Mathematical Modelling and Analysis
Volume 29, Issue 1, 109–124, 2024
https://doi.org/10.3846/mma.2024.17944

A Variational Formulation Governed by Two Bipotentials for a Frictionless Contact Model

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Received November 3, 2022; accepted January 12, 2024

Abstract. We consider a frictionless contact model whose constitutive law and contact condition are described by means of subdifferential inclusions. For this model, we deliver a variational formulation based on two bipotentials. Our formulation envisages the computation of a three-field unknown consisting of the displacement vector, the stress tensor and the normal stress on the contact zone, the contact being described by a generalized Winkler condition. Subsequently, we obtain existence and uniqueness results. Some properties of the solution are also discussed, focusing on the data dependence.

Keywords: contact condition, subdifferential inclusion, bipotentials, three-field weak solution, data dependence.

AMS Subject Classification: 49J40; 49J53; 74M15.

1 Introduction

The notion of bipotential was introduced by the pioneering work of de Saxcé and Feng (see [10]) in 1991 in order to allow the application of the classical variational principles to some boundary value problems arising in mechanics. This approach is based on convex analysis, the cornerstone being an extension of the Fenchel inequality [11]. Since then, many efforts were taken in order to develop the theory of bipotentials, as witnessed by the large number of papers published in this direction.

Several works were devoted to the understanding of the bipotential approach, by studying in which conditions a law can be expressed by means of a bipotential and also by exploring the ways to construct a class of bipotentials (see, e.g., [5, 6]). The importance of bipotentials is illustrated by the broad spectrum of applicability: plasticity, soil mechanics, dynamics of granular materials, viscoplasticity, elastostatics (see [4, 7, 12, 13, 18]).

In connection with the calculus of variations, the bipotential theory allowed to deliver two-field variational formulations for many boundary value problems. Such formulations were proposed for several models in contact mechanics, see [9, 16, 17, 18, 19, 20, 21, 22], where the existence and uniqueness of the pair solutions consisting of the displacement vector and the Cauchy stress tensor have been studied.

The advantage of the approach via bipotentials is that it facilitates the implementation of new and efficient numerical algorithms in order to approximate the solutions. The use of bipotentials in applications is particularly attractive in numerical simulations when using the finite element method and the discrete element method. Many contact problems have been recently addressed within the bipotential framework, see, e.g., [8, 14, 28].

The present work is a new contribution to the theory of the weak solvability via bipotentials in contact mechanics. We study a contact model whose constitutive law and contact condition are described by means of subdifferential inclusions. We introduce two bipotentials: the first one is in connection with the constitutive law and the second one is in connection with the contact law. These bipotentials allow us to deliver a variational formulation whose solution consists of the displacement vector, the stress tensor and the normal stress on the contact zone. We study the existence and uniqueness of the solution of the proposed variational formulation and we also study the dependence of the solution on the data.

The novelty feature of this work is the consideration of two bipotentials in the variational approach; note that all the existing results related to the weak solvability via bipotentials involve only one bipotential, which is in connection with the constitutive law. The existence and uniqueness results we get in the present paper are the first in the literature that involve a bipotential which is in connection with a boundary condition. The present work opens a door to research in this direction.

The advantage of the new approach we propose is that we can compute not only the displacement vector and the stress tensor, but also the normal stress on the contact zone.

The rest of the paper is structured as follows. Section 2 is devoted to some preliminaries and notations. In Section 3, we describe the model and in Section 4 we deliver its variational formulation via two bipotentials. The well-posedness of the model is studied in Section 5, where we establish existence and uniqueness results and we study the data dependence of the solution. Finally, Section 6 is devoted to final comments, conclusions and outlook. For the convenience of the reader, the symbols and notation used in the paper are condensed into a table included in Appendix A.

2 Preliminaries and notation

In this paper, we denote by $\mathbb{S}^3$ the space of second order symmetric tensors on $\mathbb{R}^3$. By $\cdot$ and $:$ we denote the inner product on $\mathbb{R}^3$ and $\mathbb{S}^3$, respectively, while by means of the notation $\|\cdot\|$ and $\|\cdot\|_{\mathbb{S}^3}$ we denote the Euclidean norm on $\mathbb{R}^3$ and $\mathbb{S}^3$, respectively. Every field in $\mathbb{R}^3$ or $\mathbb{S}^3$ is typeset in boldface. A complete list of symbols including also the notation we use can be found at the end of the paper.

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary Γ. We denote by $\boldsymbol{\nu}$ the unit outward normal to Γ, which is defined almost everywhere.

In our study, we shall use the Hilbert space

$$L^2_s(\Omega)^{3\times 3} = \{\boldsymbol{\mu} = (\mu_{ij}) \,:\, \mu_{ij} \in L^2(\Omega),\, \mu_{ij} = \mu_{ji} \text{ for all } i,j \in \{1,2,3\}\},$$

which is endowed with the inner product

$$(\boldsymbol{\mu}, \boldsymbol{\tau})_{L^2_s(\Omega)^{3\times 3}} = \int_\Omega \sum_{i,j=1}^{3} \mu_{ij}(\boldsymbol{x})\tau_{ij}(\boldsymbol{x})\, dx.$$

By $\boldsymbol{\varepsilon} : H^1(\Omega)^3 \to L^2_s(\Omega)^{3\times 3}$ we denote the deformation operator, a linear and continuous operator given by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big) \quad i,j \in \{1,2,3\};$$

see, e.g., [27] (p. 85).

The linear and continuous Sobolev trace operator for vector-valued functions is denoted by $\boldsymbol{\gamma} : H^1(\Omega)^3 \to L^2(\Gamma)^3$; see, e.g., Theorem 6.13 in [1].

Another useful Hilbert space is

$$V = \{\, \boldsymbol{v} \in H^1(\Omega)^3 \;:\; \boldsymbol{\gamma}\,\boldsymbol{v} = \boldsymbol{0} \text{ a.e. on } \Gamma_1\},$$

where $\Gamma_1 \subset \Gamma$ such that $meas(\Gamma_1) > 0$. This space is endowed with the inner product

$$(\cdot,\cdot)_V : V \times V \to \mathbb{R}, \quad (\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2_s(\Omega)^{3\times 3}};$$

see, e.g., [27] (pp. 86-87). We also note that $\boldsymbol{\varepsilon}(V) = \{\boldsymbol{\varepsilon}(\boldsymbol{v}) \,:\, \boldsymbol{v} \in V\}$ and

$$\boldsymbol{\varepsilon}(V)^{\perp} = \{\boldsymbol{\tau} \in L^2_s(\Omega)^{3\times 3} \,:\, (\boldsymbol{\tau}, \boldsymbol{\sigma})_{L^2_s(\Omega)^{3\times 3}} = 0 \quad \text{for all } \boldsymbol{\sigma} \in \boldsymbol{\varepsilon}(V)\}$$

are closed subspaces of $L^2_s(\Omega)^{3\times 3}$. Moreover, $L^2_s(\Omega)^{3\times 3} = \boldsymbol{\varepsilon}(V) \oplus \boldsymbol{\varepsilon}(V)^{\perp}$, see, for instance, Theorem 1.16 in [27].

We mention the following Green formula, which will be used in our variational approach:

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx + \int_\Omega \operatorname{Div} \boldsymbol{\sigma} \cdot \boldsymbol{v}\, dx = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{\gamma}\,\boldsymbol{v}\, d\Gamma \quad \text{for all } \boldsymbol{v} \in H^1(\Omega)^3, \tag{2.1}$$

with $\boldsymbol{\sigma}$ a regular enough function. For the above formula see, e.g., [27] (p. 89).

Next, we recall some elements of convex analysis.

Definition 1. Let $(X,(\cdot,\cdot)_X)$ be a Hilbert space and let $\varphi : X \to (-\infty,\infty]$. The subdifferential of φ at a point $u \in dom(\varphi)$ is the (possibly empty) set

$$\partial\varphi(u) = \{\zeta \in X \,:\, \varphi(v) - \varphi(u) \geq (\zeta, v-u)_X \quad \text{for all } v \in X\}.$$

For the above definition we refer, for instance, to [24] (p. 128). The next theorem can be found in many books, see, e.g., [24] (p. 45).

Theorem 1. *Let $(X,(\cdot,\cdot)_X)$ be a Hilbert space and let $\varphi : X \to (-\infty,\infty]$ be a proper, convex, lower semicontinuous functional. Then:*

i) for each $u, v \in X$, we have $\varphi(u) + \varphi^*(v) \geq (u,v)_X$;

ii) for each $u, v \in X$ we have the equivalences

$$v \in \partial\varphi(u) \Leftrightarrow u \in \partial\varphi^*(v) \Leftrightarrow \varphi(u) + \varphi^*(v) = (u,v)_X.$$

Herein, φ^* denotes the Fenchel conjugate of φ,

$$\varphi^* : X \to (-\infty,\infty], \quad \varphi^*(v) = \sup_{w\in X}\{(v,w)_X - \varphi(w)\}.$$

Also, we shall need the following minimization theorem.

Theorem 2. *Let X be a real reflexive Banach space and let $K \subset X$ be a nonempty, closed, convex and unbounded subset of X. Suppose $\varphi : K \to \mathbb{R}$ is coercive, convex and lower semicontinuous. Then φ is bounded from below on K and attains its infimum in K. If φ is strictly convex then φ has a unique minimizer.*

Minimization results can be found in many books, see, for instance, [24, 27].

Since the variational approach we adopt in this paper is based on the bipotential theory, we mention below the following definition, which can be found in [6].

Definition 2. Let $(X,(\cdot,\cdot)_X)$ be a Hilbert space. A bipotential is a function $B : X \times X \to (-\infty,\infty]$ with the following three properties:

i) B is convex and lower semicontinuous in each argument;

ii) for each $x, y \in X$ we have $B(x,y) \geq (x,y)_X$;

iii) for each $x, y \in X$ we have the equivalences

$$y \in \partial B(\cdot,y)(x) \Leftrightarrow x \in \partial B(x,\cdot)(y) \Leftrightarrow B(x,y) = (x,y)_X.$$

Finally, we consider a concept of *convergence of sets* originating from Mosco's theory; see, e.g., [23] for more details on this topic.

Definition 3. Let X be a Hilbert space. Let $(\mathcal{K}_n)_n \subset X$ be a sequence of nonempty subsets and $\mathcal{K} \subset X$, $\mathcal{K} \neq \emptyset$.

The sequence $(\mathcal{K}_n)_n$ converges to $\mathcal{K}$ in the sense of Mosco ($\mathcal{K}_n \xrightarrow{\mathrm{M}} \mathcal{K}$) if:

i) for each sequence $(\mu_n)_n$ such that $\mu_n \in \mathcal{K}_n$ for each $n \in \mathbb{N}$ and $\mu_n \rightharpoonup \mu$ in X, we have $\mu \in \mathcal{K}$;

ii) for every $\mu \in \mathcal{K}$, there exists a sequence $(\mu_n)_n \subset X$ such that $\mu_n \in \mathcal{K}_n$ for each $n \in \mathbb{N}$ and $\mu_n \to \mu$ in X.

Everywhere in this paper we shall use the notation "$\xrightarrow{\mathrm{M}}$" to refer to the convergence in the sense of Mosco according to Definition 3. The notation "$\rightharpoonup$" refers to the convergence in the weak topology. For the convenience of the reader, recall that a sequence $(\mu_n)_n$ in a normed space X converges weakly to μ in X (and we write $\mu_n \rightharpoonup \mu$) if $\varphi(\mu_n) \to \varphi(\mu)$ for every linear and continuous functional $\varphi : X \to \mathbb{R}$.

3 The model and working hypotheses

Our study is concerned with the following mechanical model.

Problem 1. Find $\boldsymbol{u} : \overline{\Omega} \to \mathbb{R}^3$ and $\boldsymbol{\sigma} : \overline{\Omega} \to \mathbb{S}^3$ such that

$$\operatorname{Div} \boldsymbol{\sigma}(\boldsymbol{x}) + \boldsymbol{f}_0(\boldsymbol{x}) = \boldsymbol{0} \qquad \text{in } \Omega, \tag{3.1}$$

$$\boldsymbol{\sigma}(\boldsymbol{x}) \in \partial\omega(\boldsymbol{\varepsilon}(\boldsymbol{u})(\boldsymbol{x})) \qquad \text{in } \Omega, \tag{3.2}$$

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{0} \qquad \text{on } \Gamma_1, \tag{3.3}$$

$$\boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{\nu}(\boldsymbol{x}) = \boldsymbol{f}_2(\boldsymbol{x}) \qquad \text{on } \Gamma_2, \tag{3.4}$$

$$-\sigma_\nu(\boldsymbol{x}) \in \partial\psi(u_\nu(\boldsymbol{x})), \ \ \boldsymbol{\sigma}_\tau(\boldsymbol{x}) = \boldsymbol{0} \qquad \text{on } \Gamma_3. \tag{3.5}$$

Herein, $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary Γ, partitioned in three measurable parts, Γ_1, Γ_2 and Γ_3, such that $meas(\Gamma_1) > 0$.

We denote by $\boldsymbol{u} = (u_i)$ the displacement field, by $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))$ the infinitesimal strain tensor and by $\boldsymbol{\sigma} = (\sigma_{ij})$ the Cauchy stress tensor.

The body, which occupies the domain Ω, is acted upon by forces of density $\boldsymbol{f}_0$, the behavior of the material being modeled by a constitutive law expressed as a subdifferential inclusion. The body is mechanically constrained on the boundary: on Γ_1 it is clamped and hence, the displacement field vanishes here, surface tractions of density $\boldsymbol{f}_2$ act on Γ_2, while on Γ_3 it is in frictionless contact with an obstacle, the contact being modeled by a subdifferential inclusion involving the normal components of the displacement vector and Cauchy vector, defined by the formulas $u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}$, $\sigma_\nu = (\boldsymbol{\sigma\nu}) \cdot \boldsymbol{\nu}$. Also, by $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu \boldsymbol{\nu}$, $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu \boldsymbol{\nu}$, we denote the tangential components of the displacement vector and Cauchy vector on the boundary.

Problem 1 is a contact model mathematically described by means of a boundary value problem governed by two subdifferential inclusions. Contact models governed by subdifferential inclusions have been recently studied by other authors within the framework of variational-hemivariational inequalities, see, e.g., [15, 29, 30]. In Problem 1 we have convex subdifferentials and this feature of the subdifferentials allows us to adopt an approach based on the theory of bipotentials.

In order to study this model we make the following assumptions:

(A1) $\boldsymbol{f}_0 \in L^2(\Omega)^3$ and $\boldsymbol{f}_2 \in L^2(\Gamma_2)^3$.

(A2) The constitutive function $\omega : \mathbb{S}^3 \to \mathbb{R}$ is convex and in addition, there exist α_1, β_1 such that $1 > \beta_1 \geq \alpha_1 > 0$ and $\beta_1 \|\boldsymbol{\tau}\|^2_{\mathbb{S}^3} \geq \omega(\boldsymbol{\tau}) \geq \alpha_1 \|\boldsymbol{\tau}\|^2_{\mathbb{S}^3}$ for all $\boldsymbol{\tau} \in \mathbb{S}^3$.

(A3) The function $\psi : \mathbb{R} \to \mathbb{R}$ is convex and in addition, there exist α_2, β_2 such that $1 > \beta_2 \geq \alpha_2 > 0$ and $\beta_2\, x^2 \geq \psi(x) \geq \alpha_2\, x^2$ for all $x \in \mathbb{R}$.

Remark 1. Taking into consideration the properties of the functions ω and ψ considered in assumptions (A2)–(A3), it follows that ω and ψ are continuous (see, e.g., Proposition 2.17 in [2]).

Remark 2. The Fenchel conjugates of ω and ψ, denoted by ω^* and ψ^*, are convex (see, e.g., [3]). Moreover, it can be proved by similar arguments to those presented in [16] that

$$(1-\beta_1)\|\boldsymbol{\tau}\|^2_{\mathbb{S}^3} \leq \omega^*(\boldsymbol{\tau}) \leq \frac{1}{4\alpha_1}\|\boldsymbol{\tau}\|^2_{\mathbb{S}^3} \quad \text{for all } \boldsymbol{\tau} \in \mathbb{S}^3,$$
$$(1-\beta_2)\, x^2 \leq \psi^*(x) \leq \frac{1}{4\alpha_2}\, x^2 \quad \text{for all } x \in \mathbb{R}.$$

Thus, since $\omega^* : \mathbb{S}^3 \to \mathbb{R}$ and $\psi^* : \mathbb{R} \to \mathbb{R}$ are finite and convex, then according to Proposition 2.17 in [2] they are continuous.

Examples of functions ω satisfying (A2) are the constitutive maps:

- $\omega : \mathbb{S}^3 \to \mathbb{R}, \quad \omega(\boldsymbol{\tau}) = \frac{1}{2}\mathcal{E}\boldsymbol{\tau} : \boldsymbol{\tau}$, where $\mathcal{E} : \mathbb{S}^3 \to \mathbb{S}^3$ is a fourth order tensor with the following properties:
  - i) $\mathcal{E}\boldsymbol{\sigma} : \boldsymbol{\tau} = \boldsymbol{\sigma} : \mathcal{E}\boldsymbol{\tau}$ for all $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^3$;
  - ii) there exists $m_{\mathcal{E}} \in (0,2)$ such that $\mathcal{E}\boldsymbol{\tau} : \boldsymbol{\tau} \geq m_{\mathcal{E}}\|\boldsymbol{\tau}\|^2_{\mathbb{S}^3}$ for all $\boldsymbol{\tau} \in \mathbb{S}^3$;
  - iii) there exists $c_{\mathcal{E}} \in [m_{\mathcal{E}}, 2)$ such that $\|\mathcal{E}\boldsymbol{\tau}\|_{\mathbb{S}^3} \leq c_{\mathcal{E}}\|\boldsymbol{\tau}\|_{\mathbb{S}^3}$ for all $\boldsymbol{\tau} \in \mathbb{S}^3$.
- $\omega : \mathbb{S}^3 \to \mathbb{R}, \quad \omega(\boldsymbol{\tau}) = \frac{1}{2}\mathcal{E}\boldsymbol{\tau} : \boldsymbol{\tau} + \frac{\xi}{2}\|\boldsymbol{\tau} - P_{\mathcal{K}}\boldsymbol{\tau}\|^2_{\mathbb{S}^3}$, where $\mathcal{E}$ is the fourth order tensor with the properties presented in the previous example, ξ is a positive coefficient of the material, small enough, and $P_{\mathcal{K}} : \mathbb{S}^3 \to \mathcal{K}$ is the projection operator on the set $\mathcal{K}$, which is a closed convex subset of $\mathbb{S}^3$ such that $\mathbf{0}_{\mathbb{S}^3} \in \mathcal{K}$; for more details, see [16] and also the useful reference [26].

For an example of function ψ, let us consider the function

$$\psi : \mathbb{R} \to \mathbb{R}, \quad \psi(x) = k_0 x^2/2, \qquad 0 < k_0 < 2. \tag{3.6}$$

Obviously, ψ is convex. Moreover, it satisfies the chain of inequalities in (A3) with $\alpha_2 = \beta_2 = k_0/2$. Notice that ψ is also differentiable and thus, according to a standard result in convex analysis, $\partial\psi(x) = \{\psi'(x)\} = \{k_0\, x\}$.

Therefore, by considering in the boundary condition (3.5) the function ψ defined in (3.6) we can write

$$-\sigma_\nu(\boldsymbol{x}) \in \partial\psi(u_\nu(\boldsymbol{x})) \Longleftrightarrow \sigma_\nu(\boldsymbol{x}) = -k_0\, u_\nu(\boldsymbol{x}) \quad \text{on } \Gamma_3,$$

which is the Winkler contact law. This law describes in a simplified manner the interaction between a deformable body and the soil, having extensive applications in civil engineering; for more details, see, e.g., [25] (pp. 83-84).

4 Variational formulation via two bipotentials

In order to deliver a variational formulation for our model, let $\boldsymbol{u}, \boldsymbol{\sigma}$ be regular enough functions which verify Problem 1. Taking into account (3.1), (3.3), (3.4) and (3.5), by using the Green formula (2.1), for all $\boldsymbol{v} \in V$ we can write

$$\int_{\Omega} \boldsymbol{\sigma}(\boldsymbol{x}) : \boldsymbol{\varepsilon}(\boldsymbol{v})(\boldsymbol{x})\, dx - \int_{\Gamma_3} \sigma_\nu(\boldsymbol{x})\, v_\nu(\boldsymbol{x})\, d\Gamma = \int_{\Omega} \boldsymbol{f}_0(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(\boldsymbol{x}) \cdot \boldsymbol{\gamma}\, \boldsymbol{v}(\boldsymbol{x})\, d\Gamma. \quad (4.1)$$

In order to simplify the writing, in the rest of this paper we use the following notation:

$$\overline{L} = L^2_s(\Omega)^{3\times 3} \times L^2(\Gamma_3) = \{\overline{\boldsymbol{\mu}} = (\boldsymbol{\mu}, \mu) \,:\, \boldsymbol{\mu} \in L^2_s(\Omega)^{3\times 3},\, \mu \in L^2(\Gamma_3)\}.$$

Let us define the following set:

$$\Lambda_f = \{\overline{\boldsymbol{\mu}} = (\boldsymbol{\mu}, \mu) \in \overline{L} : \int_{\Omega} \boldsymbol{\mu}(\boldsymbol{x}) : \boldsymbol{\varepsilon}(\boldsymbol{v})(\boldsymbol{x})\, dx - \int_{\Gamma_3} \mu(\boldsymbol{x})\, v_\nu(\boldsymbol{x})\, d\Gamma = (\boldsymbol{f}, \boldsymbol{v})_V \text{ for all } \boldsymbol{v} \in V\},$$

where by $\boldsymbol{f} \in V$ we denote the unique element obtained from the Riesz representation theorem applied to the linear and continuous mapping $V \ni \boldsymbol{v} \mapsto \int_{\Omega} \boldsymbol{f}_0(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(\boldsymbol{x}) \cdot \boldsymbol{\gamma}\, \boldsymbol{v}(\boldsymbol{x})\, d\Gamma$:

$$(\boldsymbol{f}, \boldsymbol{v})_V = \int_{\Omega} \boldsymbol{f}_0(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(\boldsymbol{x}) \cdot \boldsymbol{\gamma}\, \boldsymbol{v}(\boldsymbol{x})\, d\Gamma \quad \text{for all } \boldsymbol{v} \in V.$$

Notice that, according to (4.1), we have

$$\overline{\boldsymbol{\sigma}} = (\boldsymbol{\sigma}, \sigma_\nu) \in \Lambda_f. \quad (4.2)$$

Next, by (3.2) and Theorem 1 (applied to $X = \mathbb{S}^3$ and $\varphi = \omega$) we can write

$$\boldsymbol{\sigma}(\boldsymbol{x}) \in \partial\omega(\boldsymbol{\varepsilon}(\boldsymbol{u})(\boldsymbol{x})) \Leftrightarrow \omega(\boldsymbol{\varepsilon}(\boldsymbol{u})(\boldsymbol{x})) + \omega^*(\boldsymbol{\sigma}(\boldsymbol{x})) = \boldsymbol{\sigma}(\boldsymbol{x}) : \boldsymbol{\varepsilon}(\boldsymbol{u})(\boldsymbol{x}) \text{ a.e. in } \Omega. \quad (4.3)$$

We are now in the position to associate to the constitutive map ω the function $B_1 : \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{R}$ defined by

$$B_1(\boldsymbol{\sigma}, \boldsymbol{\mu}) = \omega(\boldsymbol{\sigma}) + \omega^*(\boldsymbol{\mu}) \quad \text{for all } \boldsymbol{\sigma}, \boldsymbol{\mu} \in \mathbb{S}^3.$$

On the other hand, by (3.5) and Theorem 1 (applied to $X = \mathbb{R}$ and $\varphi = \psi$) we can write

$$-\sigma_\nu(\boldsymbol{x}) \in \partial\psi(u_\nu(\boldsymbol{x})) \Leftrightarrow \psi(u_\nu(\boldsymbol{x})) + \psi^*(-\sigma_\nu(\boldsymbol{x})) = -\sigma_\nu(\boldsymbol{x}) u_\nu(\boldsymbol{x}) \text{ a.e. on } \Gamma_3. \quad (4.4)$$

We associate to the map ψ the function $B_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$B_2(x, y) = \psi(x) + \psi^*(y) \quad \text{for all } x, y \in \mathbb{R}.$$

We emphasize that, due to the convexity and lower semicontinuity of the functions ω and ψ, we can easily use Theorem 1 to prove that B_1 and B_2 are bipotentials. We define now the function

$$b : V \times \overline{L} \to \mathbb{R}, \quad b(\boldsymbol{v}, \overline{\boldsymbol{\mu}}) = \int_{\Omega} B_1(\boldsymbol{\varepsilon}(\boldsymbol{v})(\boldsymbol{x}), \boldsymbol{\mu}(\boldsymbol{x}))\, dx + \int_{\Gamma_3} B_2(v_\nu(\boldsymbol{x}), -\mu(\boldsymbol{x}))\, d\Gamma.$$

Note that by assumptions (A2)–(A3), keeping in mind Remark 2 we can see that $\omega(\boldsymbol{\varepsilon}(\boldsymbol{v})(\cdot)) \in L^1(\Omega)$ for all $\boldsymbol{v} \in V$, $\omega^*(\boldsymbol{\mu}(\cdot)) \in L^1(\Omega)$ for all $\boldsymbol{\mu} \in L^2_s(\Omega)^{3\times 3}$, $\psi(v_\nu(\cdot)) \in L^1(\Gamma_3)$ for all $\boldsymbol{v} \in V$ and $\psi^*(-\mu(\cdot)) \in L^1(\Gamma_3)$ for all $\mu \in L^2(\Gamma_3)$. Therefore, b is well defined.

Since B_1 and B_2 are bipotentials, we can write

$$\int_{\Omega} B_1(\boldsymbol{\varepsilon}(\boldsymbol{v})(\boldsymbol{x}), \boldsymbol{\mu}(\boldsymbol{x}))\, dx \geq \int_{\Omega} \boldsymbol{\mu}(\boldsymbol{x}) : \boldsymbol{\varepsilon}(\boldsymbol{v})(\boldsymbol{x})\, dx \text{ for all } \boldsymbol{v} \in V,\ \boldsymbol{\mu} \in L^2_s(\Omega)^{3\times 3}, \tag{4.5}$$

$$\int_{\Gamma_3} B_2(v_\nu(\boldsymbol{x}), -\mu(\boldsymbol{x}))\, d\Gamma \geq -\int_{\Gamma_3} \mu(\boldsymbol{x})\, v_\nu(\boldsymbol{x})\, d\Gamma \text{ for all } \boldsymbol{v} \in V,\ \mu \in L^2(\Gamma_3). \tag{4.6}$$

By adding (4.5) and (4.6) we obtain

$$b(\boldsymbol{v}, \overline{\boldsymbol{\mu}}) \geq \int_{\Omega} \boldsymbol{\mu}(\boldsymbol{x}) : \boldsymbol{\varepsilon}(\boldsymbol{v})(\boldsymbol{x})\, dx - \int_{\Gamma_3} \mu(\boldsymbol{x})\, v_\nu(\boldsymbol{x})\, d\Gamma \quad \text{for all } \boldsymbol{v} \in V,\ \overline{\boldsymbol{\mu}} \in \overline{L}. \tag{4.7}$$

Also, by using (4.3) and (4.4) we deduce that

$$b(\boldsymbol{u}, \overline{\boldsymbol{\sigma}}) = \int_{\Omega} \boldsymbol{\sigma}(\boldsymbol{x}) : \boldsymbol{\varepsilon}(\boldsymbol{u})(\boldsymbol{x})\, dx - \int_{\Gamma_3} \sigma_\nu(\boldsymbol{x})\, u_\nu(\boldsymbol{x})\, d\Gamma. \tag{4.8}$$

From (4.7) with $\overline{\boldsymbol{\mu}} = \overline{\boldsymbol{\sigma}}$ and (4.8), for all $\boldsymbol{v} \in V$ we can write

$$b(\boldsymbol{v}, \overline{\boldsymbol{\sigma}}) - b(\boldsymbol{u}, \overline{\boldsymbol{\sigma}}) \geq \int_{\Omega} \boldsymbol{\sigma}(\boldsymbol{x}) : \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u})(\boldsymbol{x})\, dx - \int_{\Gamma_3} \sigma_\nu(\boldsymbol{x})(v_\nu - u_\nu)(\boldsymbol{x})\, d\Gamma,$$

and by using (4.2) we are led to

$$b(\boldsymbol{v}, \overline{\boldsymbol{\sigma}}) - b(\boldsymbol{u}, \overline{\boldsymbol{\sigma}}) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V. \tag{4.9}$$

On the other hand, from (4.7) with $\boldsymbol{v} = \boldsymbol{u}$ and (4.8) we can write for all $\overline{\boldsymbol{\mu}} \in \overline{L}$ that

$$\begin{aligned} b(\boldsymbol{u}, \overline{\boldsymbol{\mu}}) - b(\boldsymbol{u}, \overline{\boldsymbol{\sigma}}) \geq & \int_{\Omega} \boldsymbol{\mu}(\boldsymbol{x}) : \boldsymbol{\varepsilon}(\boldsymbol{u})(\boldsymbol{x})\, dx - \int_{\Gamma_3} \mu(\boldsymbol{x})\, u_\nu(\boldsymbol{x})\, d\Gamma \\ & - \int_{\Omega} \boldsymbol{\sigma}(\boldsymbol{x}) : \boldsymbol{\varepsilon}(\boldsymbol{u})(\boldsymbol{x})\, dx + \int_{\Gamma_3} \sigma_\nu(\boldsymbol{x})\, u_\nu(\boldsymbol{x})\, d\Gamma, \end{aligned}$$

which implies

$$b(\boldsymbol{u}, \overline{\boldsymbol{\mu}}) - b(\boldsymbol{u}, \overline{\boldsymbol{\sigma}}) \geq (\boldsymbol{f}, \boldsymbol{u})_V - (\boldsymbol{f}, \boldsymbol{u})_V = 0 \quad \text{for all } \overline{\boldsymbol{\mu}} \in \Lambda_f. \tag{4.10}$$

Keeping in mind (4.9) and (4.10) we are led to the following weak formulation of Problem 1.

Problem 2. Find $\boldsymbol{u} \in V$ and $\overline{\boldsymbol{\sigma}} = (\boldsymbol{\sigma}, \sigma_\nu) \in \Lambda_f$ such that

$$\begin{aligned} & b(\boldsymbol{v}, \overline{\boldsymbol{\sigma}}) - b(\boldsymbol{u}, \overline{\boldsymbol{\sigma}}) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V \quad \text{for all } \boldsymbol{v} \in V, \\ & b(\boldsymbol{u}, \overline{\boldsymbol{\mu}}) - b(\boldsymbol{u}, \overline{\boldsymbol{\sigma}}) \geq 0 \qquad\qquad\quad \text{for all } \overline{\boldsymbol{\mu}} \in \Lambda_f. \end{aligned}$$

Any solution $(\boldsymbol{u}, \overline{\boldsymbol{\sigma}})$ of Problem 2 is called *weak solution* of Problem 1.

5 Well-posedness results

We begin this section with an existence and uniqueness result.

Theorem 3. *Under assumptions (A1)–(A3), Problem 2 has at least one solution. If, in addition, at least one of the functionals ω and ψ is strictly convex and at least one of the functionals ω^* and ψ^* is strictly convex, then Problem 2 has a unique solution.*

Proof. We emphasize that Problem 2 can be reformulated as follows: find $\boldsymbol{u} \in V$ and $\overline{\boldsymbol{\sigma}} = (\boldsymbol{\sigma}, \sigma_\nu) \in \Lambda_f$ such that

$$J(\boldsymbol{v}) - J(\boldsymbol{u}) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V \quad \text{for all } \boldsymbol{v} \in V, \tag{5.1}$$

$$J^*(\overline{\boldsymbol{\mu}}) - J^*(\overline{\boldsymbol{\sigma}}) \geq 0 \quad \text{for all } \overline{\boldsymbol{\mu}} \in \Lambda_f, \tag{5.2}$$

where

$$J : V \to \mathbb{R}, \quad J(\boldsymbol{v}) = \int_\Omega \omega(\boldsymbol{\varepsilon}(\boldsymbol{v})(\boldsymbol{x}))\, dx + \int_{\Gamma_3} \psi(v_\nu(\boldsymbol{x}))\, d\Gamma \tag{5.3}$$

$$J^* : \overline{L} \to \mathbb{R}, \quad J^*(\overline{\boldsymbol{\mu}}) = \int_\Omega \omega^*(\boldsymbol{\mu}(\boldsymbol{x}))\, dx + \int_{\Gamma_3} \psi^*(-\mu(\boldsymbol{x}))\, d\Gamma.$$

By defining the functional $J_f : V \to \mathbb{R}$, $J_f(\boldsymbol{v}) = J(\boldsymbol{v}) - (\boldsymbol{f}, \boldsymbol{v})_V$, we can see that (5.1) is equivalent to the minimization problem

$$J_f(\boldsymbol{v}) - J_f(\boldsymbol{u}) \geq 0 \quad \text{for all } \boldsymbol{v} \in V.$$

It is easy to observe that J_f is convex, lower semicontinuous and coercive, due to the properties of ω and ψ considered in (A2)–(A3). Therefore, using Theorem 2 we deduce that J_f has at least one minimum on V.

On the other hand, we observe that

$$\overline{\boldsymbol{\mu}} = (\boldsymbol{\varepsilon}(\boldsymbol{f}), 0_{L^2(\Gamma_3)}) \in \overline{L} \tag{5.4}$$

is an element of Λ_f. Therefore, the set Λ_f is nonempty. Using standard arguments we can deduce that Λ_f is also closed and convex. Moreover, Λ_f is unbounded. In order to prove this, we construct a sequence $(\overline{\boldsymbol{\mu}}_n)_n \subset \overline{L}$ as follows:

$$\overline{\boldsymbol{\mu}}_n = \overline{\boldsymbol{\mu}} + n\overline{\boldsymbol{\tau}} \quad \text{for all } n \geq 0,$$

with $\overline{\boldsymbol{\mu}}$ defined in (5.4) and $\overline{\boldsymbol{\tau}} = (\boldsymbol{\tau}, 0_{L^2(\Gamma_3)})$, where $\boldsymbol{\tau} \neq \boldsymbol{0}$ is an element of $\boldsymbol{\varepsilon}(V)^\perp$.

Since $\boldsymbol{\tau} \in \boldsymbol{\varepsilon}(V)^{\perp}$, we have

$$\int_{\Omega} \Big(\boldsymbol{\varepsilon}(\boldsymbol{f})(\boldsymbol{x}) + n\boldsymbol{\tau}(\boldsymbol{x})\Big) : \boldsymbol{\varepsilon}(\boldsymbol{v})(\boldsymbol{x})\, dx - \int_{\Gamma_3} (0+0)\, v_\nu(\boldsymbol{x})\, d\Gamma = (\boldsymbol{f}, \boldsymbol{v})_V$$

and we can see that $\overline{\boldsymbol{\mu}}_n \in \Lambda_f$ for all $n \geq 0$. Furthermore, $\|\overline{\boldsymbol{\mu}}_n\|_{\overline{L}} \to \infty$ as $n \to \infty$. Therefore, Λ_f is unbounded.

Keeping in mind Remark 2, it is easy to observe that J^* is convex and lower semicontinuous. Note that also by Remark 2, for all $\overline{\boldsymbol{\mu}} = (\boldsymbol{\mu}, \mu) \in \overline{L}$ we can write

$$\begin{aligned} J^*(\overline{\boldsymbol{\mu}}) &= \int_{\Omega} \omega^*(\boldsymbol{\mu}(\boldsymbol{x}))\, dx + \int_{\Gamma_3} \psi^*(-\mu(\boldsymbol{x}))\, d\Gamma \geq (1-\beta_1)\|\boldsymbol{\mu}\|^2_{L^2_s(\Omega)^{3\times 3}} \\ &+ (1-\beta_2)\|\mu\|^2_{L^2(\Gamma_3)} \geq \min\{1-\beta_1, 1-\beta_2\}\Big(\|\boldsymbol{\mu}\|^2_{L^2_s(\Omega)^{3\times 3}} + \|\mu\|^2_{L^2(\Gamma_3)}\Big) \\ &= \min\{1-\beta_1, 1-\beta_2\}\|\overline{\boldsymbol{\mu}}\|^2_{\overline{L}} \end{aligned}$$

and see that the functional J^* is coercive. We apply now Theorem 2 and find that J^* admits al least one minimum on Λ_f. Hence, Problem 2 admits the pair solution whose first component is the minimizer of J_f on V and whose second component is the minimizer of J^* on Λ_f. If at least one of the functionals ω and ψ is strictly convex and at least one of the functionals ω^* and ψ^* is strictly convex, then J_f and J^* have unique minimizers and Problem 2 admits a unique solution. □

We want to investigate now some properties of the solution.

Proposition 1. *Consider (A1)–(A3). Let* $\boldsymbol{f} \in V$ *be given datum and let* $(\boldsymbol{u}, \overline{\boldsymbol{\sigma}})$ *be a solution of Problem 2.*

i) There exists $c_1 > 0$ *independent of the datum such that*

$$\|\boldsymbol{u}\|_V \leq c_1 \|\boldsymbol{f}\|_V.$$

ii) There exists $c_2 > 0$ *independent of the datum such that*

$$\|\overline{\boldsymbol{\sigma}}\|_{\overline{L}} \leq c_2 \|\boldsymbol{f}\|_V.$$

Proof. To prove i), we note that (A2) and (A3) imply that the functional J defined in (5.3) vanishes in $\mathbf{0}_V$. Thus, we consider $\boldsymbol{v} = \mathbf{0}_V$ in (5.1) to write

$$\int_{\Omega} \omega(\boldsymbol{\varepsilon}(\boldsymbol{u})(\boldsymbol{x}))\, dx + \int_{\Gamma_3} \psi(u_\nu(\boldsymbol{x}))\, d\Gamma \leq (\boldsymbol{f}, \boldsymbol{u})_V.$$

Note that assumptions (A2)–(A3) also show that $\int_{\Omega} \omega(\boldsymbol{\varepsilon}(\boldsymbol{u})(\boldsymbol{x}))\, dx \geq \alpha_1 \|\boldsymbol{u}\|^2_V$ and $\int_{\Gamma_3} \psi(u_\nu(\boldsymbol{x}))\, d\Gamma \geq 0$. Thus, we are led to

$$\alpha_1 \|\boldsymbol{u}\|^2_V \leq \|\boldsymbol{f}\|_V\, \|\boldsymbol{u}\|_V.$$

As a result, $\|\boldsymbol{u}\|_V \leq c_1\, \|\boldsymbol{f}\|_V$, where $c_1 = 1/\alpha_1$.

To prove *ii*), we take in the inequality (5.2) the element $\overline{\boldsymbol{\mu}} = (\boldsymbol{\varepsilon}(\boldsymbol{f}), 0_{L^2(\Gamma_3)}) \in \Lambda_f$ defined in (5.4) to write

$$\int_\Omega \omega^*(\boldsymbol{\varepsilon}(\boldsymbol{f})(\boldsymbol{x}))\,dx + \int_{\Gamma_3} \psi^*(0)\,d\Gamma - \int_\Omega \omega^*(\boldsymbol{\sigma}(\boldsymbol{x}))\,dx - \int_{\Gamma_3} \psi^*(-\sigma_\nu(\boldsymbol{x}))\,d\Gamma \geq 0.$$

We emphasize that Remark 2 implies $\int_\Omega \omega^*(\boldsymbol{\varepsilon}(\boldsymbol{f})(\boldsymbol{x}))\,dx \leq \frac{1}{4\alpha_1}\|\boldsymbol{f}\|_V^2$. Hence, since $\int_{\Gamma_3} \psi^*(0)\,d\Gamma = 0$, we obtain that

$$\frac{1}{4\alpha_1}\|\boldsymbol{f}\|_V^2 \geq \int_\Omega \omega^*(\boldsymbol{\sigma}(\boldsymbol{x}))\,dx + \int_{\Gamma_3} \psi^*(-\sigma_\nu(\boldsymbol{x}))\,d\Gamma.$$

Using again Remark 2, we can write

$$\begin{aligned}\frac{1}{4\alpha_1}\|\boldsymbol{f}\|_V^2 &\geq (1-\beta_1)\|\boldsymbol{\sigma}\|^2_{L^2_s(\Omega)^{3\times 3}} + (1-\beta_2)\|\sigma_\nu\|^2_{L^2(\Gamma_3)} \\ &\geq \min\{1-\beta_1, 1-\beta_2\}\|\overline{\boldsymbol{\sigma}}\|^2_{\overline{L}}.\end{aligned}$$

Thus, $\|\overline{\boldsymbol{\sigma}}\|_{\overline{L}} \leq c_2\|\boldsymbol{f}\|_V$ with $c_2 = 1/(2\sqrt{\alpha_1 \min\{1-\beta_1, 1-\beta_2\}})$. $\square$

In our study on the data dependence we shall need the following lemma.

Lemma 1. *Let $(\boldsymbol{f}_n)_n \subset V$ be a sequence such that $\boldsymbol{f}_n \to \boldsymbol{f}$ in V. Then, $\Lambda_{f_n} \xrightarrow{M} \Lambda_f$.*

Proof. Let us check the conditions in Definition 3. To start, we prove that for every sequence $(\overline{\boldsymbol{\mu}}_n)_n$ such that $\overline{\boldsymbol{\mu}}_n \in \Lambda_{f_n}$ for each $n \in \mathbb{N}$ and $\overline{\boldsymbol{\mu}}_n = (\boldsymbol{\mu}_n, \mu_n) \rightharpoonup \overline{\boldsymbol{\mu}} = (\boldsymbol{\mu}, \mu)$ in $\overline{L}$, we have $\overline{\boldsymbol{\mu}} \in \Lambda_f$.

Let $\overline{\boldsymbol{\mu}}_n \in \Lambda_{f_n}$ be such that $\overline{\boldsymbol{\mu}}_n \rightharpoonup \overline{\boldsymbol{\mu}}$ in $\overline{L}$. As

$$\int_\Omega \boldsymbol{\mu}_n(\boldsymbol{x}) : \boldsymbol{\varepsilon}(\boldsymbol{v})(\boldsymbol{x})\,dx - \int_{\Gamma_3} \mu_n(\boldsymbol{x})\,v_\nu(\boldsymbol{x})\,d\Gamma = (\boldsymbol{f}_n, \boldsymbol{v})_V \quad \text{for all } \boldsymbol{v} \in V,$$

by passing to the limit for $n \to \infty$, we obtain

$$\int_\Omega \boldsymbol{\mu}(\boldsymbol{x}) : \boldsymbol{\varepsilon}(\boldsymbol{v})(\boldsymbol{x})\,dx - \int_{\Gamma_3} \mu(\boldsymbol{x})\,v_\nu(\boldsymbol{x})\,d\Gamma = (\boldsymbol{f}, \boldsymbol{v})_V \text{ for all } \boldsymbol{v} \in V,$$

which concludes that $\overline{\boldsymbol{\mu}} \in \Lambda_f$.

We prove now that for every $\overline{\boldsymbol{\mu}} = (\boldsymbol{\mu}, \mu) \in \Lambda_f$, there exists a sequence $(\overline{\boldsymbol{\mu}}_n)_n$ such that $\overline{\boldsymbol{\mu}}_n \in \Lambda_{f_n}$ for each $n \in \mathbb{N}$ and $\overline{\boldsymbol{\mu}}_n \to \overline{\boldsymbol{\mu}}$ in $\overline{L}$.

Let $\overline{\boldsymbol{\mu}} = (\boldsymbol{\mu}, \mu) \in \Lambda_f$ be arbitrarily fixed. Let us construct a sequence $(\overline{\boldsymbol{\mu}}_n)_n$ as follows: for each positive integer n,

$$\overline{\boldsymbol{\mu}}_n = \overline{\boldsymbol{\mu}} - (\boldsymbol{\varepsilon}(\boldsymbol{f}), 0_{L^2(\Gamma_3)}) + (\boldsymbol{\varepsilon}(\boldsymbol{f}_n), 0_{L^2(\Gamma_3)}) = (\boldsymbol{\mu} - \boldsymbol{\varepsilon}(\boldsymbol{f}) + \boldsymbol{\varepsilon}(\boldsymbol{f}_n), \mu).$$

Since for all $\boldsymbol{v} \in V$ it holds

$$\int_\Omega \Big(\boldsymbol{\mu}(\boldsymbol{x}) - \boldsymbol{\varepsilon}(\boldsymbol{f})(\boldsymbol{x}) + \boldsymbol{\varepsilon}(\boldsymbol{f}_n)(\boldsymbol{x})\Big) : \boldsymbol{\varepsilon}(\boldsymbol{v})(\boldsymbol{x})\,dx - \int_{\Gamma_3} \mu(\boldsymbol{x})\,v_\nu(\boldsymbol{x})\,d\Gamma = (\boldsymbol{f}_n, \boldsymbol{v})_V,$$

it is easy to observe that $\overline{\boldsymbol{\mu}}_n \in \Lambda_{f_n}$ for each positive integer n. Moreover, since $\boldsymbol{f}_n \to \boldsymbol{f}$ in V we deduce that $\overline{\boldsymbol{\mu}}_n \to \overline{\boldsymbol{\mu}}$ in $\overline{L}$. $\square$

In what follows, we will see how the weak solution of Problem 1 behaves when we modify the data. The behavior is illustrated by the following theorem.

Theorem 4. *We admit assumptions (A1)–(A3), and, in addition, we assume that at least one of the functionals ω and ψ is strictly convex and at least one of the functionals ω^* and ψ^* is strictly convex. The operator*

$$S : V \to V \times \overline{L}, \qquad S(\boldsymbol{f}) = (\boldsymbol{u}, \overline{\boldsymbol{\sigma}})$$

associated to Problem 2 is demicontinuous.

Proof. Let $(\boldsymbol{f}_n)_n \subset V$ be a convergent sequence to $\boldsymbol{f}$ in V. Let n be a positive integer and let $(\boldsymbol{u}_n, \overline{\boldsymbol{\sigma}}_n)$ be the unique solution of Problem 2 corresponding to $\boldsymbol{f}_n$. We denote by $(\boldsymbol{u}, \overline{\boldsymbol{\sigma}})$ the unique solution of Problem 2 corresponding to $\boldsymbol{f}$.

Since $(\boldsymbol{f}_n)_n \subset V$ is a convergent sequence, there exists $M > 0$ such that

$$\|\boldsymbol{f}_n\|_V \leq M \quad \text{for all } n \in \mathbb{N}.$$

Using Proposition 1 for the data $\boldsymbol{f}_n$, we obtain that $\Big((\boldsymbol{u}_n, \overline{\boldsymbol{\sigma}}_n)\Big)_n$ is a bounded sequence in $V \times \overline{L}$. Thus, there exists a subsequence $\Big((\boldsymbol{u}_{n'}, \overline{\boldsymbol{\sigma}}_{n'})\Big)_{n'}$ and an element $(\widetilde{\boldsymbol{u}}, \widetilde{\overline{\boldsymbol{\sigma}}}) \in V \times \overline{L}$ such that

$$(\boldsymbol{u}_{n'}, \overline{\boldsymbol{\sigma}}_{n'}) \rightharpoonup (\widetilde{\boldsymbol{u}}, \widetilde{\overline{\boldsymbol{\sigma}}}) \text{ in } V \times \overline{L} \text{ as } n' \to \infty,$$

which implies that

$$\boldsymbol{u}_{n'} \rightharpoonup \widetilde{\boldsymbol{u}} \quad \text{in } V, \quad \overline{\boldsymbol{\sigma}}_{n'} \rightharpoonup \widetilde{\overline{\boldsymbol{\sigma}}} \quad \text{in } \overline{L}.$$

Since

$$J(\boldsymbol{v}) - J(\boldsymbol{u}_{n'}) \geq (\boldsymbol{f}_{n'}, \boldsymbol{v} - \boldsymbol{u}_{n'})_V \quad \text{for all } \boldsymbol{v} \in V,$$

we can take the limsup as $n' \to \infty$ to obtain that

$$J(\boldsymbol{v}) - J(\widetilde{\boldsymbol{u}}) \geq (\boldsymbol{f}, \boldsymbol{v} - \widetilde{\boldsymbol{u}})_V \quad \text{for all } \boldsymbol{v} \in V. \tag{5.5}$$

On the other hand, we know that the following inequality holds

$$J^*(\overline{\boldsymbol{\mu}}_{n'}) - J^*(\overline{\boldsymbol{\sigma}}_{n'}) \geq 0 \quad \text{for all } \overline{\boldsymbol{\mu}}_{n'} \in \Lambda_{f_{n'}}.$$

We want to prove that

$$J^*(\overline{\boldsymbol{\mu}}) - J^*(\widetilde{\overline{\boldsymbol{\sigma}}}) \geq 0 \quad \text{for all } \overline{\boldsymbol{\mu}} \in \Lambda_{f}.$$

For this purpose, let $\overline{\boldsymbol{\mu}} \in \Lambda_f$. Notice that applying Lemma 1 we have

$$\Lambda_{f_{n'}} \xrightarrow{\mathrm{M}} \Lambda_f. \tag{5.6}$$

Therefore, there exists $(\widetilde{\overline{\boldsymbol{\mu}}}_{n'})_{n'} \subset \overline{L}$ such that $\widetilde{\overline{\boldsymbol{\mu}}}_{n'} \in \Lambda_{f_{n'}}$ for each $n' \in \mathbb{N}$ and $\widetilde{\overline{\boldsymbol{\mu}}}_{n'} \to \overline{\boldsymbol{\mu}}$ in $\overline{L}$. Hence, we can write

$$J^*(\widetilde{\overline{\boldsymbol{\mu}}}_{n'}) - J^*(\overline{\boldsymbol{\sigma}}_{n'}) \geq 0,$$

and passing to the limsup as $n' \to \infty$ in the above inequality, since J^* is upper semicontinuous and weakly lower semicontinuous (being convex and lower semicontinuous), we get

$$J^*(\overline{\boldsymbol{\mu}}) - J^*(\widetilde{\overline{\boldsymbol{\sigma}}}) \geq 0. \tag{5.7}$$

In addition, as $\overline{\boldsymbol{\sigma}}_{n'} \rightharpoonup \widetilde{\overline{\boldsymbol{\sigma}}}$ and $\overline{\boldsymbol{\sigma}}_{n'} \in \Lambda_{f_{n'}}$ for each $n' \in \mathbb{N}$, we have from (5.6) that $\widetilde{\overline{\boldsymbol{\sigma}}} \in \Lambda_f$.

Therefore, keeping in mind (5.5) and (5.7) we deduce that $(\widetilde{\boldsymbol{u}}, \widetilde{\overline{\boldsymbol{\sigma}}})$ is a solution of Problem 2 corresponding to $\boldsymbol{f}$. But since $(\boldsymbol{u}, \overline{\boldsymbol{\sigma}})$ is the unique solution of Problem 2 corresponding to $\boldsymbol{f}$, we deduce that $\widetilde{\boldsymbol{u}} = \boldsymbol{u}$, $\widetilde{\overline{\boldsymbol{\sigma}}} = \overline{\boldsymbol{\sigma}}$ and

$$\boldsymbol{u}_{n'} \rightharpoonup \boldsymbol{u} \text{ as } n' \to \infty; \quad \overline{\boldsymbol{\sigma}}_{n'} \rightharpoonup \overline{\boldsymbol{\sigma}} \text{ as } n' \to \infty.$$

Thus, the weak limits are independent of the subsequences. Consequently, the entire sequences $(\boldsymbol{u}_n)_n$ and $(\overline{\boldsymbol{\sigma}}_n)_n$ are weakly convergent to $\boldsymbol{u}$ and $\overline{\boldsymbol{\sigma}}$, respectively. Then,

$$(\boldsymbol{u}_n, \overline{\boldsymbol{\sigma}}_n) \rightharpoonup (\boldsymbol{u}, \overline{\boldsymbol{\sigma}}).$$

Therefore, $S(\boldsymbol{f}_n) \rightharpoonup S(\boldsymbol{f})$, which concludes that S is demicontinuous. □

6 Final comments, conclusions and outlook

In order to highlight the relevance of our study we briefly comment on some possible variational formulations under appropriate hypotheses. Firstly, we emphasize that by following a classical approach we can arrive to the primal variational formulation of Problem 1,

$$\boldsymbol{u} \in V, \quad J(\boldsymbol{v}) - J(\boldsymbol{u}) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V \quad \text{for all } \boldsymbol{v} \in V,$$

with J defined in (5.3). Such a variational formulation allows to compute the displacement field $\boldsymbol{u}$.

On the other hand, by introducing a single bipotential which is in connection with the constitutive law, if instead of (A3) we assume that ψ is a seminorm then with a similar technique to that used in [21] we can deliver for Problem 1 a two-field variational formulation of the form below

$$\begin{aligned} b_1(\boldsymbol{v}, \boldsymbol{\sigma}) - b_1(\boldsymbol{u}, \boldsymbol{\sigma}) + j_\psi(\boldsymbol{v}) - j_\psi(\boldsymbol{u}) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V \quad & \text{for all } \boldsymbol{v} \in V, \\ b_1(\boldsymbol{u}, \boldsymbol{\mu}) - b_1(\boldsymbol{u}, \boldsymbol{\sigma}) \geq 0 \quad & \text{for all } \boldsymbol{\mu} \in \Lambda, \end{aligned}$$

where

- $j_\psi : V \to \mathbb{R}, \quad j_\psi(\boldsymbol{v}) = \int_{\Gamma_3} \psi(v_\nu(\boldsymbol{x}))\, d\Gamma$;
- $b_1 : V \times L^2_s(\Omega)^{3\times 3} \to \mathbb{R}$, $b_1(\boldsymbol{v}, \boldsymbol{\mu}) = \int_\Omega B_1(\boldsymbol{\varepsilon}(\boldsymbol{v})(\boldsymbol{x}), \boldsymbol{\mu}(\boldsymbol{x}))\, dx$ with B_1 defined in (4.5);

- $\Lambda = \{\boldsymbol{\mu} \in L^2_s(\Omega)^{3\times 3} \,:\, (\boldsymbol{\mu}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2_s(\Omega)^{3\times 3}} + j_\psi(\boldsymbol{v}) \geq (\boldsymbol{f}, \boldsymbol{v})_V$ for all $\boldsymbol{v} \in V\}$.

This second variational formulation allows the computation of the pair $(\boldsymbol{u}, \boldsymbol{\sigma})$.

In the present paper, the working hypotheses allow us to introduce two bipotentials. Using this approach, we are able to find not only the displacement vector $\boldsymbol{u}$ and the stress tensor $\boldsymbol{\sigma}$ but also the normal stress σ_ν on the contact zone.

Our work represents a first step in the direction of the weak solvability via bipotentials which are in connection with boundary conditions. It would be interesting to explore this new path by formulating via bipotentials arising from the boundary conditions other mathematical models. Also, finding appropriate numerical algorithms in order to approximate the weak solutions is a very challenging task.

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Appendix A

List of symbols

$\mathbb{N}$	set of positive integers, $\mathbb{R}$ set of real numbers
$\mathbb{R}^3$	3-dimensional Euclidean space
$\mathbb{S}^3$	space of second order symmetric tensors on $\mathbb{R}^3$
$(\cdot,\cdot)_H$	inner product in the Hilbert space H
$\cdot$	inner product on $\mathbb{R}^3$, $:$ inner product on $\mathbb{S}^3$
$\|\cdot\|_B$	norm in the Banach space B
$\|\cdot\|$	Euclidean norm on $\mathbb{R}^3$, $\|\cdot\|_{\mathbb{S}^3}$ Euclidean norm on $\mathbb{S}^3$
Ω	a bounded domain in $\mathbb{R}^3$ (i.e. an open, bounded, connected set)
Γ	boundary of the domain Ω
$\overline{\Omega}$	closure of Ω in $\mathbb{R}^3$, i.e. $\overline{\Omega} = \Omega \cup \Gamma$
Γ_i	a part of the boundary, $i \in \{1,2,3\}$
$meas(\Gamma_i)$	surface measure of Γ_i
$\boldsymbol{\nu}$	unit outward normal vector to Γ
$\boldsymbol{u}$	displacement vector
u_ν	normal component of the vector $\boldsymbol{u}$, i.e. $u_\nu = \boldsymbol{u}\cdot\boldsymbol{\nu}$
$\boldsymbol{u}_\tau$	tangential component of the vector $\boldsymbol{u}$, i.e. $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu\boldsymbol{\nu}$
$\boldsymbol{\sigma}$	Cauchy stress tensor
σ_ν	normal component of the Cauchy vector $\boldsymbol{\sigma\nu}$, i.e. $\sigma_\nu = \boldsymbol{\sigma\nu}\cdot\boldsymbol{\nu}$
σ_τ	tangential part of the Cauchy vector $\boldsymbol{\sigma\nu}$, i.e. $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu\boldsymbol{\nu}$
Div	divergence operator for tensors
$\boldsymbol{\gamma}$	trace operator for vector-valued functions
$\partial\psi$	subdifferential of the function ψ
$\rightharpoonup$	weak convergence, $\xrightarrow{\mathrm{M}}$ Mosco convergence
$a.e.$	almost everywhere
$P_{\mathcal{K}}$	projection operator on a nonempty convex subset $\mathcal{K}$
$\oplus$	direct sum
(x,y)	pair of components x and y
$A^\perp$	orthogonal complement of a subset A of a Hilbert space
$\emptyset$	empty set