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# Identification of a Time-Dependent Source Term in a Nonlocal Problem for Time Fractional Diffusion Equation*

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**Abstract.** This paper is concerned with the inverse problem of recovering the time dependent source term in a time fractional diffusion equation, in the case of non-local boundary condition and integral overdetermination condition. The boundary conditions of this problem are regular but not strongly regular. The existence and uniqueness of the solution are established by applying generalized Fourier method based on the expansion in terms of root functions of a spectral problem, weakly singular Volterra integral equation and fractional type Gronwall's inequality. Moreover, we show its continuous dependence on the data.

**Keywords:** inverse source problem, fractional diffusion equation, not strongly regular boundary condition, generalized Fourier method, weakly singular Volterra integral equation.

**AMS Subject Classification:** 35R30; 34A12.

## 1 Introduction

Differential equations involving fractional time derivatives of order less than 1, namely time fractional diffusion equation (TFDe) have become an essential tool in modeling slow diffusion (subdiffusion) processes in chemistry, physics, viscoelasticity, biology, nuclear power engineering [7,26,27]. The problems with

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TFDe should be more difficult since some standard methods cannot be applied for a non-classical derivative. The difficulty comes from the definition of the fractional-order derivatives, which is essentially an integral with the kernel of weak singularity.

It is important to emphasize that the first theoretical results for the inverse problem of determining the coefficients in TFDe are obtained in [10, 20, 22, 33, 39]. Inverse source problems for TFDe have been intensively investigated by many researchers under various initial, boundary and overdetermination conditions. The inverse problem of finding of a space-dependent source term from final temperature distribution was considered in [4, 6, 14, 15, 21, 22, 30, 37] and recovering a space-dependent source term from total energy measurement has been discussed in [8, 24, 25]. For a TFDe inverse problem of identification a time-dependent source term from temperature measurement at the selected point in the spatial domain was considered in [5, 19] and determining a time-dependent source term from integral type overdetermination condition was studied in [2, 3, 16, 18]. The initial and boundary data identification from final measurements in the initial boundary value problem for time-fractional heat equation was studied in [1, 21] and [9], respectively.

In studying inverse source problem for TFDe there are several approaches in the literature, one of which is the generalized Fourier method. The papers [1, 2, 3, 4, 18, 22, 30] consider the inverse source problems in point of view of spectral analysis that the temperature distribution is expanded in terms of root functions (the system of eigenfunction and associated function) of a spectral problem with the boundary conditions which are used in related problem. If the boundary conditions includes some nonlocal characters then the classical self-adjoint eigenfunction expansion results can not be applied in auxiliary spectral problem and it neeeds additional eigenfunction expansion investigation [28]. Non-self-adjoint operators appear e.g. in the modeling of processes with dissipation [17, 36]. In many cases a nonlocal condition is more realistic in handling physical problems than the classical local conditions, which motivates the investigation of nonlocal boundary-value problems.

In this paper, we are concerned with an inverse source problem of recovering a time-dependent source term $a(t)$ and $u(x,t)$ for the following TFDe,

$$D^q_{0+}(u(x,t)-u(x,0)) = u_{xx} + a(t)f(x,t), \qquad (x,t) \in \Omega_T, \tag{1.1}$$

supplemented with the initial condition

$$u(x,0) = \varphi(x), \quad 0 \le x \le 1, \tag{1.2}$$

and the nonlocal family of boundary conditions

$$u(0,t) = u(1,t), \quad \alpha u_x(0,t) = u_x(1,t)\ ,\ 0 \le t \le T\ , \tag{1.3}$$

where $0 < q < 1$, $\Omega_T = \{(x,t) : 0 < x < 1, 0 < t \le T\}$ , $f(x,t), \varphi(x)$ are given functions and $\alpha$ is real constant.

The direct problem is to determine the solution $u(x,t)$ that satisfies (1.1)–(1.3), when the function $a(t)$ is known. The structure of the source term

$a(t)f(x,t)$ in (1.1) appears in microwave heating process, in which $a(t)$ is proportional to power of external energy source and $f(x,t)$ is local conversion rate of microwave energy. The external energy is supplied to a target at a controlled level by the microwave generating equipment. Inverse source problem for such a model gives an idea of how total energy content might be externally controlled. However, we are interested in finding the pair of solution $\{a(t), u(x,t)\}$ from (1.1)–(1.3) with integral overdetermination condition

$$\int_0^1 u(x,t)dx = E(t), \ 0 \leq t \leq T, \tag{1.4}$$

where $E(t)$ are given functions. The integral condition (1.5) arises naturally and can be used as supplementary information in the identification of the source term. Such type of condition can describe several physical phenomena in context of chemical engineering, thermoelasticity, heat conduction and diffusion process, fluid flow in porous media [18].

Compared with the inverse source problem for TFDe with local boundary conditions, many less results are available for the nonlocal problems.

The paper [22] investigated first an inverse problem of finding the temperature distribution and the space dependent heat source from final data. The obtained results were extended to the two-dimensional problem in [23]. For the case $\alpha = 0$ in (1.3), the authors in [2] determined a source term independent of the space variable and solution for TFDe from integral type extra condition. The well-posedness of the inverse problem in more general case was proved by using eigenfunction expansion of a non-self adjoint spectral problem along the generalized Fourier method in [18]. For a two parameter TFDe, the authors in [15] constructed the series representations of the solution and space dependent source term by utilizing solvability of a $2 \times 2$ linear system with a coefficient matrix involving Mittag-Leffler functions. Recovering time dependent coefficient of a generalized TFDe from energy measurement and space dependent source term from final data was performed in [13] and [14], respectively. In all the previous works cited above, the problems considered involve the special case of (1.3): $\alpha = 0$ in [2] and $\alpha = 1$ in [13, 14, 15, 22, 23]. The general case $\alpha \neq 0$ for boundary condition and weakly singular Volterra integral equation approach for unique determination of solely time dependent source term is considered for the first time in our work.

The rest of the manuscript is organized as follows: in Section 2, we present the mathematical formulation of auxiliary spectral problem and construction of the bi-orthogonal system. In Section 3, existence and uniqueness of the solution of the inverse time-dependent source problem is proved by generalized Fourier method and Volterra type integral equation with the kernel may have a diagonal singularity. Moreover, the continuous dependence upon the data of the solution of the inverse problem is also shown by utilizing the fractional type Gronwall's inequality in this section. Finally, in Section 4, we recall some basic definitions and facts on fractional calculus and present some necessary lemmas needed for further investigations.

## 2 Auxiliary spectral problem and construction of the bi-orthogonal system

The spectral problem for the initial boundary value problem (1.1)–(1.3) is given by

$$\begin{cases} X''(x) + \lambda X(x) = 0, \ 0 < x < 1, \\ X(0) = X(1), \quad \alpha X^{'}(0) = X^{'}(1). \end{cases} \tag{2.1}$$

If $\alpha \neq -1$, then the boundary conditions in (2.1) are regular but not strengthened regular [28]. In this case the problem (2.1) has double eigenvalues $\lambda_n = (2\pi n)^2$ (except for the first $\lambda_0 = 0$). The set of eigenfunctions and eigenvalues of the problem (2.1) are the following:

$$X_0(x) = 0, \ \lambda_0 = 0; \ X_{2n-1}(x) = 4\cos(2\pi nx), \ \lambda_n = (2\pi n)^2, \ n \geq 1. \tag{2.2}$$

It is well-known that the main difficulty in applying the Fourier method is its basisness, i.e., the expansion in terms of eigenfunctions of the auxiliary spectral problem. The set of eigenfunctions $\{X_n(x)\}$ $(n = 0, 1, 3, 5, ...)$ of the spectral problem (2.1) is not complete in the space $L_2(0,1)$. The set of eigenfunctions with the associated eigenfunctions is supplemented to make the set complete on $L_2(0,1)$. We use the equation

$$\begin{cases} -X''_{2n}(x) = \lambda_n X_{2n}(x) + \sqrt{\lambda_n} X_{2n-1}(x), \ 0 < x < 1, \\ X_{2n}(0) = X_{2n}(1), \quad \alpha X^{'}_{2n}(0) = X^{'}_{2n}(1), \end{cases}$$

to avoid the problem of the choice of associated functions for their construction.

Then, associated functions are: $X_{2n}(x) = \frac{2}{1-\alpha}(1 - (1-\alpha)x)\sin(2\pi nx)$, $n = 1, 2, ....$ This set of functions $\{X_0(x), X_{2n-1}(x), X_{2n}(x)\}$ form a Riesz basis in $L_2(0,1)$. The system $\{Y_0(x), Y_{2n-1}(x), Y_{2n}(x)\}$ $(n = 1, 2, ...)$, which is bi-orthogonal to the system $\{X_0(x), X_{2n-1}(x), X_{2n}(x)\}$ $(n = 0, 1, 2, ...)$ has the form

$$\begin{cases} Y_0(x) = \frac{\alpha + (1-\alpha)}{1+\alpha} x, \\ Y_{2n-1}(x) = \frac{\alpha + (1-\alpha)}{1+\alpha} x \cos(2\pi nx), \\ Y_{2n}(x) = 2\frac{1-\alpha}{1+\alpha} \sin(2\pi nx). \end{cases} \tag{2.3}$$

In the rest of the paper, we will consider the following class of functions to investigate well-posedness of the inverse source problem (1.1)–(1.4): $\Phi^4[0,1] := \{\psi(x) \in C^4[0,1]; \psi(0) = \psi(1), \alpha\psi^{'}(0) = \psi^{'}(1), \psi^{''}(0) = \psi^{''}(1)\}$.

**Lemma 1.** *( [32]) If $\varphi(x) \in \Phi^4[0,1]$, $f(x,t) \in \Phi^4[0,1], \forall t \in [0,T]$, then we have:*

$$\lambda_n^2(\varphi, Y_{2n}) = (\varphi^4, Y_{2n}), \ \ \lambda_n^2(f, Y_{2n}) = (f^4, Y_{2n}), \ \ n \geq 1,$$

$$\sum_{n=1}^{\infty} \left| \sqrt{\lambda_n}(f, Y_{2n}) \right| \leq c_1 \|f\|_{C^4[0,1]}, \ \sum_{n=1}^{\infty} \left| \sqrt{\lambda_n}(\varphi, Y_{2n}) \right| \leq c_2 \|\varphi\|_{C^4[0,1]},$$

*where $c_1$ and $c_2$ are some positive constants.*

## 3 Well-posedness of the inverse problem

A solution of the inverse problem (1.1)–(1.4), which we called the classical solution, is a set of functions $\{a(t), u(x,t)\}$ satisfying $a(t) \in C[0,T]$, $u(.,t) \in C^2([0,1],\mathbb{R})$ and $D^q_{0+}(u(x,.) - u(x,0)) \in C[(0,T],\mathbb{R}]$.

The choice of the term $D^q_{0+}(u(x,t) - u(x,0))$ instead of the usual term $D^q_{0+}u(x,t)$ is not only to avoid the singularity at zero, but also to include certain initial conditions [31].

In this part, we will examine the existence, uniqueness and stability of the solution of the inverse initial-boundary value problem for the TFDe (1.1) with time-dependent source term. For the proof, we will need to use the eigenfunctions expansion method, some properties of the Volterra integral equation with weak singular kernel and fractional type Gronwall inequality.

**Theorem 1.** *(Existence and uniqueness) Suppose that the following conditions hold:*

$(A_1)$ $\varphi(x) \in \Phi^4[0,1]$;

$(A_2)$ $E(t) \in C[0,T]\backslash C^1[0,T]$; $D^q_{0+}(E(t) - E(0)) \in C[0,T]$; $E(0) = \int_0^1 \varphi(x)dx$;

$(A_3)$ $f(x,t) \in C(\overline{\Omega_T})$; $f(x,t) \in \Phi^4[0,1]$, $\forall t \in [0,T]$; $f(0,t) = f(1,t)$, $\alpha f_x(0,t) = f_x(1,t)$, $f_{xx}(0,t) = f_{xx}(1,t)$, $\int_0^1 f(x,t)dx \neq 0 \, \forall t \in [0,T]$.

*Then, there exists a unique classical solution of the inverse problem* (1.1)–(1.4) *in* $\Omega_T$.

*Proof.* For arbitrary $a(t) \in C[0,T]$, by applying standard procedure of the generalized Fourier method the solution $u(x,t)$ of problem (1.1)–(1.3) can be expanded to series form as follows:

$$u(x,t) = v_0(t)X_0(x) + \sum_{n=1}^{\infty}[v_{1n}(t)X_{2n-1}(x) + v_{2n}(t)X_{2n}(x)], \tag{3.1}$$

where the functions $v_0(t)$, $v_{1n}(t)$, $v_{2n}(t)$, $n = 1, 2, ...$ are to be determined. Let $\{f_0(t), f_{1n}(t), f_{2n}(t)\}$ and $\{\varphi_0, \varphi_{1n}, \varphi_{2n}\}$ be the coefficients of the series expansion of $f(x,t)$ and $\varphi(x)$, respectively, i.e.,

$$f_0(t) = \int_0^1 f(x,t)Y_0(x)dx, \quad f_{1n}(t) = \int_0^1 f(x,t)Y_{2n-1}(x)dx,$$
$$f_{2n}(t) = \int_0^1 f(x,t)Y_{2n}(x)dx, \quad \varphi_0 = \int_0^1 \varphi(x)Y_0(x)dx,$$
$$\varphi_{1n} = \int_0^1 \varphi(x)Y_{2n-1}(x)dx, \quad \varphi_{2n} = \int_0^1 \varphi(x)Y_{2n}(x)dx.$$

By virtue of the bi-orthogonal system (2.2)–(2.3) and using (1.1), we can easily see that $v_0(t)$, $v_{1n}(t)$, $v_{2n}(t)$, $n = 1, 2, ...$ satisfy the following linear fractional differential equations, respectively,

$$D^q_{0+}(v_0(t) - v_0(0)) = a(t)f_0(t), \quad v_0(0) = v_0, \tag{3.2}$$
$$D^q_{0+}(v_{2n}(t) - v_{2n}(0)) + \lambda_n v_{2n}(t) = a(t)f_{2n}(t), \quad v_{2n}(0) = v_{2n}, \tag{3.3}$$

$$D^q_{0+}(v_{1n}(t) - v_{1n}(0)) + \lambda_n v_{1n}(t) + \sqrt{\lambda_n} v_{2n}(t) = a(t) f_{1n}(t), \; v_{1n}(0) = v_{1n}, \quad (3.4)$$

where $\lambda_n$ are eigenvalues of (2.1). By taking the fractional integral $I^q_{0+}$ of (3.2), we get the solution $v_0(t)$ of (3.2) as

$$v_0(t) = \varphi_0 + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} a(\tau) f_0(\tau) d\tau.$$

According to (4.1), it can easily be seen that the solution $v_{2n}(t)$, $n = 1, 2, \ldots$ of (3.3) is

$$v_{2n}(t) = \varphi_{2n} e_q(t, \lambda_n) + \int_0^t e_{q,q}(t-\tau, \lambda_n) a(\tau) f_{2n}(\tau) d\tau, \quad (3.5)$$

where $e_q(t, \lambda) := E_q(-\lambda t^q)$ and $e_{q,q}(t, \lambda) := t^{q-1} E_{q,q}(-\lambda t^q)$.

According to (4.1), (3.5), the solution $v_{1n}(t), n{=}1, 2, ...$ of (3.4) is given by

$$v_{1n}(t) = \varphi_{1n} e_q(t, \lambda_n) + \mathcal{H}(t) * e_{q,q}(t, \lambda_n),$$

where $'*'$ is the convolution integral,

$$\mathcal{H}(t) = -\sqrt{\lambda_n}[\varphi_{2n} e_q(t, \lambda_n) + r(t) f_{2n}(t) * e_{q,q}(t, \lambda_n)] + a(t) f_{1n}(t).$$

The formulas (3.1) and (1.4) yield a following Volterra integral equation of the first kind with respect to $a(t)$ :

$$\int_0^t K(t, \tau) a(\tau) d\tau + F(t) = E(t), \quad (3.6)$$

where

$$K(t, \tau) = \sum_{n=1}^{\infty} \left[ \frac{2}{\sqrt{\lambda_n}} f_{2n}(\tau) e_{q,q}(t-\tau, \lambda_n) \right],$$

$$F(t) = 2\left[\varphi_0 + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} a(\tau) f_0(\tau) d\tau \right] + \sum_{n=1}^{\infty} \left[ \frac{2}{\sqrt{\lambda_n}} \varphi_{2n} e_q(t, \lambda_n) \right].$$

Further, the Equation (3.6) yields the following Volterra integral equation of the second kind by taking fractional derivative $D^q_{0+}$:

$$\int_0^t D^q_\tau K(t, \tau) a(\tau) d\tau + a(t) \lim_{\tau \to t-0} I^{q-1}_\tau K(t, \tau) + D^q_{0+}(F(t) - F(0)) = D^q_{0+}(E(t) - E(0)). \quad (3.7)$$

By using the properties $(iii)$ in Proposition 1, it is easy to show that

$$D^q_{0+}(F(t) - F(0)) = 2a(t) f_0(t) + \sum_{n=1}^{\infty} \left[ 2\sqrt{\lambda_n} \varphi_{2n} e_q(t, \lambda_n) \right], \quad (3.8)$$

$$D^q_\tau K(t, \tau) = \sum_{n=1}^{\infty} \left[ 2\sqrt{\lambda_n} e_{q,q}(t-\tau, \lambda_n) f_{2n}(\tau) \right] \quad (3.9)$$

and

$$a(t) \lim_{\tau\to t-0} I_\tau^{q-1} K(t,\tau) + D_{0+}^q (F(t) - F(0))$$
$$= a(t) \int_0^1 f(x,t)dx + \sum_{n=1}^{\infty} \left[ 2\sqrt{\lambda_n}\varphi_{2n} e_q(t,\lambda_n) \right] \tag{3.10}$$

According to (3.7)–(3.10), we obtain the Volterra integral equation of the second kind with respect to $a(t)$ in the form

$$a(t) = P(t) + \int_0^t Q(t,\tau)a(\tau)d\tau \tag{3.11}$$

with the free term $P$ and kernel $Q$

$$P(t) = \frac{D_{0+}^q(E(t) - E(0)) - \sum\limits_{n=1}^{\infty} \left[2\sqrt{\lambda_n}\varphi_{2n} e_q(t,\lambda_n)\right]}{\int_0^1 f(x,t)dx}, \quad Q(t,\tau) = -\frac{D_\tau^q K(t,\tau)}{\int_0^1 f(x,t)dx}.$$

According to the Lemmas 2 and 3 we estimate the kernel of (3.11) in the following form:

$$|Q(t,\tau)| = |D_\tau^q K(t,\tau)| / \left| \int_0^1 f(x,t)dx \right| \le \frac{C}{(t-\tau)^\nu}, \tag{3.12}$$

where

$$C = \frac{2c_1 \max\limits_{t\in[0,T]} \|f(.,t)\|_{C^4[0,1]}}{\Gamma(q) \min\limits_{t\in[0,T]} \left| \int_0^1 f(x,t)dx \right|}, \quad \nu = 1 - q.$$

Because the kernel $Q(t,\tau)$ belongs to the class $S^\nu$ with $0 < \nu < 1$, Volterra integral equation (3.11) is weakly singular. Then, it has a unique solution $a \in C[0,T]$ according by Lemmas 5 and 6.

First, let us show that the solution of inverse problem (1.1)–(1.4) is unique. To prove the uniqueness of the solution, we suppose that $\{a(t), u(x,t)\}$ and $\{r(t), v(x,t)\}$ are two solution sets of inverse problem (1.1)–(1.4). According to form of solutions (3.1) and (3.11), we have

$$\begin{aligned}
u(x,t) - v(x,t) = &\sum_{n=1}^{\infty} \left( \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f_0(\tau)[a(\tau) - r(\tau)]d\tau \right) X_0(x) \\
&+ \sum_{n=1}^{\infty} \left( \int_0^t e_{q,q}(t-\tau,\lambda_n) f_{2n}(\tau)[a(\tau) - r(\tau)]d\tau \right) X_{2n} \\
&+ \sum_{n=1}^{\infty} \Big( \sqrt{\lambda_n}[(a(t) - r(t)) f_{2n}(t) * e_{q,q}(t,\lambda_n)] \\
&+ (a(t) - r(t)) f_{1n}(t) * e_{q,q}(t,\lambda_n) \Big) X_{2n-1}(x),
\end{aligned} \tag{3.13}$$

$$a(t) - r(t) = \int_0^t Q(t,\tau) \left[a(\tau) - r(\tau)\right] d\tau. \tag{3.14}$$

Now, we denote $R(t)=|a(t)-r(t)|$, thus (3.12) and (3.14) imply the inequality

$$R(t) \leq \frac{\epsilon}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} R(\tau) d\tau,$$

where $\epsilon = 2c_1 \max\limits_{t\in[0,T]} \|f(.,t)\|_{C^4[0,1]} / \min\limits_{t\in[0,T]} \left|\int_0^1 f(x,t)dx\right|$. In view of the Lemma 7 we have $R(t)\leq 0$ that yields $a(t)=r(t)$. Consequently, we have $u=v$ after inserting $a=r$ in (3.13).

So far, we have proved the uniqueness of the solution of the inverse problem. Because the solution $u(x,t)$ is formally given by the series form (3.1), we need to show that the series corresponding to $u(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$ and $D^q_{0+}(u(x,.)-u(x,0))$ represent continuous functions. Under the assumptions $(A_1)$–$(A_3)$ and Lemma 1, for all $(x,t)\in\overline{\Omega_T}$, the series corresponding to $u(x,t)$ is bounded above by the series

$$\begin{aligned} &|\varphi_0| + \frac{T^q}{q\Gamma(q)} N\,|f_0| + \sum_{n=1}^{\infty}\left[\frac{1}{\lambda_n^2}\varphi_{2n}^{(4)} + \frac{1}{\lambda_n^3} N F_{2n}^{(4)}\right] \\ &+ \sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}\left[\varphi_{1n}^{(4)} + \frac{1}{\sqrt{\lambda_n}}\varphi_{2n}^{(4)} + \frac{1}{\lambda_n\sqrt{\lambda_n}} N F_{2n}^{(4)} + \frac{1}{\sqrt{\lambda_n}} N F_{1n}^{(4)}\right], \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} &\max_{t\in[0,T]} |r(t)| = N, F_{1n}^{(i)} = \max_{t\in[0,T]}\left|\int_0^1 \frac{\partial^i f(x,t)}{\partial x^i} Y_{2n-1}(x)dx\right|, \\ &F_{2n}^{(i)} = \max_{t\in[0,T]}\left|\int_0^1 \frac{\partial^i f(x,t)}{\partial x^i} Y_{2n}(x)dx\right|, \\ &\varphi_{1n}^{(i)} = \max_{t\in[0,T]}\left|\int_0^1 \frac{\partial^i f(x,t)}{\partial x^i} \varphi_{2n-1}(x)dx\right|,\ \varphi_{2n}^{(i)} = \max_{t\in[0,T]}\left|\int_0^1 \frac{\partial^i f(x,t)}{\partial x^i} \varphi_{2n}(x)dx\right|, \end{aligned}$$

$n=1,2,...(i=0,1,2,3,4)$.

The majorizing series (3.15) is convergent by using Lemma 1, Schwarz inequality and $p$-series test. This implies that by the Weierstrass M-test, the series (3.1) is uniformly convergent in the rectangle $\overline{\Omega_T}$ and therefore, the solution $u(x,t)$ is continuous in the rectangle $\overline{\Omega_T}$.

The majorizing series for $x$-partial derivative is

$$\begin{aligned} &\sum_{n=1}^{\infty}\sqrt{\lambda_n}\left[\frac{1}{\lambda_n^2}\varphi_{2n}^{(4)} + \frac{1}{\lambda_n^3} N F_{2n}^{(4)}\right] + \sum_{n=1}^{\infty}\frac{\alpha}{\lambda_n^2}(\sqrt{\lambda_n}+1) \\ &\times\left[\varphi_{1n}^{(4)} + \frac{1}{\sqrt{\lambda_n}}\varphi_{2n}^{(4)} + \frac{1}{\lambda_n\sqrt{\lambda_n}} N F_{2n}^{(4)} + \frac{1}{\sqrt{\lambda_n}} N F_{1n}^{(4)}\right]. \end{aligned} \tag{3.16}$$

The majorizing series (3.16) is convergent by employing Lemma 1, Schwarz inequality and $p$-series test. Hence, by the Weierstrass M-test, the series obtained for $x$-partial derivatives of (3.1) is uniformly convergent in the rectangle $\overline{\Omega_T}$. Therefore, their sum $u_x(x,t)$ is continuous in $\overline{\Omega_T}$. Similarly, we can show that the series corresponding to $xx$-partial derivatives of (3.1) is uniformly convergent and $u_{xx}(x,t)$ represents continuous function.

Now, it remains to show that $q$-fractional derivative of the series $u(x,t) - u(x,0)$ represents continuous function on $\Omega_T$. We will show that for any $\epsilon > 0$ and $t \in [\epsilon, T]$, the following series is generated according to Lemma 4

$$\begin{aligned} &D^q_{0+}(v_0(t) - v_0(0))\ X_0(x) \\ &+ \sum_{n=1}^{\infty}[D^q_{0+}(v_{1n}(t) - v_{1n}(0))X_{2n-1}(x) + D^q_{0+}(v_{2n}(t) - v_{2n}(0))X_{2n}(x)], \end{aligned}$$

corresponding to $q$-fractional derivative of the function $u(x,t) - u(x,0)$ is uniformly convergent. Now, we can see that Equations (3.2)–(3.4) yield

$$\begin{aligned} &D^q_{0+}(v_0(t) - v_0(0)) = r(t)f_0(t), \\ &D^q_{0+}(v_{2n}(t) - v_{2n}(0)) = -\lambda_n v_{2n}(t) + r(t)f_{2n}(t), \\ &D^q_{0+}(v_{1n}(t) - v_{1n}(0)) = -\lambda_n v_{1n}(t) - \sqrt{\lambda_n} v_{2n}(t) + r(t)f_{1n}(t). \end{aligned}$$

We have the following estimates

$$\begin{aligned} |D^q_{0+}(v_{2n}(t) - v_{2n}(0))| \le& |\varphi_{2n}|\lambda_n e_q(\varepsilon, \lambda_n) + 2N\frac{1}{\lambda_n^2}F^{(4)}_{2n}, \\ |D^q_{0+}(v_{1n}(t) - v_{1n}(0))| \le& 2\,|\varphi_{2n}|\,\sqrt{\lambda_n} e_q(\varepsilon, \lambda_n) + 2N\frac{1}{\sqrt{\lambda_n}\lambda_n^2}F^{(4)}_{2n} \\ &+ |\varphi_{1n}|\,\lambda_n e_q(\varepsilon, \lambda_n) + (\sqrt{\lambda_n} + 1)N\frac{1}{\lambda_n^2}F^{(4)}_{1n} \end{aligned}$$

and we obtain a majorant series as follows

$$\begin{aligned} &\sum_{n=1}^{\infty} |\varphi_{2n}|\,\lambda_n e^{-\frac{\lambda_n}{\Gamma(1+q)}\varepsilon^q} + 2\,|\varphi_{2n}|\,\sqrt{\lambda_n} e^{-\frac{\lambda_n}{\Gamma(1+q)}\varepsilon^q} + 2N\frac{1}{\lambda_n^2}F^{(4)}_{2n} \\ &+ 2N\frac{1}{\sqrt{\lambda_n}\lambda_n^2}F^{(4)}_{2n} + |\varphi_{1n}|\,\lambda_n e^{-\frac{\lambda_n}{\Gamma(1+q)}\varepsilon^q} + (\sqrt{\lambda_n} + 1)N\frac{1}{\lambda_n^2}F^{(4)}_{1n}. \end{aligned}$$

Consequently, $D^q_{0+}(u(x,t) - u(x,0))$ is uniformly convergent in the rectangle $\Omega_T$. □

*Remark 1.* It is important to note the phenomenon that the smoothness of coefficient of (1.1) does not imply smoothness of the solution $u$ in the closed domain. It is discussed in more detail in [38] that $\frac{\partial}{\partial t}u(x,t)$ blows up as $t \to 0^+$ which forces, for example, the overdetermination data $E(t)$ to belong to the class $(A_2)$.

The following result on continuously dependence on the data of solution of the inverse problem (1.1)–(1.4) holds.

**Theorem 2.** *(Lipschitz stability) Let $\Im$ be the class of triples in the form of $\{f, \varphi, E\}$, which satisfies the assumptions $(A_1)$–$(A_3)$ of Theorem 1 and*

$$0 < N_0 \leq \min_{t\in[0,T]} \left| \int_0^1 f(x,t)dx \right|, \quad \|f\|_{C^{4,0}(\overline{\Omega_T})} \leq N_1,$$
$$\|\varphi\|_{C^4[0,1]} \leq N_2, \qquad \|E\|_{C_q[0,T]} \leq N_3$$

*for some positive constants $N_i, i = 0,1,2,3$.*

*Then, the solution pair $\{a(t), u(x,t)\}$ of the inverse problem* (1.1)–(1.4) *depends continuously upon the data in $\Im$.*

*Proof.* Let $\digamma = \{f, \varphi, E\}$ and $\tilde{\digamma} = \left\{\widetilde{f}, \widetilde{\varphi}, \widetilde{E}\right\} \in \Im$ be two sets of data. Let us denote $\|\digamma\| = \|f\|_{C^{4,0}(\overline{\Omega_T})} + \|\varphi\|_{C^4[0,1]} + \|E\|_{C_q[0,T]}$, where $\|E\|_{C_q[0,T]} \equiv \max_{t\in[0,T]} \left|D_{0+}^q(E(t) - E(0))\right|$. Let $(a,u)$ and $(\widetilde{a}, \widetilde{u})$ be the solutions of the inverse problem (1.1)–(1.4) corresponding to the data $\digamma$ and $\tilde{\digamma}$, respectively.

According to (3.8) we have

$$a(t) = P(t) + \int_0^t Q(t,\tau)a(\tau)d\tau, \quad \widetilde{a}(t) = \widetilde{P}(t) + \int_0^t \widetilde{Q}(t,\tau)\widetilde{a}(\tau)d\tau. \tag{3.17}$$

Let us estimate the difference $a - \widetilde{a}$. From (3.17) we obtain

$$a(t) - \widetilde{a}(t) = P(t) - \widetilde{P}(t) + \int_0^t \left[Q(t,\tau) - \widetilde{Q}(t,\tau)\right] r(\tau)d\tau + \int_0^t \widetilde{Q}(t,\tau)\left[a(\tau) - \widetilde{a}(\tau)\right]d\tau, \tag{3.18}$$

where

$$Q(t,\tau) - \widetilde{Q}(t,\tau) = \Big(\int_0^1 f(x,t)dx \int_0^1 \widetilde{f}(x,t)dx\Big)^{-1}$$
$$\times \Big[\int_0^1 \widetilde{f}(x,t)dx(D_\tau^q K(t,\tau) - D_\tau^q \widetilde{K}(t,\tau)) + D_\tau^q \widetilde{K}(t,\tau)\Big(\int_0^1 f(x,t)dx - \int_0^1 \widetilde{f}(x,t)dx\Big)\Big].$$

Let

$$\epsilon_1 =: \left\|P - \widetilde{P}\right\|_{C([0,T])} + \frac{T^q}{q\Gamma(q)} \frac{2N_1c_1}{N_0^2} \left\|f - \widetilde{f}\right\|_{C^{4,0}(\overline{\Omega_T})} \|a\|_{C([0,T])}.$$

Then, denoting $R(t) =: |a(t) - \widetilde{a}(t)|$, according to (3.18) we have the inequality

$$R(t) \leq \epsilon_1 + \frac{\epsilon_2}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} R(\tau)d\tau,$$

where $\epsilon_2 = \frac{N_1c_1}{N_0}$. Then, a weakly singular Gronwall's inequality, see Lemma 7, implies that

$$R(t) \leq \epsilon_1 E_q(\epsilon_2 t^q), \quad t \in [0,T].$$

Finally, using (3.11) and (3.18), we obtain

$$\|a - \widetilde{a}\|_{C([0,T])} \leq \epsilon_3 \left(\left\|P - \widetilde{P}\right\|_{C([0,T])} + \|a\|_{C([0,T])} \frac{T^q}{q\Gamma(q)} \frac{2N_1c_1}{N_0^2} \left\|f - \widetilde{f}\right\|_{C^{4,0}(\overline{\Omega_T})}\right), \tag{3.19}$$

where $\epsilon_3 = E_q(\epsilon_2 T^q)$. By virtue of the expression for $P(t) - \widetilde{P}(t)$, one can estimate that

$$\left\|P - \widetilde{P}\right\|_{C([0,T])} \leq M_1 \left\|f - \widetilde{f}\right\|_{C^{4,0}(\overline{\Omega_T})} + M_2 \left\|\varphi - \widetilde{\varphi}\right\|_{C^4[0,1]} + M_3 \left\|E - \widetilde{E}\right\|_{C_q([0,T])}, \tag{3.20}$$

where $M_1 = \frac{c_2 N_2 + \frac{T^q}{q\Gamma(1-q)} N_3}{N_0^2}$, $M_2 = \frac{c_2 N_1}{N_0^2}$, $M_3 = \frac{T^q}{q\Gamma(1-q)} \frac{N_1}{N_0^2}$.

By using the inequality (3.20), from (3.19) we get

$$\begin{aligned} \left\|a - \widetilde{a}\right\|_{C([0,T])} \leq & M_4 \left( \left\|f - \widetilde{f}\right\|_{C^{4,0}(\overline{\Omega_T})} + \left\|\varphi - \widetilde{\varphi}\right\|_{C^4[0,1]} \right. \\ & \left. + \left\|E - \widetilde{E}\right\|_{C_q([0,T])} \right) = M_4 \left\|\mathcal{F} - \widetilde{\mathcal{F}}\right\|, \end{aligned}$$

where $M_4 = \max(\epsilon_3 M_2, \epsilon_3 M_3, \epsilon_3 M_1 + \epsilon_3 \left\|r\right\|_{C([0,T])} \frac{T^q}{q\Gamma(q)} \frac{2N_1 c_1}{N_0^2})$. This shows that $a(t)$ depends continuously upon the input data. From (3.13), a similar estimate is also obtained for the difference $u - \widetilde{u}$ as

$$\left\|u - \widetilde{u}\right\|_{C(\overline{\Omega_T})} \leq M_5 \left\|\mathcal{F} - \widetilde{\mathcal{F}}\right\|.$$

This completes the proof of Theorem 2. $\square$

## 4 Conclusions

The paper considers an inverse source problem of identification of the time-dependent source term from the energy measurement for TFDe with a general form of nonlocal boundary conditions which is regular but not strongly regular. The well-posedness of the inverse problem is proved by means of Fourier expansion method, some properties of Volterra integral equation with weak singular kernel and fractional type Gronwall's inequality. In all the previous works cited in introduction, the problems studied involve the special case of (1.3): $\alpha = 0$ in [2] and $\alpha = 1$ in [13, 14, 15, 22, 23]. The general case $\alpha \neq 0$ for boundary condition and weakly singular Volterra integral equation approach for unique determination of solely time dependent source term is considered for the first time in this work. This approach can be extended by considering different type of fractional derivative, boundary and overdetermination conditions, which are the line of future investigations. Another future line of investigation is to extend the fractional derivative to multi-dimensional and nonlinear evolutional equations, see [30] and references therein. The paper [30] demonstrates the importance of addressing the given inverse problem for establishing the existence of a solution to the nonstationary Navier–Stokes equations with a prescribed flow rate within an infinite cylinder.

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## Appendix

### Notes on fractional calculus

In this part, we recall some basic definitions and facts on fractional calculus in [11,31,34,35] and present some necessary lemmas for further investigations. Consider the following initial value problem, existence and uniqueness result for such problem is given in [11], for a linear fractional differential equation with order $0 < q < 1$,

$$\begin{cases} D^q_{0+}[u(t) - u(0)] + \lambda u(t) = h(t), \quad t > 0, \\ u(0) = u_0, \end{cases} \tag{4.1}$$

where $D^q_{0+}$ refers to the the Riemann-Liouville fractional derivative of order $q$ $(0 < q < 1)$ in the time variable defined by

$$D^q_{0+} u(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t \frac{u(\tau)}{(t-\tau)^q} d\tau.$$

By using the Laplace transform, the solution of IVP (2.1) is given as

$$u(t) = u_0 E_{q,1}(-\lambda t^q) + \int_0^t (t-\tau)^{q-1} E_{q,q}(-\lambda(t-\tau)^q)h(\tau)d\tau,$$

where $E_{q,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(qk+\beta)}$, $q>0$, $\beta>0$, $E_{q,1}(z) = E_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(qk+1)}$, $q > 0$ are two parameter and one parameter Mittag-Leffler function, respectively. Let us introduce the functions

$$e_q(t,\lambda) := E_q(-\lambda t^q), \quad e_{q,q}(t,\lambda) := t^{q-1} E_{q,q}(-\lambda t^q),$$

where $\lambda \in \mathbb{R}_+$. Then, the following statements for the Mittag-Leffler type functions $e_q(t,\lambda)$ and $e_{q,q}(t,\lambda)$ hold.

**Proposition 1.** *( [31])*

*i) For $0 < q < 1$, $\lambda \in \mathbb{R}_+$ the function $e_q(t,\lambda)$ is a monotonically decreasing function.*

*ii) The function $e_q(t,\lambda)$ has the estimates $e_q(t,\lambda) \simeq e^{-\frac{\lambda}{\Gamma(q+1)}t^q}$ for $t \ll 1$ and $e_q(t,\lambda) \simeq \frac{1}{\Gamma(1-q)\lambda t^q}$ for $t \gg 1$.*

*iii)*

$$D_{0+}^q(e_{q,q}(t,\lambda)) = -\lambda e_{q,q}(t,\lambda),$$
$$D_{0+}^q(e_q(t,\lambda) - e_q(0,\lambda)) = -\lambda e_q(t,\lambda), \ \ I_{0+}^{1-q}(e_{q,q}(t,\lambda)) = e_q(t,\lambda),$$

*where $I_{0+}^{\gamma}$ is the fractional integral of order $\gamma > 0$ for an integrable function $f$, which is defined by $I_{0+}^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s)ds$.*

Taking into also account the monotonically decreasing character of $E_{q,q}(-\lambda t^q)$ ( [35]), where $\lambda \in \mathbb{R}_+$, the following statement holds true.

**Lemma 2.** *For $0 < q < 1$, Mittag-Leffler type function $E_{q,q}(-\lambda t^q)$ satisfies*

$$0 \le E_{q,q}(-\lambda t^q) \le 1/\Gamma(q), \quad t \in [0,\infty), \ \lambda \ge 0.$$

Now, we give the following lemma which is necessary for further development.

**Lemma 3.** *For $0 < q < 1$, $\lambda \in \mathbb{R}_+$, we have*

$$\int_{t_0}^t (t-\tau)^{q-1} E_{q,q}(-\gamma(t-\tau)^q)d\tau = \frac{1}{\gamma}(1 - E_q(-\gamma(t-t_0)^q)).$$

*Proof.* By applying change of variable $z = t - \tau$ in the above integral and using $\frac{d}{dt}E_q(-\gamma t^q) = -\gamma t^{q-1}E_{q,q}(-\gamma t^q)$, [34] , we have

$$\int_{t_0}^t (t-\tau)^{q-1} E_{q,q}(-\gamma(t-\tau)^q)d\tau = \int_0^{t-t_0} z^{q-1} E_{q,q}(-\gamma z^q)dz$$
$$= -\frac{1}{\gamma}\int_0^{t-t_0} \frac{d}{dz}E_q(-\gamma z^q)dz = \frac{1}{\gamma}(1 - E_q(-\gamma(t-t_0)^q)).$$

□

We will also need to recall the following result.

**Lemma 4.** *( [34], Lemma 15.2) Let $f_i$ be a sequence of functions defined on the interval $(a, b]$. Suppose the following conditions holds:*

*(i) the fractional derivative $D^q_{0+}\ f_i(t)$, for a given $q > 0$, exists for all $i \in \mathbb{N}, t \in (a, b]$;*

*(ii) both series $\sum\limits_{i=1}^{\infty} f_i(t)$ and $\sum\limits_{i=1}^{\infty} D^q_{0+}\ f_i(t)$ are uniformly convergent on the interval $[a + \epsilon, b]$ for any $\epsilon > 0$.*

*Then, the function defined by the series $\sum\limits_{i=1}^{\infty} f_i(t)$ is $q$ differentiable and satisfies $D^q_{0+} \sum\limits_{i=1}^{\infty} f_i(t) = \sum\limits_{i=1}^{\infty} D^q_{0+} f_i(t)$.*

**Weakly singular Volterra integral equation**

In this part, we present some basic results on Volterra type integral equation with the kernel may have a diagonal singularity. For details see [12, 29].

Consider the Volterra integral equation

$$u(t) = \int_0^t Q(t, \tau) u(\tau) d\tau + f(t),\ 0 \le t \le 1. \tag{4.2}$$

Denote $\Delta = \{(t, \tau) : 0 \le \tau < t \le 1\}$ and introduce the class $S^\nu$of kernels $Q(t, \tau)$ that are defined and continuous on $\Delta$ and satisfy for $(t, \tau) \in \Delta$ the inequality

$$|Q(t, \tau)| \le c\,(t - \tau)^{-\nu},\ \nu > 0,\ c = const > 0.$$

The kernel $Q(t, \tau) \in S^\nu$ is weakly singular if $\nu < 1$. The weak singularity of the kernel implies that the corresponding integral operator is compact in the space $C[0, T]$. More precisely, the following statement holds true.

**Lemma 5.** *( [29]) Let $Q(t, \tau) \in S^\nu$ and $\nu < 1$. Then, the Volterra integral operator $B$ defined by $(Br)(t) = \int_0^t Q(t, \tau) r(\tau) d\tau$ maps $C[0, T]$ into itself and $B : C[0, T] \to C[0, T]$ is compact.*

The proof of Lemma 5 is standard a detailed argument can be found in [29]. A consequence of Lemma 5 is the following result.

**Lemma 6.** *( [29]) Let $f \in C[0, 1]$ and $Q(t, \tau) \in S^\nu$ with $\nu < 1$. Then, Equation (4.2) has a unique solution $u \in C[0, 1]$.*

We will also need to recall the results on weakly singular version of the Gronwall's inequality.

**Lemma 7.** *( [12], page 537) Let $T, \varepsilon, M \in \mathbb{R}_+$ and $0 < q < 1$. Moreover, assume that $\delta : [0, T] \to \mathbb{R}$ is a continuous function satisfying the inequality*

$$|\delta(t)| \le \varepsilon + \frac{M}{\Gamma(q)} \int_0^t (t - \tau)^{-\nu} |\delta(\tau)| d\tau,\ \textit{with}\ \nu = 1 - q$$

*for all $t \in [0, T]$. Then, $|\delta(t)| \le \varepsilon E_q(M t^q)$ for $t \in [0, T]$.*