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Singular Nonlinear Problems with Natural Growth in the Gradient

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Abstract. In this paper, we consider the equation $-\text{div}\,(a(x,u,Du) = H(x,u,Du) + \frac{a_0(x)}{|u|^\theta} + \chi_{\{u\neq 0\}}\, f(x)$ in Ω, with boundary conditions $u = 0$ on $\partial\Omega$, where Ω is an open bounded subset of $\mathbb{R}^N$, $1 < p < N$, $-\text{div}(a(x,u,Du))$ is a Leray-Lions operator defined on $W_0^{1,p}(\Omega)$, $a_0 \in L^{N/p}(\Omega)$, $a_0 > 0$, $0 < \theta \leq 1$, $\chi_{\{u\neq 0\}}$ is a characteristic function, $f \in L^{N/p}(\Omega)$ and $H(x,s,\xi)$ is a Carathéodory function such that $-c_0\, a(x,s,\xi)\xi \leq H(x,s,\xi)\,\text{sign}(s) \leq \gamma\, a(x,s,\xi)\xi$ a.e. $x \in \Omega, \forall s \in \mathbb{R}, \forall \xi \in \mathbb{R}^N$. For $\|a_0\|_{N/p}$ and $\|f\|_{N/p}$ sufficiently small, we prove the existence of at least one solution u of this problem which is moreover such that the function $\exp(\delta|u|) - 1$ belongs to $W_0^{1,p}(\Omega)$ for some $\delta \geq \gamma$. This solution satisfies some a priori estimates in $W_0^{1,p}(\Omega)$.

Keywords: nonlinear problems, existence, singularity.

AMS Subject Classification: 35J60; 35J75.

1 Introduction

In this paper, we consider the nonlinear problem

$$\begin{cases} u \in W_0^{1,p}(\Omega), \\ -\text{div}(a(x,u,Du)) = H(x,u,Du) + \dfrac{a_0(x)}{|u|^\theta} + \chi_{\{u\neq 0\}}\, f(x) \quad \text{in } \mathcal{D}'(\Omega), \end{cases} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $1 < p < N$, $-\text{div}\,(a(x,u,Du))$ is a Leray-Lions operator defined on $W_0^{1,p}(\Omega)$ and $H(x,u,Du)$ is a Carathéodory function with natural growth in $|Du|^p$, and more precisely satisfies:

$$|H(x,s,\xi)| \leq c|\xi|^p.$$

for some positive constant c. We assume that $a_0 \in L^{N/p}(\Omega)$, $a_0 > 0$, $0 < \theta \leq 1$, $\chi_{\{u \neq 0\}}$ is a characteristic function and $f \in L^{N/p}(\Omega)$.

When a_0 and f are sufficiently small, that is they satisfy the smallness condition (2.8), we prove in the present paper the existence of at least one solution u of (1.1) such that the function: $(e^{\mu|u|} - 1) \in W_0^{1,p}(\Omega)$, with

$$\left\| \mu^{-1}(e^{\mu|u|} - 1) \right\|_{W_0^{1,p}(\Omega)} \leq Z_\delta, \tag{1.2}$$

where μ is in an interval $(0, \mu_0)$, which depends on the norms of a_0 and f, the bound of H and the coercivity of a, and the nonnegative constant Z_δ is given in (3.18).

A similar result has been proved in [18] in the quasilinear case $p = 2$ and where the function $a(x, s, \xi)$ is assumed to have the form $a(x, s, \xi) = A(x)\xi$, with $A(x)$ being a matrix bounded entries and coercive. In that setting the change of unknown function $w = \mu^{-1}(e^{\mu|u|} - 1)\,\mathrm{sign}(u)$ transforms equation (2.1) in a quasilinear equation with a quadratic term which satisfies a "sign condition".

The proof used in the present paper follows along the lines of the proof in [12, 13, 18, 19] and can be obtained as follows.

We first consider a sequence of problems which approximate (2.1), obtained by truncation of the functions $H(x, s, \xi)$, $a_0(x)$ and $f(x)$ at level n, with $n \in \mathbb{N}^\star$, thanks to Leray-Lions theorem (see [20,21]), this approximation guarantees the existence of solution u_n of (3.14).

Once the solution of the approximate problem has been obtained, we perform the change of unknown function $w_n = \mu^{-1}(e^{\mu|u_n|} - 1)\,\mathrm{sign}(u_n)$, we then obtain equation (3.5) which is equivalent to (3.3). Thanks to the smallness condition, we obtain the a priori estimate of w_n which does not depend on n, and we extract a subsequence denoted w_n such that w_n weakly converges in $W_0^{1,p}(\Omega)$ to some w, and we prove that w_n strongly converges in $W_0^{1,p}(\Omega)$. By equivalent Theorem 2, we obtain the strong convergence of u_n. Another difficulty is the passage to the limit of the singular term, for that we use the method introduced in [14], and more precisely we treat a control of strong $\int_{\{|u_n| \leq \nu\}} \frac{a_n(x)}{(|u_n|+1/n)^\theta} \varphi \, dx$ when ν is small. Compared to the results obtained in the latest papers, we prove in the present paper, as said above, the existence of at least one solution of (2.1) in the case (2.6) (i.e., $a_0 \geq 0$) when a_0 and f satisfy the smallness condition (2.8), but our result is obtained in the general case of a nonlinearity $H(x, s, \xi)$ which satisfies only (2.5) with $f \in L^{N/p}(\Omega)$ and with a_0 to $L^{N/p}(\Omega)$.

Let us begin some review of the literature, the problem (2.1) has been extensively studied by many authors in the case

$$a(x, u, Du) = A(x)Du, \quad H(x, u, Du) = 0, \quad f(x) = 0$$

and $a_0(x)$ is smooth (see, for instance, [1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 22, 23, 25]). In [16, 17], the authors studied a singular elliptic problem whose model is

$$\begin{cases} -\Delta u = |\nabla u|^2/|u|^\theta + f(x) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\theta \in (0,1)$ and the datum f has no constant sign and belongs to $L^m(\Omega)$, with $m \geq \frac{N}{2}$, and prove the existence of a solution. In [25], the author considered the following two classes of singular boundary value problems

$$\begin{cases} -\Delta u \mp \lambda|\nabla u|^2/|u|^\theta = f(x)u^{-\alpha}, \\ u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$, $\theta > 0$, $\alpha > -1$ and the datum f satisies some property.

In [14] and [15], the authors proved the existence of at least one nonnegative solution and a stability result for the following problem

$$\begin{cases} -\text{div}\,(A(x)Du) = f(x)g(u) + l(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $A(x) \in L^\infty(\Omega)^{N\times N}$ is a coercive matrix, $g : [0,+\infty) \to [0,+\infty)$ is continuous and $0 \leq g(s) \leq 1/s^\theta + 1,\ \forall s > 0,\ 0 < \theta \leq 1$; and $f,\ l \in L^r(\Omega)$, where r satisfies some conditions. In [8], the authors studied the existence and nonexistence results for problems whose model is

$$-\Delta u = f(x)/u^\theta \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $\theta > 0$ and f is a nonegative function on Ω and belongs to some Lebesgue spaces. For this, they have introduced an approximate problem by treating the singular term $\frac{1}{u^\theta}$ and construct an increasing sequence $(u_n)_{n\in\mathbb{N}}$ of solutions to the nonsingular problem

$$\begin{cases} -\text{div}\,(A(x)Du_n) = f_n(x)/(u_n + 1/n)^\theta & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f_n(x) = \min(f(x), n)$. The sequence u_n satisfies, for any $\omega \subset\subset \Omega$, and

$$u_n \geq u_{n-1} \geq \cdots \geq u_1 \geq C_\omega, \quad \forall x \in \omega.$$

The authors discussed in [1] the solution of the elliptic problem, with a gradient term and a singular nonlinearity

$$\begin{cases} -\Delta u = |\nabla u|^q + f/g(u) & \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega \quad \text{and} \quad u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded regular domain, $g:\ \mathbb{R}_+ \to \mathbb{R}$ is a continuous increasing function with additional hypotheses given, $1 < q \leq 2$ and f is a measurable nonnegative function and obtained optimal conditions on g, q which allow to get the existence positive solution for the largest possible class of datum f.

2 Existence result and comments

As said in this introduction we study in this paper the existence of the solutions to the following singular nonlinear problem

$$\begin{cases} u \in W_0^{1,p}(\Omega), \\ -\text{div}\, a(x,u,Du) = H(x,u,Du) + \dfrac{a_0(x)}{|u|^\theta} + \chi_{\{u\neq 0\}}\, f(x) \quad \text{in } \mathcal{D}'(\Omega), \end{cases} \tag{2.1}$$

where

$$\Omega \quad \text{is a bounded open subset of } \mathbb{R}^N \text{ and } 1 < p < N. \tag{2.2}$$

The function $a : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^N$ is a Carathéodory function which also satisfies, for a.e. $x \in \Omega$, any $s \in \mathbb{R}$ and any $\xi, \xi' \in \mathbb{R}^N$, with $\xi \neq \xi'$:

$$\begin{cases} \big(a(x,s,\xi) - a(x,s,\xi')\big)(\xi - \xi') > 0, \\ a(x,s,\xi)\xi \geq \alpha|\xi|^p, \\ |a(x,s,\xi)| \leq \beta\big(b(x) + |s|^{p-1} + |\xi|^{p-1}\big), \end{cases} \tag{2.3}$$

for a given constant $\alpha > 0$, a constant $\beta > 0$, a nonnegative function $b \in L^{N/(p-1)}(\Omega)$, and which satisfies

$$a(x,-s,-\xi) = -a(x,s,\xi) \quad \text{a.e. } x \in \Omega,\ \forall s \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N, \tag{2.4}$$

the nonlinearity $H(x,s,\xi)$ is a Carathéodory function with a natural growth in ξ, and more precisely satisfies

$$\begin{cases} -c_0\, a(x,s,\xi)\xi \leq H(x,s,\xi)\,\mathrm{sign}(s) \leq \gamma\, a(x,s,\xi)\xi, \\ \qquad\qquad\qquad\qquad \text{a.e. } x \in \Omega, \forall s \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N, \\ \text{where } \gamma > 0 \text{ and } c_0 \geq 0, \end{cases} \tag{2.5}$$

the function sign defined by

$$\mathrm{sign}(s) = \begin{cases} +1, & \text{if } s > 0, \\ 0, & \text{if } s = 0, \\ -1, & \text{if } s < 0, \end{cases}$$

the coefficient a_0, the exponent θ and f satisfy

$$a_0 \in L^{N/p}(\Omega), \quad a_0 \geq 0, \quad a_0 \not\equiv 0, \quad 0 < \theta \leq 1, \quad f \in L^{N/p}(\Omega). \tag{2.6}$$

Since $N > p$, $p^\star$ is the Sobolev's exponent defined by $\frac{1}{p^\star} = \frac{1}{p} - \frac{1}{N}$, and let $C_{N,p}$ be the Sobolev's constant defined as the best constant such that

$$\|\varphi\|_{p^\star} \leq C_{N,p}\|D\varphi\|_p, \quad \forall \varphi \in W_0^{1,p}(\Omega). \tag{2.7}$$

Since Ω is a bounded, we equip the space of $W_0^{1,p}(\Omega)$ with the norm

$$\|u\|_{W_0^{1,p}(\Omega)} = \|Du\|_{(L^p(\Omega))^N}.$$

Finally, we assume that $\|a_0\|_{N/p}$ and $\|f\|_{W^{-1,p'}(\Omega)}$ are sufficiently small (see Remark 2, below), and more precisely that

$$C_\lambda(\theta)\,\|a_0\|_{N/p} + \|f\|_{N/p} \leq \left(\frac{p-1}{\gamma}\right)^{p-1} \frac{\alpha}{C_{N,p}^p}, \tag{2.8}$$

where $C_\lambda(\theta)$ is defined by

$$C_\lambda(\theta) = \left(\lambda/\ln(1+\lambda)\right)^\theta, \quad \text{where } \lambda = \gamma/(p-1).$$

Our main result is the following.

Theorem 1. *Assume that* (2.2), (2.3), (2.5), (2.6) *hold true. Assume moreover that* (2.8) *holds true, Then, there exists at least one solution of* (2.1)*, which further that:*

$$\begin{aligned} &(e^{\delta|u|}-1)\in W_0^{1,p}(\Omega), \quad \forall\delta\geq\gamma \quad \textit{such that} \\ &C_\mu(\theta)\,\|a_0\|_{N/p}+\|f\|_{N/p}\leq\frac{\alpha}{\mu^{p-1}C_{N,p}^p}, \end{aligned} \tag{2.9}$$

where $C_\mu(\theta)$ is the constant defined by

$$C_\mu(\theta)=(\mu/\ln(1+\mu))^\theta\,. \tag{2.10}$$

Remark 1. In the case where the function $H(x,s,\xi)=H(x,\xi)$ does not depend on s, assumption (2.5) is satisfied if and only if

$$|H(x,\xi)|\leq c\,a(x,s,\xi)\xi,$$

for some $c>0$. When $\gamma=0$ in (2.5), the nonlinearity function $H(x,\xi)$ satisfies a sign condition and existence result can be proved for every $f\in W^{-1,p'}(\Omega)$.

Remark 2. In this Remark, we consider that the open set Ω, the functions a and H are fixed and the functions a_0 and f as parameters. Our set of assumptions on these parameters is made of the smallness condition (2.8). Indeed, if, for example, a_0 is sufficiently small such that it satisfies

$$\|a_0\|_{N/p}\leq\frac{\alpha}{\mu^{p-1}\,C_\lambda(\theta)\,C_{N,p}^p}\quad(C_\lambda(\theta)\geq1),$$

then the smallness condition (2.8) is satisfied if $\|f\|_{N/p}$ is sufficiently small.

Similarly, if, for example, f is sufficiently small that it satisfies

$$\|f\|_{N/2}\leq\alpha/(\gamma C_N^2),$$

then, the smallness condition (2.8) is satisfied if $\|a_0\|_{N/p}$ is sufficiently small.

Remark 3. The smallness condition (2.8) shows that the parameter δ is bounded and satisfies to the following condition

$$0<\gamma\leq\delta\leq(p-1)\,\alpha^{\frac{1}{p-1}}/\Big(C_{N,p}^{\frac{p}{p-1}}\,\|a_0\|_{N/p}^{\frac{1}{p-1}}\Big).$$

Remark 4. The definition of the constant Z_δ which appears in (1.2) and (3.18) is given in (the technical) Appendix (see Lemma 1). This definition is based on the properties of the family of functions Φ_δ (see (3.47)) which look like convex parabolas (see Figure 1): Z_δ is the unique value satisfying $\Phi_\delta(Z_\delta)=0$.

3 Proof of Theorem 1

The proof will be done in seven steps.

Step 1: *Approximation and change of unknown function.*

For $n \in \mathbb{N}^\star$ we set $T_n(s) = \min\{n, \max\{s, -n\}\}$ and $G_n(s) = s - T_n(s)$, we consider two sequences a_n and f_n such that

$$a_n(x) = T_n(a_0(x)), \quad f_n(x) = T_n(f(x)). \tag{3.1}$$

For $n \in \mathbb{N}^\star$, the function $H_n(x, s, \xi)$ is defined by

$$H_n(x, s, \xi) = H(x, s, \xi)/\Big(1 + \frac{1}{n}|H(x, s, \xi)|\Big). \tag{3.2}$$

$H_n(x, s, \xi)$ satisfies: $|H_n(x, s, \xi)| \leq H(x, s, \xi)$ as well as (2.5). Since $a_n(x)$, $f_n(x)$ and $H_n(x, s, \xi)$ are bounded in $L^\infty(\Omega)$, a classical result of J. Leray and J.-L. Lions [20] and [21] asserts that the following approximate problem (3.3) has at least one solution.

$$\begin{cases} u_n \in H_0^1(\Omega), \\ -\text{div}(a(x, u_n, Du_n)) = H_n(x, u_n, Du_n) + \dfrac{a_n(x)}{(|u_n| + \frac{1}{n})^\theta} + \chi_{\{u_n \neq 0\}} f_n(x). \end{cases} \tag{3.3}$$

Moreover, since $a_n(x)$, $f_n(x)$ and $H_n(x, u_n, Du_n)$ are bounded in $L^\infty(\Omega)$, any solution of (3.3) actually belongs to $L^\infty(\Omega)$ for each given n.

Let $\delta > 0$ be fixed and satisfies $\gamma \leq \delta$ such that

$$C_\mu(\theta)\, \|a_0\|_{N/p} + \|f\|_{N/p} \leq \frac{\alpha}{\mu^{p-1} C_{N,p}^p}, \quad \mu = \frac{\delta}{p-1}.$$

If we formally define the function w_n by

$$w_n = \varphi(u_n), \tag{3.4}$$

where $\varphi(s) = \mu^{-1}(e^{\mu|s|} - 1)\,\text{sign}(s)$, where $\mu = \delta/(p-1)$, we have at least formally

$$\begin{aligned} & w_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \\ & e^{\mu u_n} = 1 + \mu|w_n|, \quad u_n = \frac{1}{\mu}\ln(1 + \mu|w_n|), \quad Dw_n = e^{\mu|u_n|} Du_n, \end{aligned}$$

and

$$\begin{aligned} & -e^{\delta|u_n|}\,\text{div}(a(x, u_n, Du_n)) = \\ & \qquad -\text{div}(e^{\delta|u_n|} a(x, u_n, Du_n)) + \delta\, e^{\delta|u_n|}\,\text{sign}(u_n)(a(x, u_n, Du_n)) Du_n, \end{aligned}$$

where $-e^{\delta|u_n|}\,\text{div}(a(x, u_n, Du_n))$ is the distribution defined by

$$_{\mathcal{D}'(\Omega)}\Big\langle -e^{\delta|u_n|}\,\text{div}(a(x, u_n, Du_n)), \varphi \Big\rangle_{C_0^\infty(\Omega)} = \int_\Omega a(x, u_n, Du_n) D(e^{\delta|u_n|}\varphi)$$

for any $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Since $e^{\mu|u_n|} = 1 + \mu|w_n|$, we deduce that w_n is, at least formally, a solution (see Theorem 2) of the following problem

$$\begin{cases} -\text{div}(\hat{a}(x, w_n, Dw_n) = K_\delta(x, w_n, Dw_n)\,\text{sign}(w_n) \\ \quad +(1 + \mu|w_n|)^{p-1}\chi_{\{w_n \neq 0\}}\, f_n + \dfrac{(1 + \mu|w_n|)^{p-1}}{(\mu^{-1}\ln(1 + \mu|w_n|) + \frac{1}{n})^\theta}\, a_n, \\ w_n = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{3.5}$$

where $\hat{a} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function defined by

$$\hat{a}(x,s,\xi) = (1+\mu|s|)^{p-1}\, a\left(x, \frac{1}{\mu}\ln(1+\mu|s|)\operatorname{sign}(s), \frac{\xi}{1+\mu|s|}\right). \tag{3.6}$$

Note that the function $\hat{a}$ satisfies Leray-Lions conditions (2.3). The function $K_\delta : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is defined by the following formulas

$$\begin{aligned} K_\delta(x,s,\xi) =&(1+\mu|s|)^{p-1}\Big(H_n\Big(x, \frac{1}{\mu}\ln(1+\mu|s|)\operatorname{sign}(s), \frac{\xi}{1+\mu|s|}\Big)\operatorname{sign}(s) \\ &-\delta a\Big(x, \frac{1}{\mu}\ln(1+\mu|s|)\operatorname{sign}(s), \frac{\xi}{1+\mu|s|}\Big)\frac{\xi}{1+\mu|s|}\Big). \end{aligned} \tag{3.7}$$

Note also that the functions $K_\delta(x,w,Dw)$ and $K_\delta(x,w,Dw)\operatorname{sign}(w)$ are correctly defined and are measurable functions when $w \in W^{1,p}(\Omega)$ (see Remark 3.2 in [19]). When $\gamma \le \delta$, this computation in particular implies that

$$K_\delta(x,s,\xi) \le 0 \quad \text{a.e. } x \in \Omega, \quad \forall s \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^N. \tag{3.8}$$

From (2.3) and (2.5) one has

$$|K_\delta(x,s,\xi)| \le (c_0+\delta)(1+\mu|s|)^{p-1} a\left(x, \frac{1}{\mu}\ln(1+\mu|s|)\operatorname{sign}(s), \frac{\xi}{1+\mu|s|}\right)\frac{\xi}{1+\mu|s|},$$
$$\text{a.e. } x \in \Omega, \quad \forall s \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^N.$$

Step 2: *A priori estimate.*

Since the right hand side of (3.5) belongs to $L^1(\Omega)$, we take w_n as a test function in (3.5) and $K_\delta(x,s,\xi) \le 0$ (see (3.8)), therefore,

$$\begin{aligned} \int_\Omega \hat{a}(x,w_n,Dw_n)Dw_n\,dx \le& \int_\Omega (1+\mu|w_n|)^{p-1} w_n\, \chi_{\{w_n\neq 0\}}\, f_n(x)\,dx \\ &+ \int_\Omega \frac{(1+\mu|w_n|)^{p-1}w_n}{\left(\mu^{-1}\ln(1+\mu|w_n|)+\frac{1}{n}\right)^\theta} a_n(x)\,dx. \end{aligned} \tag{3.9}$$

By the coercivity condition on $\hat{a}$ and $|\chi_{\{w_n\neq 0\}}| \le 1$, we get

$$\begin{aligned} \alpha\|Dw_n\|_p^p \le& \int_\Omega (1+\mu|w_n|)^{p-1}|w_n|\,|f_n(x)|\,dx \\ &+ \int_\Omega \frac{(1+\mu|w_n|)^{p-1}w_n}{\left(\mu^{-1}\ln(1+\mu|w_n|)+1/n\right)^\theta} a_n(x)\,dx. \end{aligned}$$

Using Hölder's and Sobolev's inequalities (2.7) this implies that

$$\int_\Omega (1+\mu|w_n|)^{p-1}w_n\, f_n(x)\,dx \le \|f\|_{N/p}\|w_n\|_{p^\star}\|(1+\mu|w_n|)\|_{p^\star}^{p-1}. \tag{3.10}$$

Now, we split Ω into $\Omega = \{|w_n| \le 1\} \cup \{|w_n| > 1\}$ and writing the last term of the right-hand side of (3.9) as

$$\begin{aligned}\int_\Omega \frac{(1+\mu|w_n|)^{p-1}w_n}{(\mu^{-1}\ln(1+\mu|w_n|)+1/n)^\theta}\,a_n(x)\,dx \\ =\int_{\{|w_n|\le 1\}}\frac{(1+\mu|w_n|)^{p-1}w_n}{(\mu^{-1}\ln(1+\mu|w_n|)+1/n)^\theta}\,a_n(x)\,dx \\ +\int_{\{|w_n|>1\}}\frac{(1+\mu|w_n|)^{p-1}w_n}{(\mu^{-1}\ln(1+\mu|w_n|)+1/n)^\theta}\,a_n(x)\,dx,\end{aligned}\tag{3.11}$$

we need $\|a_n\|_{N/p}\le\|a_0\|_{N/p}$, $\|f_n\|_{N/p}\le\|f\|_{N/p}$ and let us that the function $F(x)=x/\ln(1+x)$ is increasing on $\mathbb{R}_+^\star$, we get

$$\begin{aligned}\int_{\{|w_n|\le 1\}}\frac{(1+\mu|w_n|)^{p-1}w_n}{(\mu^{-1}\ln(1+\mu|w_n|)+1/n)^\theta}\,a_n(x)\,dx \\ \le C_\mu(\theta)|\Omega|^{\frac{\theta}{p^\star}}\,\|a_0\|_{N/p}\,\|w_n\|_{p^\star}^{1-\theta}\,\|(1+\mu|w_n|)\|_{p^\star}^{p-1},\end{aligned}\tag{3.12}$$

where $C_\mu(\theta)$ is the positive constant defined in (2.10) and

$$\begin{aligned}\int_{\{|w_n|>1\}}\frac{(1+\mu|w_n|)^{p-1}w_n}{(\mu^{-1}\ln(1+\mu|w_n|)+1/n)^\theta}\,a_n(x)\,dx \\ \le C_\mu(\theta)\,\|a_0\|_{N/p}\,\|w_n\|_{p^\star}\,\|(1+\mu|w_n|)\|_{p^\star}^{p-1}.\end{aligned}\tag{3.13}$$

From (3.10)–(3.13), we have

$$\begin{aligned}\alpha\|Dw_n\|_p^p\le C_\mu(\theta)|\Omega|^{\frac{\theta}{p^\star}}\,\|a_0\|_{N/p}\,\|w_n\|_{p^\star}^{1-\theta}\|(1+\mu|w_n|)\|_{p^\star}^{p-1} \\ +\left(C_\mu(\theta)\,\|a_0\|_{N/p}+\|f\|_{N/p}\right)\|w_n\|_{p^\star}\,\|(1+\mu|w_n|)\|_{p^\star}^{p-1}.\end{aligned}\tag{3.14}$$

Using that

$$\|(1+\mu|w_n|)\|_{p^\star}\le|\Omega|^{\frac{1}{p^\star}}+\mu\|w_n\|_{p^\star}\le|\Omega|^{\frac{1}{p^\star}}+\mu C_{N,p}\|Dw_n\|_p,\tag{3.15}$$

from (3.14) and (3.15), we have

$$\begin{aligned}\alpha\|Dw_n\|_p^p\le C_{N,p}^{1-\theta}C_\mu(\theta)|\Omega|^{\frac{\theta}{p^\star}}\,\|a_0\|_{N/p}\,\|Dw_n\|_p^{1-\theta}(|\Omega|^{\frac{1}{p^\star}}+\mu C_{N,p}\|Dw_n\|_p)^{p-1} \\ +C_{N,p}\left(C_\mu(\theta)\,\|a_0\|_{N/p}+\|f\|_{N/p}\right)\|Dw_n\|_p\,(|\Omega|^{\frac{1}{p^\star}}+\mu C_{N,p}\|Dw_n\|_p)^{p-1}.\end{aligned}\tag{3.16}$$

To treat the a priori estimation, we have two cases to study:

Case 1: $\mathbf{1<p\le 2}$.

Observe that $0<\alpha=p-1\le 1$ and for every a, $b>0$, one has

$$(a+b)^\alpha\le a^\alpha+b^\alpha.\tag{3.17}$$

From (3.16), (3.17) and dividing by $\|Dw_n\|_p^{1-\theta}$ (note that the result remains true in the case when $\|Dw_n\|_p=0$), we have

$$\begin{aligned}&\left(\alpha-\mu^{p-1}C_{N,p}^p(C_\mu(\theta)\|a_0\|_{\frac{N}{p}}+\|f\|_{\frac{N}{p}})\right)\|Dw_n\|_p^{p-1+\theta} \\ &\le\mu^{p-1}C_{N,p}^{p-\theta}C_\mu(\theta)|\Omega|^{\frac{\theta}{p^\star}}\,\|a_0\|_{N/p}\,\|Dw_n\|_p^{p-1}+C_{N,p}|\Omega|^{\frac{p-1}{p^\star}} \\ &\quad\times\left(C_\mu(\theta)\|a_0\|_{\frac{N}{p}}+\|f\|_{\frac{N}{p}}\right)\|Dw_n\|_p^\theta+C_{N,p}^{1-\theta}C_\mu(\theta)|\Omega|^{\frac{p-1+\theta}{p^\star}}\,\|a_0\|_{N/p}.\end{aligned}$$

Case 2: $\mathbf{p > 2}$.
From (3.16) and dividing by $\|Dw_n\|_p^{\frac{1-\theta}{p-1}}$, we have

$$\Big(\alpha^{\frac{1}{p-1}} - \mu C_{N,p}^{\frac{p}{p-1}} (C_\mu(\theta)\|a_0\|_{N/P} + \|f\|_{N/p})^{\frac{1}{p-1}}\Big) \|Dw_n\|_p^{\frac{p-1+\theta}{p-1}}$$
$$\le \mu\, C_{N,p}^{\frac{p-\theta}{p-1}} C_\mu^{\frac{1}{p-1}}(\theta)\, |\Omega|^{\frac{\theta}{(p-1)p^\star}}\, \|a_0\|_{N/p}^{\frac{1}{p-1}} \|Dw_n\|_p$$
$$+\, C_{N,p}^{\frac{1}{p-1}} |\Omega|^{\frac{1}{p^\star}} \big(C_\mu(\theta)\,\|a_0\|_{N/p} + \|f\|_{N/p}\big)^{\frac{1}{p-1}} \|Dw_n\|_p^{\frac{\theta}{p-1}}$$
$$+\, C_{N,p}^{\frac{1-\theta}{p-1}}\, C_\mu^{\frac{1}{p-1}}(\theta)\, |\Omega|^{\frac{p-1+\theta}{p^\star}} \|a_0\|_{N/p}^{\frac{1}{p-1}}.$$

In view of the definition of (3.47) below of the function Φ_δ (see also Figure 1), we have proved if w_n is any solution of (3.5), one has

$$\Phi_\delta(\|Dw_n\|_p) \le 0, \quad \text{if } \gamma \le \delta,$$

this implies that

$$\|Dw_n\|_p \le Z_\delta, \quad \text{(does not depend on } n), \quad \text{for } \gamma \le \delta, \tag{3.18}$$

where the constant $Z_\delta > 0$ satisfies $\Phi_\delta(Z_\delta) = 0$.

Since $u_n = \mu^{-1}(\ln(1+\mu|w_n|))\,\mathrm{sign}(w_n)$, and from (3.18) implies that u_n is bounded in $H_0^1(\Omega)$.

Step 3: *Proof of regularity result of* (2.9).

Extracting a subsequence, still denoted by u_n, we have, for some $u \in H_0^1(\Omega)$ and $w \in H_0^1(\Omega)$.

$$u_n \rightharpoonup u, \;\; w_n \rightharpoonup w \quad \text{weakly in } H_0^1(\Omega), \;\; \text{a.e. in } \Omega, \tag{3.19}$$

where

$$w = \varphi(u) = \mu^{-1}(e^{\mu|u|} - 1)\,\mathrm{sign}(u).$$

We note that u and w do not belong to $L^\infty(\Omega)$ in general.

Let us consider another δ, say δ', which also satisfies

$$\gamma \le \delta' \quad \text{such that} \quad C_{\mu'}(\theta)\,\|a_0\|_{N/p} + \|f\|_{N/p} \le \alpha/\big({\mu'}^{p-1} C_{N,p}^p\big), \tag{3.20}$$

where $\mu' = \delta'/(p-1)$. The above a priori estimate (3.18) again shows that w_n' defined by

$$w_n' = \mu'^{-1}(e^{\mu'|u_n|} - 1)\,\mathrm{sign}(u_n),$$

is bounded in $W_0^{1,p}(\Omega)$, this proves that u is such that

$$(e^{\mu'|u|} - 1)\,\mathrm{sign}(u) \in W_0^{1,p}(\Omega), \quad \forall \delta' \text{ such that } \;\gamma \le \delta' \text{ satisfies (3.20)},$$

that is (2.9).

Step 4: *An estimate for* $\int_{|w_n|>k} |Dw_n|^p$.

Since $G_k(w_n) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, the use of $G_k(w_n)$ as a test function in (3.5) is licit. This gives

$$\begin{cases} \displaystyle\int_\Omega \hat{a}(x, w_n, Dw_n) DG_k(w_n)\, dx - \int_\Omega K_\delta(x, w_n, Dw_n) \operatorname{sign}(w_n)\, G_k(w_n)\, dx \\ \quad \displaystyle= \int_\Omega \frac{(1+\mu|w_n|)^{p-1}\, G_k(w_n)}{(\mu^{-1}\ln(1+\mu|w_n|)+1/n)^\theta}\, a_n(x)\, dx \\ \quad\quad \displaystyle+ \int_\Omega (1+\mu|w_n|)^{p-1} G_k(w_n) \chi_{\{w_n\neq 0\}}\, f_n(x)\, dx. \end{cases}$$

Using the coercivity (2.3) of the function $\hat{a}$, $\operatorname{sign}(s)\, G_k(s) \geq 0$ and $K_\delta(x,s,\xi) \leq 0$, we have

$$\begin{aligned} &\alpha \lim_n \int_\Omega |DG_k(u_n)|^p dx \\ &\leq \int_\Omega \frac{(1+\mu|w|)^{p-1}\, G_k(w)}{(\mu^{-1}\, \ln(1+\mu|w_n|)^\theta}\, a_0\, dx + \int_\Omega (1+\mu|w|)^{p-1}\, G_k(w) \chi_{\{w\neq 0\}}\, f\, dx, \end{aligned}$$

from which we deduce that

$$\limsup_n \int_\Omega |DG_k(u_n)|^p\, dx \;\to\; 0 \quad \text{as } k \to +\infty. \tag{3.21}$$

Step 5: *Strong convergence of $DT_k(w_n)$ in $(L^p(\Omega))^N$.*

In this step, we will fix $k > 0$ and prove that

$$DT_k(w_n) \to DT_k(w) \quad \text{strongly in } (L^p(\Omega))^N, \quad \text{as } n \to +\infty.$$

In order to prove this result we use a technique which goes to Bensoussan et al. [5], let k be fixed, we define

$$z_n = T_k(w_n) - T_k(w), \tag{3.22}$$

and we choose an increasing, C^1 function $\psi : \mathbb{R} \to \mathbb{R}$ such that

$$\psi(0) = 0, \quad \psi'(s) - (c_0+\delta)|\psi(s)| \geq 1/2, \quad \forall s \in \mathbb{R}, \tag{3.23}$$

where c_0 is the constant which appears in the left-hand side of assumption (2.5) on H and we get for example $\psi(s) = se^{\lambda s^2}$ with $\lambda = (c_0+\delta)^2/4$.

Since $z_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, and $\psi(0) = 0$, the function $\psi(z_n)$ belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. The use of $\psi(z_n)$ as a test function in (3.5) is licit. This gives

$$\begin{cases} \displaystyle\int_\Omega \hat{a}(x, w_n, Dw_n) Dz_n \psi'(z_n) dx - \int_\Omega K_\delta(x, w_n, Dw_n) \operatorname{sign}(w_n) \psi(z_n) dx \\ \qquad\qquad \displaystyle= \int_\Omega \frac{(1+\mu|w_n|)^{p-1}\, \psi(z_n)}{\left(\frac{1}{n} + \mu^{-1}\ln(1+\mu|w_n|)\right)^\theta}\, a_n(x)\, dx \\ \qquad\qquad\quad \displaystyle+ \int_\Omega (1+\mu|w_n|)^{p-1} \psi(z_n) \chi_{\{w_n\neq 0\}}\, f_n(x)\, dx. \end{cases} \tag{3.24}$$

Splitting Ω into $\Omega = \{|w_n| \le k\} \cup \{|w_n| > k\}$, and using:

$$Dw_n = DT_k(w_n) + DG_k(w_n) = Dz_n + DT_k(w) + DG_k(w_n). \tag{3.25}$$

The first term of the left-hand side of (3.24) reads as

$$\begin{cases} \displaystyle\int_\Omega \hat{a}(x, w_n, Dw_n) Dz_n \psi'(z_n)\, dx \\ \displaystyle= \int_{\{|w_n| \le k\}} (\hat{a}(x, T_k w_n, DT_k(w_n)) - \hat{a}(x, T_k w_n, DT_k(w))) Dz_n \psi'(z_n)\, dx \\ \displaystyle\quad + \int_{\{|w_n| \le k\}} \hat{a}(x, T_k w_n, DT_k(w))) Dz_n \psi'(z_n)\, dx \\ \displaystyle\quad + \int_{\{|w_n| > k\}} \hat{a}(x, w_n, Dw_n) Dz_n \psi'(z_n)\, dx. \end{cases} \tag{3.26}$$

The second term of the left-hand side of (3.24) reads as

$$\begin{cases} \displaystyle\int_\Omega K_\delta(x, w_n, Dw_n) \operatorname{sign}(w_n) \psi(z_n)\, dx \\ \displaystyle\qquad = \int_{\{|w_n| > k\}} K_\delta(x, w_n, Dw_n) \operatorname{sign}(w_n) \psi(z_n)\, dx \\ \displaystyle\qquad\quad + \int_{\{|w_n| \le k\}} K_\delta(x, w_n, Dw_n) \operatorname{sign}(w_n) \psi(z_n)\, dx, \end{cases} \tag{3.27}$$

the first term of the right-hand side of (3.27), we claim that

$$\int_{\{|w_n| > k\}} K_\delta(x, w_n, Dw_n) \operatorname{sign}(w_n) \psi(z_n)\, dx \le 0, \tag{3.28}$$

indeed in $\{|w_n| > k\}$, the integrand is negative since on the first hand the function $K_\delta(x, w_n, Dw_n) \le 0$ in view of (3.8) and $\delta \ge \gamma$, and since on the other hand we have

$$\operatorname{sign}(w_n)\, \psi(z_n) \ge 0 \quad \text{in } \{|w_n| > k\},$$

indeed in $\{|w_n| > k\}$, one has $z_n = T_k(w_n) - T_k(w) = k \operatorname{sign}(w_n) - T_k(w)$, and therefore $\operatorname{sign}(z_n) = \operatorname{sign}(w_n)$; this implies

$$\operatorname{sign}(w_n)\, \psi(z_n) = \operatorname{sign}(z_n)\, \psi(z_n) = |\psi(z_n)| \quad \text{in } \{|w_n| > k\},$$

which proves (3.28). The second term of the right-hand side of (3.27), in view of Remark 3.1 of [19], (3.6) and $\delta \ge \gamma$, we obtain

$$|K_\delta(x, w_n, Dw_n) \operatorname{sign}(w_n) \psi(z_n)| \le (c_0 + \delta)\, |\psi(z_n)|\, \hat{a}(x, w_n, Dw_n)\, Dw_n.$$

Since in view of (3.25) one has

$$Dw_n = Dz_n + DT_k(w) \quad \text{in} \quad \{|w_n| \le k\},$$

and implies that

$$\begin{cases} \displaystyle\int_{\{|w_k|\le k\}} K_\delta(x,w_n,Dw_n)\,\mathrm{sign}(w_n)\,\psi(z_n)\,dx \\ \displaystyle\le \int_{\{|w_n|\le k\}} (c_0{+}\delta)|\psi(z_n)|\,\left(\hat a(x,T_kw_n,DT_k(w_n)){-}\hat a(x,T_kw_n,DT_k(w))\right) Dz_n\,dx \\ \displaystyle\quad + \int_{\{|w_n|\le k\}} (c_0+\delta)|\psi(z_n)|\,\hat a(x,T_kw_n,DT_k(w_n))DT_k(w)\,dx \\ \displaystyle\quad + \int_{\{|w_n|\le k\}} (c_0+\delta)|\psi(z_n)|\,\hat a(x,T_kw_n,DT_k(w))Dz_n\,dx. \end{cases} \tag{3.29}$$

From (3.23), (3.24), (3.26), (3.27) and (3.29), we deduce that

$$\begin{cases} \displaystyle\frac{1}{2}\int_{\{|w_n|\le k\}} (\hat a(x,T_kw_n,DT_k(w_n)) - \hat a(x,T_kw_n,DT_k(w)))Dz_n\,dx \\ \displaystyle\le -\int_{|w_n|\le k} \hat a(x,T_kw_n,DT_k(w)))Dz_n\psi'(z_n)\,dx \end{cases}$$

$$\begin{cases} \displaystyle\quad -\int_{|w_n|\ge k} \hat a(x,w_n,Dw)Dz_n\psi'(z_n)\,dx \\ \displaystyle\quad + \int_{\{|w_n|\le k\}} (c_0+\delta)|\psi(z_n)|\,\hat a(x,T_kw_n,DT_k(w_n))DT_k(w)\,dx \\ \displaystyle\quad + \int_{\{|w_n|\le k\}} (c_0+\delta)|\psi(z_n)|\,\hat a(x,T_kw_n,DT_k(w))Dz_n\,dx \\ \displaystyle\quad + \int_{\{|w_n|> k\}} K_\delta(x,w_n,Dw_n)\,\mathrm{sign}(w_n)\psi(z_n)\,dx \\ \displaystyle\quad + \int_\Omega \frac{(1+\mu|w_n|)^{p-1}\,a_n(x)}{\left(\frac{1}{n}+\mu^{-1}\ln(1+\mu|w_n|)\right)^\theta}\,\psi(z_n)\,dx \\ \displaystyle\quad + \int_\Omega (1+\mu|w_n|)^{p-1}\chi_{\{w_n\ne 0\}}\,f_n(x)\,\psi(z_n)\,dx. \end{cases} \tag{3.30}$$

We claim that each term of the right-hand side of (3.30) tends to zero as n tends to infinity. Since $\psi'(z_n) - (c_0+\delta)\,|\psi(z_n)| \ge 1/2$ by (3.23), and since the function $\hat a$ is monotone and coercive (see (2.3)), this will imply that

$$z_n \underset{n}{\to} 0 \quad \text{in} \quad H_0^1(\Omega) \quad \text{strongly},$$

or in the other terms (see the definition (3.22) of z_n) that

$$T_k(w_n) \underset{n}{\to} T_k(w) \quad \text{in} \quad W_0^{1,p}(\Omega) \quad \text{strongly}.$$

In order to prove the claim, let us recall that in view of (3.19) and of the definition (3.22) of z_n one has

$$z_n \underset{n}{\rightharpoonup} 0 \;\text{ in }\; W_0^{1,p}(\Omega) \;\text{ weakly, }\; L^\infty(\Omega) \;\text{ weakly star and a.e. in }\; \Omega.$$

Since $\psi(0) = 0$, $\psi(z_n)$ tends to zero almost everywhere in Ω and in $L^\infty(\Omega)$ weakly star as n tends to infinity, which in turn implies that

$$Dz_n\,\psi'(z_n) = D\psi(z_n) \rightharpoonup 0 \quad \text{in} \quad (L^p(\Omega))^N \quad \text{weakly} \quad \text{as} \quad n \to +\infty.$$

This implies that the first term of the right-hand side of (3.30) tends to zero as k tends to infinity. Indeed, z_n (and thus $\psi'(z_n)$ is bounded in $L^\infty(\Omega)$ and

$$Dz_n = DT_k(w_n) - DT_k(w) \text{ tends to zero weakly in } (L^p(\Omega))^N,$$

and using Vitali's theorem, we have $\chi_{\{|w_n|\le k\}}\hat{a}(x, T_k w_n, DT_k(w))\psi'(z_n)$ tends strongly to $\chi_{\{|w|\le k\}}\hat{a}(x, T_k w, DT_k(w))\psi'(0)$ in $(L^{p'}(\Omega))^N$.

For the second term of the right-hand side of (3.30), we note that

$$\chi_{\{|w_n|>k\}}Dz_n = -\chi_{\{|w_n|>k\}}DT_k(w) \to 0 \quad \text{strongly in } (L^p(\Omega))^N,$$

as $\psi'(z_n)$ is bounded in $L^\infty(\Omega)$ and from (2.3) the fact $\hat{a}(x, w_n, Dw)\psi'(z_n)$ is bounded in $(L^{p'}(\Omega))^N$. This implies that the second term of the right-hand side of (3.30) tends to zero.

For the third term of the right-hand side of (3.30) tends to zero, since $DT_k(w)\psi(z_n)$ converges strongly to $DT_k(w)\psi(0) = 0$ in $(L^p(\Omega))^N$, while $\hat{a}(x, T_k(w_n), DT_k(w_n))$ is bounded in $(L^{p'}(\Omega))^N$.

For the fourth term of the right-hand side of (3.30) tends to zero, since $Dz_n = DT_k(w_n) - DT_k(w)$ converges to zero weakly in $(L^p(\Omega))^N$, while by Vitali's theorem and $\psi(0) = 0$, $\chi_{\{|w_n|\le k\}}\hat{a}(x, T_k(w_n), DT_k(w))|\psi(z_n)|$ converges strongly to zero in $(L^{p'}(\Omega))^N$. Together with condition (3.28), this implies that the fifth term of the right-hand side of (3.30) is negative.

For the sixth term of the right-hand side of (3.30), we consider for almost every x in the set $\{w = 0\}$ the assertions w_n and $\psi(z_n) = \psi(T_k(w_n))$ converge to zero almost everywhere in Ω and observe that $|T_k(w_n)| \le |w_n|$, thus there exists $n_0 \in \mathbb{N}$, such that for all $n \ge n_0$ implies that $|w_n| \le 1$, and

$$\frac{(1+\mu|w_n|)^{p-1}\,\psi(z_n)}{\left(\mu^{-1}\ln(1+\mu|w_n|) + \frac{1}{n}\right)^\theta}\, a_n(x) \le (1+\mu)^{p-1} C_\mu(\theta) e^\lambda\, a_0(x).$$

By Lebesque's theorem, this implies that

$$\frac{(1+\mu|w_n|)^{p-1}\,\psi(z_n)}{(\mu^{-1}\ln(1+\mu|w_n|) + 1/n)^\theta}\, a_n(x) \quad \text{converges to zero in } L^1(\{w=0\}). \tag{3.31}$$

On the other hand, we consider for almost every x in the set $\{w \ne 0\} \cap \{|w| \le 1\}$, since z_n converges almost everywhere to zero, then there exists $n_1 \in \mathbb{N}$, such that for all $n \ge n_1$ implies that

$$|z_n| = |T_k(w_n) - T_k(w)| \le |w|/2 \text{ and } |w_n| \ge |w|/2$$

and

$$\frac{(1+\mu|w_n|)^{p-1}\,\psi(z_n)}{(\mu^{-1}\ln(1+\mu|w_n|) + 1/n)^\theta}\, a_n(x) \le (1+\mu)^{p-1} C_\mu(\theta) e^\lambda\, a_0(x).$$

By Lebesque's theorem, this implies that

$$\frac{(1+\mu|w_n|)^{p-1}\,\psi(z_n)}{\left(\mu^{-1}\ln(1+\mu|w_n|) + \frac{1}{n}\right)^\theta}\, a_n(x) \underset{n}{\to} 0 \text{ in } L^1(\{w \ne 0\} \cap \{|w| \le 1\}). \tag{3.32}$$

Finally, we consider for almost every x in the set $\{|w| > 1\}$, since z_n converges almost everywhere to zero and $\psi(0) = 0$ and by Vitali's theorem, then

$$\frac{(1+\mu|w_n|)^{p-1}\,\psi(z_n)}{\left(\mu^{-1}\ln(1+\mu|w_n|)+\frac{1}{n}\right)^{\theta}}\,a_n(x) \underset{n}{\rightarrow} 0 \text{ in } L^1(\{|w_n|>1\}). \tag{3.33}$$

Collecting the results on (3.31)–(3.33) implies that for k fixed:

$$\frac{(1+\mu|w_n|)^{p-1}\,\psi(z_n)}{\left(\mu^{-1}\ln(1+\mu|w_n|)+\frac{1}{n}\right)^{\theta}}\,a_n(x) \underset{n}{\rightarrow} 0 \text{ in } L^1(\Omega).$$

For the seventh term of the right-hand side of (3.30) tends to zero, indeed by Vitali's theorem, the strong convergence (3.1) in $L^{N/p}(\Omega)$ of f_n, the $L^{p^\star}(\Omega)$ boundness on w_n and $\psi(0) = 0$ imply that

$$(1+\mu|w_n|)^{p-1}\chi_{\{w_n\neq 0\}}\,f_n(x)\,\psi(z_n) \underset{n}{\rightarrow} 0 \text{ strongly in } L^1(\Omega).$$

Passing to the limit in (3.30), we get

$$\int_{\{|w_n|\leq k\}} (\hat{a}(x,T_k w_n, DT_k(w_n)) - \hat{a}(x,T_k w_n, DT_k(w)))Dz_n\,dx \underset{n}{\rightarrow} 0. \tag{3.34}$$

By the growth condition of (2.3) on a (thus on $\hat{a}$), and by Vitali's theorem, we have

$$\int_{\{|w_n|>k\}} (\hat{a}(x,T_k w_n, DT_k(w_n)) - \hat{a}(x,T_k w_n, DT_k(w)))Dz_n \underset{n}{\rightarrow} 0. \tag{3.35}$$

Combining (3.34) and (3.35), we have

$$\int_{\Omega} (\hat{a}(x,T_k w_n, DT_k(w_n)) - \hat{a}(x,T_k w_n, DT_k(w)))Dz_n\,dx \underset{n}{\rightarrow} 0.$$

Thanks to the assumption (2.3) and by a result of Browder, we deduce

$$z_n = T_k(w_n) - T_k(w) \rightarrow 0 \quad \text{strongly in } W_0^{1,p}(\Omega). \tag{3.36}$$

Step 6: *Strong convergence of u_n in $W_0^{1,p}(\Omega)$.*

Taking into account

$$Dw_n - Dw = (DT_k(w_n) - DT_k(w)) + (DG_k(w_n) - DG_k(w)),$$

from (3.21) and (3.36), we have $Dw_n \rightarrow Dw$ strongly in $(L^p(\Omega))^N$, since we have $u_n = \mu^{-1}\ln(1+\mu|w_n|)$, it follows that

$$u_n \rightarrow u \quad \text{strongly in } W_0^{1,p}(\Omega).$$

Step 7: *Control of strong $\int_{\{|u_n|\leq\nu\}} \frac{a_n(x)}{(|u_n|+1/n)^{\theta}}\varphi\,dx$ when ν is small.*

In this step, we claim that

$$\lim_n \int_\Omega \frac{a_n(x)}{(|u_n|+\frac{1}{n})^\theta}\varphi\,dx = \int_\Omega \frac{a_0(x)}{|u|^\theta}\varphi\,dx, \quad \forall\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega).$$

First we observe that

$$\int_\Omega \frac{a_n(x)}{(|u_n|+1/n)^\theta}\varphi\,dx = \int_\Omega a(x,u_n,Du_n)D\varphi\,dx - \int_\Omega H_n(x,u_n,Du_n)\varphi\,dx - \int_\Omega (\chi_{\{|u_n|\neq 0\}}\,f_n)(x)\varphi\,dx.$$

Taking into account the growth conditions (2.3) and Hölder's inequality, we get

$$\int_\Omega \frac{a_n(x)}{(|u_n|+\frac{1}{n})^\theta}\varphi\,dx \le \beta\,\|b\|_{N/(p-1)}|\Omega|^{\frac{N-p+1}{N}}\,\|\varphi\|_\infty + \beta\|\varphi\|_p \times \left(\|u_n\|_p^{p-1} + \|Du_n\|_p^{p-1}\right) + (c_0+\delta)\|\varphi\|_\infty\|Du_n\|_p^p + \beta\,\|\varphi\|_\infty\|f_n\|_{N/p}|\Omega|^{\frac{N-p}{p}}. \tag{3.37}$$

From now on, we consider a nonnegative $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, applying Fatou's Lemma to the left-hand side of (3.37), we have.

$$\int_\Omega \frac{a_0(x)}{|u|^\theta}\,\varphi\,dx \le C_\varphi, \tag{3.38}$$

where C_φ is a positive constant and does not depend to n.

Hence $0 \le \frac{a_0(x)}{|u|^\theta}\,\varphi \in L^1(\Omega)$, for any nonnegative $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. As consequence, $1/|s|^\theta$ is unbounded as s tends to 0, we deduce that

$$\{u=0\} \subset \{a_0 = 0\},$$

up to set of zero Lebesgue measure.

From now on, we consider a nonnegative function $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, and choosing it was test function in the weak formulation, we have

$$\int_\Omega a(x,u_n,Du_n)\,D\varphi\,dx = \int_\Omega H_n(x,u_n,Du_n)\,\varphi\,dx + \int_\Omega \frac{a_n(x)}{(|u_n|+1/n)^\theta}\varphi dx + \int_\Omega (\chi_{\{u_n\neq 0\}}\,f_n)(x)\varphi dx, \tag{3.39}$$

we want to pass to the limit in the second right-hand side of (3.39) as n tends to infinity. For $\nu>0$ fixed, we consider the second right-hand side of (3.39)

$$\int_\Omega \frac{a_n(x)}{(|u_n|+\frac{1}{n})^\theta}\varphi dx = \int_{|u_n|\le\nu} \frac{a_n(x)}{(|u_n|+\frac{1}{n})^\theta}\varphi dx + \int_{|u_n|>\nu} \frac{a_n(x)}{(|u_n|+\frac{1}{n})^\theta}\,\varphi\,dx. \tag{3.40}$$

Applying Lemma 1.1 of [24], we have that for $\nu>0$, $V_\nu(u_n)$ belongs to $W_0^{1,p}(\Omega)$, where $V_\nu :]-\infty,+\infty[\ \to\ [0,+\infty[$ is defined by

$$V_\nu(s) = \begin{cases} 0, & s < -2\nu, \\ 2+s/\nu, & -2\nu \le s < -\nu, \\ 1, & -\nu \le s \le \nu, \\ 2-s/\nu, & \nu < s < 2\nu, \\ 0, & s \ge 2\nu. \end{cases}$$

Since $V_\nu(u_n) \in W_0^{1,p}(\Omega)$, the use of $(V_\nu(u_n)\,\varphi)$ as test function in (3.3) is licit. This gives

$$\int_{|u_n|\le\nu} \frac{a_n(x)}{(|u_n|+1/n)^\theta}\,\varphi\,dx \le \int_\Omega a(x,u_n,Du_n)\,D(V_\nu(u_n)\,\varphi)dx - \int_\Omega H_n(x,u_n,Du_n)V_\nu(u_n)\,\varphi\,dx - \int_\Omega (\chi_{\{u_n\neq 0\}}\,f_n)(x)\,V_\nu(u_n)\,\varphi\,dx. \quad (3.41)$$

The first term of the right-hand side of (3.41) can be written

$$\int_\Omega a(x,u_n,Du_n)\,D(V_\nu(u_n)\varphi)\,dx = \int_\Omega a(x,u_n,Du_n)\,D\varphi\,V_\nu(u_n)\,dx.$$

Indeed, splitting Ω into $\Omega = \{|u_n| \le \nu\} \cup \{|u_n| > \nu\}$ and using (2.4), we get

$$\int_\Omega a(x,u_n,Du_n)Du_n\,V_\nu'(u_n)\,\varphi\,dx = 0.$$

Since $V_\nu(u_n)\,D\varphi$ converges to $V_\nu(u)\,D\varphi$ strongly in $L^p(\Omega)^N$, as n tends to infinity, while the Carathéodory function $a(x,u_n,Du_n)$ converges to $a(x,u,Du)$ strongly in $L^p(\Omega)^N$, we obtain

$$\lim_n \int_\Omega a(x,u_n,Du_n)D\varphi\,V_\nu(u_n)\,dx = \int_\Omega a(x,u,Du)\,D\varphi\,V_\nu(u)\,dx. \quad (3.42)$$

In the second term of the right-hand side of (3.41), $\varphi V_\nu(u_n)$ is bounded in $L^\infty(\Omega)$ and

$$H_n(u_n,Du_n)\,\varphi V_\nu(u_n) \le \|\varphi\|_\infty(c_0+\gamma)|Du_n|^2,$$

which implies that the functions $H_n(u_n,Du_n)\,\varphi V_\nu(u_n)$ are equiintegrable since Du_n strongly converges to Du in $L^p(\Omega)^N$, we have

$$\lim_n \int_\Omega H_n(u_n,Du_n)\,\varphi V_\nu(u_n)\,dx = \int_\Omega H(u,Du)\,\varphi V_\nu(u)\,dx. \quad (3.43)$$

In the third term of the right-hand side of (3.41), the functions $\chi_{\{u_n\neq 0\}}$, f_n, φ, $V_\nu(u_n)$ are equiintegrable, since f_n strongly converges in $L^{N/p}(\Omega)$ and $V_\nu(u_n)$ converges to $V_\nu(u)$ strongly in $L^{p^\star}(\Omega)$. Thus, Vitali's theorem implies that

$$\lim_n \int_\Omega \chi_{\{u_n\neq 0\}}\,f_n\,\varphi V_\nu(u_n)\,dx = \int_\Omega \chi_{\{u\neq 0\}}\,f\;\varphi V_\nu(u)\,dx. \quad (3.44)$$

Together with (3.41), the three limits (3.42)–(3.44) imply that

$$\begin{cases} \lim_n \int_{|u_n|\le\nu} \frac{a_n(x)}{(|u_n|+1/n)^\theta}\,\varphi\,dx \le \int_\Omega a(x,u,Du)\,D\varphi\;V_\nu(u)\,dx \\ \qquad + \int_\Omega H(u,Du)\;\varphi V_\nu(u)\,dx + \int_\Omega \chi_{\{u\neq 0\}}\,f\varphi V_\nu(u)\,dx. \end{cases} \quad (3.45)$$

Since $V_\nu(u)$ converges to $\chi_{\{u=0\}}$ a.e. in Ω as $\nu \to 0$ and $u \in W_0^{1,p}(\Omega)$, then,

$$\big(a(x,u,Du)\,Du\,D\varphi + H(u,Du)\,\varphi + \chi_{\{u\neq 0\}}\,f\,\varphi\big)\;V_\nu(u) \underset{\nu\to 0}{\to} 0 \quad \text{a.e. in } \Omega.$$

Applying the Lebesgue's dominated convergence Theorem on the right-hand side of (3.45), we obtain that

$$\lim_{\nu\to 0}\lim_{n}\int_{\{|u_n|\le\nu\}}\frac{a_n(x)}{(|u_n|+1/n)^\theta}\,\varphi\,dx=0.$$

Finally, let us pass to limit in n for $\nu>0$ fixed in the second term of the right-hand side of (3.40)

$$\int_{\{|u_n|>\nu\}}\frac{a_n(x)}{(|u_n|+1/n)^\theta}\,\varphi\,dx=\int_{\Omega}\frac{a_n(x)}{(|u_n|+1/n)^\theta}\,\chi_{\{|u_n|>\nu\}}\,\varphi\,dx.$$

Observing that we need to select $\nu\notin\{\nu\,:\,\text{meas}\{|u(x)|=\nu\}>0\}$ which is at most countable set, we have

$$\frac{a_n(x)}{(|u_n|+1/n)^\theta}\,\varphi\to\frac{a_0(x)}{|u|^\theta}\,\varphi\ \text{ a.e. on }\Omega,\quad\frac{a_n(x)}{(|u_n|+1/n)^\theta}\,\varphi\le\frac{a_0(x)}{|u|^\theta}\,\varphi\in L^1(\Omega).$$

By Lebesgue's theorem, we have

$$\lim_{n}\int_{\{|u_n|>\nu\}}\frac{a_n(x)}{(|u_n|+1/n)^\theta}\,\varphi\,dx=\int_{\{|u|>\nu\}}\frac{a_0(x)}{|u|^\theta}\,\varphi\,dx,\quad\forall\nu\notin\mathcal{C}.\tag{3.46}$$

Moreover, it follows by (3.38) that $\forall\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega),\ \varphi\ge 0$

$$\lim_{\nu\to 0}\lim_{n}\int_{\{|u_n|>\nu\}}\frac{a_n(x)}{(|u_n|+1/n)^\theta}\,\varphi\,dx=\int_{\{|u|>0\}}\frac{a_0(x)}{|u|^\theta}\,\varphi\,dx.$$

Moreover, decomposing any $\varphi=\varphi^+-\varphi^-$ and observing that (3.46) is linear in φ, we deduce that (3.46) holds for every $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

As $u_n\to u$ strongly in $W_0^{1,p}(\Omega)$, it is then easy to pass to the limit in the approximate equation (3.3). This proves that u is a solution of (2.1). The proof of Theorem 1 is then complete. □

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Appendix

An equivalence result

Theorem 2. *[12] Assume that* (2.3), (2.5), (2.6), (3.1)*, and* (3.2) *hold true, and let* $\delta > 0$ *be fixed. Let the function* K_δ *be defined in* (3.7)*. If* u_n *is any solution of* (2.1) *then the function* w_n *defined by* (3.4) *is a solution of* (3.5).

Conversely, if w_n *is any solution of* (3.5)*, then the function* u_n *is a solution of* (3.3).

Definition of Z_δ

The goal of this Subsection is to define the constant Z_δ (with $\delta = \mu(p-1)$) which appear in Theorem 1. We will prove the following result.

Lemma 1. *For* $\delta \geq 0$*, let* $\varPhi_\delta : \mathbb{R}^+ \to \mathbb{R}$ *(see Figure 1) be the function defined by*

$$\varPhi_\delta(X) = \begin{cases} \varPhi_\delta^{(1)}(X), & 1 < p \leq 2, \\ \varPhi_\delta^{(21)}(X), & p > 2, \end{cases} \tag{3.47}$$

where

$$\begin{cases} \varPhi_\delta^{(1)}(X) = \left(\alpha - \mu^{p-1} C_{N,p}^{p} (C_\mu(\theta) \|a_0\|_{\frac{N}{p}} + \|f\|_{\frac{N}{p}})\right) X^{p-1+\theta} \\ -\mu^{p-1} C_{N,p}^{p-\theta} C_\mu(\theta) |\varOmega|^{\frac{\theta}{p^\star}} \|a_0\|_{N/p} X^{p-1} \\ - C_{N,p} |\varOmega|^{\frac{p-1}{p^\star}} \left(C_\mu(\theta) \|a_0\|_{\frac{N}{p}} + \|f\|_{\frac{N}{p}} \right) X^{\theta} - C_{N,p}^{1-\theta} C_\mu(\theta) |\varOmega|^{\frac{p-1+\theta}{p^\star}} \|a_0\|_{N/p}, \end{cases}$$

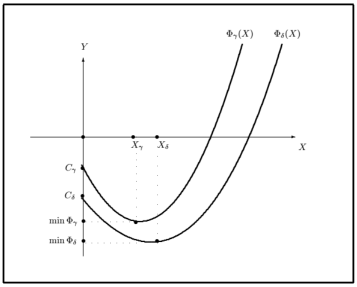

Figure 1. The graphs of the functions $\Phi_\delta(X)$ and $\Phi_\gamma(X)$.

and

$$\begin{cases} \Phi_\delta^{(2)}(X) = \left(\alpha^{\frac{1}{p-1}} - \mu C_{N,p}^{\frac{p}{p-1}} (C_\mu(\theta)\|a_0\|_{N/P} + \|f\|_{N/p})^{\frac{1}{p-1}}\right) X^{\frac{p-1+\theta}{p-1}} \\ -\mu\, C_{N,p}^{\frac{p-\theta}{p-1}} C_\mu^{\frac{1}{p-1}}(\theta)\,|\Omega|^{\frac{\theta}{(p-1)p^\star}}\,\|a_0\|_{N/p}^{\frac{1}{p-1}} X \\ -\, C_{N,p}^{\frac{1}{p-1}} |\Omega|^{\frac{1}{p^\star}} \left(C_\mu(\theta)\,\|a_0\|_{N/p} + \|f\|_{N/p}\right)^{\frac{1}{p-1}} X^{\frac{\theta}{p-1}} \\ -\, C_{N,p}^{\frac{1-\theta}{p-1}}\, C_\mu^{\frac{1}{p-1}}(\theta)\,|\Omega|^{\frac{p-1+\theta}{p^\star}}\,\|a_0\|_{N/p}^{\frac{1}{p-1}}, \end{cases}$$

where θ *satisfies* (2.6)*, namely* $0 < \theta < 1$*,* $C_\mu(\theta)$ *is the constant satisfying* (2.10) *and where* $C_{N,p}$ *is the best constant in the Sobolev's inequality* (2.7)*. Then, for* $\delta \geq \gamma$*, there exists a unique number* Z_δ *such that*

$$\Phi_\delta(Z_\delta) = 0, \quad \text{and} \quad \forall X \leq Z_\delta \;:\; \Phi_\delta(X) \leq 0.$$

Proof. Let us now study the family of functions $\Phi_\delta(X) : \mathbb{R}^+ \to \mathbb{R}$ defined by (3.47), from the smallness condition relative to δ (see 2.9), implies that

$$\alpha - \mu^{p-1} C_{N,p}^p (C_\mu(\theta)\|a_0\|_{\frac{N}{p}} + \|f\|_{\frac{N}{p}}) \geq 0 \quad \text{for } 1 < p \leq 2,$$

and

$$\alpha^{\frac{1}{p-1}} - \mu C_{N,p}^{\frac{p}{p-1}} (C_\mu(\theta)\|a_0\|_{N/P} + \|f\|_{N/p})^{\frac{1}{p-1}} \geq 0 \quad \text{for } p > 2.$$

Each function Φ_δ look like the restriction to $\mathbb{R}^+$ of a "convex parabola", when $0 < \gamma \leq \delta$. This "convex parabola" has a unique minimizer in X_δ of the function Φ_δ, and the minimum of Φ_δ, namely $\Phi_\delta(X_\delta)$ is negative and using the intermediate value theorem, then there exists Z_δ such that $\Phi_\delta(Z_\delta) = 0$. □