p \in \{0.1\%, 1\%\}$, $\beta \in \{0, 10^{-9}, 10^{-8}, 10^{-7}, 10^{-6}\}$.

p	β	RMSE(a)	RMSE(b)	Iter
0.1%	0	0.2686	0.3231	40
	10^{-9}	0.0537	0.0629	20
	10^{-8}	0.0269	0.0320	20
	10^{-7}	0.0372	0.0418	20
1%	0	1.3429	1.8279	90
	10^{-8}	0.1343	0.1587	30
	10^{-7}	0.1074	0.1265	30
	10^{-6}	0.1263	0.1402	30

7 Conclusions

In the work, the classical solvability of a nonlinear inverse boundary value problem for a 2D parabolic equation with nonlocal conditions was studied. First, the considered problem was reduced to an auxiliary inverse boundary value problem in a certain sense and its equivalence to the original problem is shown. Then using the Fourier method and contraction mappings principle, the existence and uniqueness theorem for auxiliary problem is proved. Further, on the basis of the equivalency of these problems, the existence and uniqueness theorem for the classical solution of the original inverse coefficient problem is established. Moreover, the discretization of the forward problem was solved based on the ADE technique. The non-linear optimization problem was numerically solved by the MATLAB *lsqnonlin* routine. The investigated problem was ill-posed, therefore, the Tikhonov regularization was applied in order to tackle the stability. The approximation results show that ADE is an accurate, stable and robust regularization method for reconstructing the timewise lowest and force terms from knowledge of additional measurements.

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