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Asymptotic Analysis of Sturm–Liouville Problem with Dirichlet and Nonlocal Two-Point Boundary Conditions

Artūras Štikonas[a] **and Erdoğan Şen**[b]

[a] *Institute of Applied Mathematics, Vilnius University*
Naugarduko g. 24, LT-03225 Vilnius, Lithuania
[b] *Tekirdag Namik Kemal University*
Kampus str. 1, TR-59030 Tekirdag, Turkey
E-mail(*corresp.*): arturas.stikonas@mif.vu.lt
E-mail: esen@nku.edu.tr

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Abstract. In this study, we obtain asymptotic expansions for eigenvalues and eigenfunctions of the one–dimensional Sturm–Liouville equation with one classical Dirichlet type boundary condition and two-point nonlocal boundary condition. We analyze the characteristic equation of the boundary value problem for eigenvalues and derive asymptotic expansions of arbitrary order. We apply the obtained results to the problem with two-point nonlocal boundary condition.

Keywords: Sturm–Liouville problem, Dirichlet condition, two-point nonlocal conditions, asymptotics of eigenvalues and eigenfunctions.

AMS Subject Classification: 34B24; 34L20; 35R10.

1 Introduction

Consider the following one-dimensional Sturm–Liouville equation

$$-u''(t) + q(t)u(t) = \lambda u(t), \quad t \in [0,1], \tag{1.1}$$

where the real-valued function $q \in C[0,1]$; $\lambda = s^2$ is a complex spectral parameter and $s = x + \imath y$; $x, y \in \mathbb{R}$. We will use notation $Q(t) = \frac{1}{2}\int_0^t q(\tau)d\tau$ and $q_0 := 2\int_0^1 |q(\tau)|\, d\tau$.

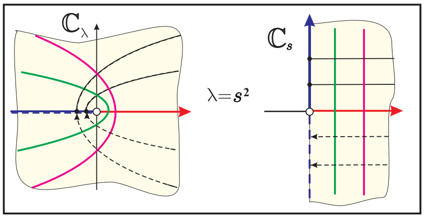


Figure 1. Domain $\mathbb{C}_s$.

Remark 1. In this article, $s \in \mathbb{C}_s := \mathbb{R}_s \cup \mathbb{C}_s^+ \cup \mathbb{C}_s^-$, where $\mathbb{R}_s := \mathbb{R}_s^- \cup \mathbb{R}_s^+ \cup \mathbb{R}_s^0$, $\mathbb{R}_s^- := \{s = x + \imath y \in \mathbb{C} : x = 0, y > 0\}$, $\mathbb{R}_s^+ := \{s = x + \imath y \in \mathbb{C} : x > 0, y = 0\}$, $\mathbb{R}_s^0 := \{s = 0\}$, $\mathbb{C}_s^+ := \{s = x + \imath y \in \mathbb{C} : x > 0, y > 0\}$ and $\mathbb{C}_s^- := \{s = x + \imath y \in \mathbb{C} : \ x > 0, \ y < 0\}$. Then a map $\lambda = s^2$ is the bijection between $\mathbb{C}_s$ and $\mathbb{C}_\lambda := \mathbb{C}$ [36] (see Figure 1).

In this study, we shall investigate Sturm–Liouville Problem (SLP) that consists of Equation (1.1) on $[0, 1]$ with one classical (local) Dirichlet type Boundary Condition (BC)

$$u(0) = 0, \tag{1.2}$$

another two-point Nonlocal Boundary Condition (NBC) [5, 22, 23]

$$\text{(Case 1)} \qquad u'(1) = \gamma u(\xi), \qquad \xi \in [0, 1], \tag{1.3$_1$}$$

$$\text{(Case 2)} \qquad u'(1) = \gamma u'(\xi), \qquad \xi \in [0, 1), \tag{1.3$_2$}$$

$$\text{(Case 3)} \qquad u(1) = \gamma u(\xi), \qquad \xi \in [0, 1), \tag{1.3$_3$}$$

where $\gamma \in \mathbb{R}$. In Case 1 and Case 3 for $\xi = 0$ we have the same problem as in the case $\gamma = 0$. A more comprehensive list can be found in the survey article [33]. In the (classical) case $\gamma = 0$ we have two local BCs.

We denote: $a_k := k\pi$ in Cases 1, 2; $a_k := (k - 1/2)\pi$ in Case 3 and $I_k := \big(a_k, \, a_{k+1}\big)$, $k \in \mathbb{N}$.

Asymptotic formulas for eigenvalues and eigenfunctions for Sturm–Liouville equation (1.1) with local BCs are investigated in the classical books [17, 18, 38]. These results were generalized for tasks with retarded argument [3, 21, 24, 27] and for some other local BCs [1, 20] and Sturm–Liouville Problem (SLP) with eigenparameter in BCs [7, 9, 11, 12, 13]. Asymptotical analysis of eigenvalues and eigenfunctions of SLPs with periodic BCs was obtained in [2, 6, 10].

Nonlocal Boundary Value Problems (BVP) are widely used for mathematical modelling of various processes of physics, ecology, chemistry and industry, when it is impossible to determine the boundary values of the unknown function. The bibliography on the subject of nonlocal BVPs is very extensive and we refer to the list of the works in [8, 14, 33]. Characteristic curves and Green's functions for problems with NBCs and $q \not\equiv 0$ were investigated in [25, 26, 32]. Until this time, there were only few works about asymptotic properties of eigenvalues and eigenfunctions with potential function $q(x)$ in Equation (1.1) and

NBCs. We will note papers [28, 34, 35] where the asymptotic properties are studied for some NBCs.

In [28], under the condition $q \in C[0,1]$, for sufficiently large k and $|\gamma| < 1$ it is derived that the asymptotic expansions

$$s_k = x_k + \mathcal{O}(k^{-1}), \quad u_k(t) = -\sin(x_k t)x_k^{-1} + \mathcal{O}(k^{-2}) \tag{1.4}$$

are valid for eigenvalues and eigenfunctions, respectively, for the SLP (1.1), (1.2), (1.3_3), where x_k, $k \in \mathbb{N}$, are the positive roots of $\sin x - \gamma \sin(\xi x) = 0$. Under the condition $q \in C^1[0,1]$, it is obtained that the asymptotic expansions

$$s_k = x_k + Q_1(x_k)x_k^{-1} + \mathcal{O}(k^{-2}), \tag{1.5}$$

$$u_k(t) = -\sin(x_k t)x_k^{-1} + \big(Q(t) - tQ_1(x_k)\big)\cos(x_k t)x_k^{-2} + \mathcal{O}(k^{-3}) \tag{1.6}$$

are valid for eigenvalues and eigenfunctions, respectively, for the SLP (1.1)–(1.2), (1.3_3), where

$$Q_1(s) := \big(Q(1)\cos s - \gamma Q(\xi)\cos(\xi s)\big)\big(\cos s - \gamma\xi\cos(\xi s)\big)^{-1}.$$

We will generalize formulas (1.4)–(1.6) for $q \in C^r[0,1]$, where $r \in \mathbb{N}$, for BC (1.3_3), and will derive analogous formulas for BCs (1.3_1),(1.3_2).

Spectral asymptotics of eigenvalues and eigenfunctions of SLP with Dirichlet BC (1.2) and integral NBC

$$u(1) = \gamma\int_\alpha^\beta u(t)dt, \quad \gamma \in \mathbb{R}, \quad [\alpha,\beta] \subset [0,1],$$

have been investigated recently [34]. For sufficiently large k it is derived that the asymptotic expansions (1.4) are valid for all $\gamma \in \mathbb{R}$, where $x_k = \pi k$, $k \in \mathbb{N}$. Under the condition $q \in C^1[0,1]$, it is obtained that the asymptotic formulas (1.4)–(1.5) are valid, where $Q_1(x) = Q(1) + (-1)^{k+1}\gamma\cos(\beta x) + (-1)^k\gamma\cos(\alpha x)$. In [34] asymptotical expansions for equation (1.1) with conditions $u(0) = 0$, $u'(0) = -1$ were derived for all $r \in \mathbb{N}$. We will use them to write the asymptotic formulas for the functions describing the characteristic equations in the case of BC (1.2)–(1.3).

The investigation of SLP with two-points NBC (1.3) we started in [35], where Neumann BC $u'(0) = 0$ were used instead of Dirichlet BC (1.2). More about non-trivial solutions of local and nonlocal Neumann boundary-value problems one can find in [16] and in [15] for parameter-dependent higher order problems. We investigated a characteristic equation of BVP and derive asymptotic expansions of arbitrary order if $q \in C^r[0,1]$. We will try to obtain analogous formulas in the case of Dirichlet BC. Many of the formulations and proofs are similar, but the specific functions and the first terms of the asymptotic expansions differ. We will try to compare these two cases.

The article is organized as follows. The statement of the problem and a literature review are given in Section 1. In Section 2, we present results about solution of Initial Value Problem (IVP) and formulas for its asymptotics. In Section 3, some results about the case $q \equiv 0$ are presented. In Section 4, we analyze the characteristic equation of the BVP (1.1)–(1.3). In Section 5, we investigate the distribution of eigenvalues and obtain asymptotic expansions for eigenvalues and eigenfunctions. Also, we calculate normalized eigenfunctions.

2 Solution of Initial Value Problem and its asymptotics

In this section, we present some statements about solution of Initial Value Problem (IVP). These statements were proved in [34]. We will use them for investigation asymptotic expansions for SLP (1.1)–(1.3). Additionally, we introduce some notation related to our asymptotical analysis of this problem.

Let $\lambda = s^2$, $s \in \mathbb{C}_s$ and $\omega_s(t)$ be a solution of Equation (1.1) satisfying the initial conditions

$$\omega_s(0) = 0, \qquad \omega_s'(0) = -1.$$

According to [18, Theorem 1.1 in Chapter I], this IVP determines a unique solution of (1.1) on $[0,1]$. The function $\omega_s(t) = \omega(t,s)$ is an analytic (holomorphic) function of s. We will use notation for derivatives $\omega_s'(t) := \partial\omega(t,s)/\partial t$, $(\omega_s)_s^{(l)}(t,s) := \partial^l\omega(t,s)/\partial s^l$, $(\omega_s')_s^{(l)}(t,s) := \partial^{l+1}\omega(t,s)/(\partial t\partial s^l)$.

Under the condition that $q \in C^r[0,1]$, $r \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, asymptotic expansions may be obtained for $\omega_s(t)$ [34]. We define functions $p_1^0(t) = -1$,

$$p_{i+1}^0(t) = -\frac{1}{2}\int_0^t q(\tau)p_i^0(\tau)\,d\tau - \sum_{j=2}^{i} \frac{(qp_{j-1}^0)^{(i-j)}(t) + (-1)^i(qp_{j-1}^0)^{(i-j)}(0)}{2^{i-j+2}} \tag{2.1}$$

for $i = \overline{1,r}$.

Lemma 1. (See [34, Lemma 7].) *Let $s \in \mathbb{C}_s$ and $q \in C^r[0,1]$. Then for $|s| \geq q_0$ we have the asymptotic expansions*

$$(\omega_s)_s^{(l)}(t,s) = -\sum_{j=1}^{r+1} p_j^l(t)\cos\Big(st + \frac{\pi}{2}(j-l)\Big)s^{-j} + \mathcal{O}\big(s^{-(r+2)}\mathrm{e}^{(r+2)|y|t}\big),$$
$$(\omega_s')_s^{(l)}(t,s) = -\sum_{j=0}^{r} \bar{p}_j^l(t)\cos\Big(st + \frac{\pi}{2}(j-l)\Big)s^{-j} + \mathcal{O}\big(s^{-(r+1)}\mathrm{e}^{(r+2)|y|t}\big) \tag{2.2}$$

for $l \in \mathbb{N}_0$, where $p_i^k(t) = (1-i)p_{i-1}^{k-1}(t) - tp_i^{k-1}(t)$, $i = \overline{1, r+1}$, $\bar{p}_0^k(t) = -t\bar{p}_0^{k-1}(t)$, $\bar{p}_i^k(t) = (1-i)\bar{p}_{i-1}^{k-1}(t) - t\bar{p}_i^{k-1}(t)$, $i = \overline{1,r}$, $k \in \mathbb{N}$, $\bar{p}_i^0(t) = p_i^{0\prime}(t) - p_{i+1}^0(t)$, $i = \overline{1,r}$, $\bar{p}_0^0(t) = 1$, and $p_j^0(t)$ is calculated by (2.1).

Now we will consider real λ. If $s \in \mathbb{R}_s^-$, i.e., $s = \imath y$, $y > 0$, $q \in C[0,1]$, then

$$\omega_{\imath y}(t) = -\sinh(yt)y^{-1} + \mathcal{O}\big(y^{-2}\mathrm{e}^{yt}\big) = -y^{-1}\mathrm{e}^{yt}/2 + \mathcal{O}\big(y^{-2}\mathrm{e}^{yt}\big),$$
$$\omega_{\imath y}'(t) = -\cosh(yt) + \mathcal{O}\big(y^{-1}\mathrm{e}^{yt}\big) = -\mathrm{e}^{yt}/2 + \mathcal{O}\big(y^{-1}\mathrm{e}^{yt}\big). \tag{2.3}$$

If $s \in \mathbb{R}_s^+$, i.e., $s = x$, $x > 0$, $q \in C[0,1]$, then

$$\omega_x(t) = -\sin(xt)x^{-1} + \mathcal{O}\big(x^{-2}\big), \;\; \omega_x'(t) = -\cos(xt) + \mathcal{O}\big(x^{-1}\big).$$

We define functions ($n_1,\dots,n_m$ are nonnegative integers)

$$R_{m+1}(t,x) = -\sum_{\substack{n_1+\dots+n_m=l,\ j>0,\\ j+n_1+2n_2+\dots+mn_m=m+1}} \frac{Q_1^{n_1}(x)\cdots Q_m^{n_m}(x)}{n_1!\cdots n_m!} p_j^l(t)\cos\biggl(xt+\frac{\pi}{2}(j-l)\biggr),$$

$$\bar{R}_m(t,x) = -\sum_{\substack{n_1+\dots+n_m=l,\ j\geq 0\\ j+n_1+2n_2+\dots+mn_m=m}} \frac{Q_1^{n_1}(x)\cdots Q_m^{n_m}(x)}{n_1!\cdots n_m!} \bar{p}_j^l(t)\cos\biggl(xt+\frac{\pi}{2}(j-l)\biggr),$$

$m=\overline{0,r}$.

Corollary 1. (See [34, Corollary 2].) Let $x\in\mathbb{R}_s^+$, $\delta\in\mathbb{R}$, $q\in C^r[0,1]$, $Q_j(x)$, $j=\overline{1,r}$ are bounded functions. If $s=x+\delta$, $\delta=\sum_{j=1}^r Q_j(x)x^{-j}+\mathcal{O}(x^{-(r+1)})$, then we have the following asymptotic expansions

$$\omega_s(t)=\sum_{j=1}^{r+1} R_j(t,x)x^{-j}+\mathcal{O}\bigl(x^{-(r+2)}\bigr),\ \omega_s'(t)=\sum_{j=0}^{r} \bar{R}_j(t,x)x^{-j}+\mathcal{O}\bigl(x^{-(r+1)}\bigr).$$

Now, we write explicit formulas in the cases $q\in C[0,1]$ and $q\in C^1[0,1]$.

Lemma 2. (See [28, Lemma 2, Lemma 3], [34, Lemma 5].) *Let $s\in\mathbb{C}_s$ and $q\in C[0,1]$. Then there exists $q_0>0$ such that for $|s|\geq q_0$, we have the asymptotic expansions*

$$\omega_s(t)=-\sin(st)s^{-1}+\mathcal{O}\bigl(s^{-2}\mathrm{e}^{|y|t}\bigr),\quad (\omega_s)_s'(t,s)=-t\cos(st)s^{-1}+\mathcal{O}\bigl(s^{-2}\mathrm{e}^{|y|t}\bigr),$$
$$(\omega_s)_s''(t,s)=t^2\sin(st)s^{-1}+\mathcal{O}\bigl(s^{-2}\mathrm{e}^{|y|t}\bigr),$$
$$\omega_s'(t)=-\cos(st)+\mathcal{O}\bigl(s^{-1}\mathrm{e}^{|y|t}\bigr),\quad (\omega_s')_s'(t,s)=t\sin(st)+\mathcal{O}\bigl(s^{-1}\mathrm{e}^{|y|t}\bigr).$$

These formulas hold uniformly for $0\leq t\leq 1$.

Corollary 2. (See [34, Corollary 1].) Let $x\in\mathbb{R}_s^+$, $\delta\in\mathbb{R}$, $q\in C[0,1]$. If $s=x+\delta$, $\delta=\mathcal{O}(x^{-1})$, then we have the following asymptotic expansions:

$$\omega_s(t)=-\sin(xt)x^{-1}+\mathcal{O}\bigl(x^{-2}\bigr),\qquad \omega_s'(t)=-\cos(xt)+\mathcal{O}\bigl(x^{-1}\bigr).$$

Lemma 3. (See [34, Lemma 8].) *Let $s\in\mathbb{C}_s$ and $q\in C^1[0,1]$. Then for $|s|\geq q_0$, we have the asymptotic expansions*

$$\begin{aligned}
\omega_s(t)&=-\sin(st)s^{-1}+Q(t)\cos(st)s^{-2}+\mathcal{O}\bigl(s^{-3}\mathrm{e}^{3|y|t}\bigr),\\
(\omega_s)_s'(t,s)&=-t\cos(st)s^{-1}+\bigl(1-tQ(t)\bigr)\sin(st)s^{-2}+\mathcal{O}\bigl(s^{-3}\mathrm{e}^{3|y|t}\bigr),\\
(\omega_s)_s''(t,s)&=t^2\sin(st)s^{-1}+t\bigl(2-tQ(t)\bigr)\cos(st)s^{-2}+\mathcal{O}\bigl(s^{-3}\mathrm{e}^{3|y|t}\bigr),\\
\omega_s'(t)&=-\cos(st)-Q(t)\sin(st)s^{-1}+\mathcal{O}\bigl(s^{-2}\mathrm{e}^{3|y|t}\bigr),\\
(\omega_s')_s'(t,s)&=t\sin(st)-tQ(t)\cos(st)s^{-1}+\mathcal{O}\bigl(s^{-2}\mathrm{e}^{3|y|t}\bigr).
\end{aligned}$$

Remark 2. (See [35, Lemma 10].) For solution $\omega_s(t)$ of equation (1.1) satisfying the initial conditions

$$\omega_s(0)=1,\quad \omega_s'(0)=0 \tag{2.4}$$

in the case $q \in C^1[0,1]$ we have the asymptotic expansions

$$\begin{aligned}
\omega_s(t) &= \cos(st) + Q(t)\sin(st)s^{-1} + \mathcal{O}\big(s^{-2}\mathrm{e}^{3|y|t}\big),\\
(\omega_s)'_s(t,s) &= -\,t\sin(st) + tQ(t)\cos(st)s^{-1} + \mathcal{O}\big(s^{-2}\mathrm{e}^{3|y|t}\big),\\
(\omega_s)''_s(t,s) &= -\,t^2\cos(st) - t^2Q(t)\sin(st)s^{-1} + \mathcal{O}\big(s^{-2}\mathrm{e}^{3|y|t}\big),\\
\omega'_s(t) &= -\,s\sin(st) + Q(t)\cos(st) + \mathcal{O}\big(s^{-1}\mathrm{e}^{3|y|t}\big),\\
(\omega'_s)'_s(t,s) &= -\,st\cos(st) - \big(1 + tQ(t)\big)\sin(st) + \mathcal{O}\big(s^{-1}\mathrm{e}^{3|y|t}\big).
\end{aligned}$$

As we can see, the asymptotic formulas for these two IVP are slightly different.

Corollary 3. (See [34, Corollary 3].) Let $x \in \mathbb{R}$, $\delta \in \mathbb{R}$, $q \in C^1[0,1]$, $Q_1(x)$ is bounded function. If $s = x + \delta$, $\delta = Q_1(x)x^{-1} + \mathcal{O}(x^{-2})$, then we have the following asymptotic expansions

$$\begin{aligned}
\omega_s(t) &= -\sin(xt)x^{-1} + \big(Q(t) - tQ_1(x)\big)\cos(xt)x^{-2} + \mathcal{O}\big(x^{-3}\big),\\
\omega'_s(t) &= -\cos(xt) - \big(Q(t) - tQ_1(x)\big)\sin(xt)x^{-1} + \mathcal{O}\big(x^{-2}\big).
\end{aligned}$$

Remark 3. (See [35, Corollary 3].) Let the conditions of Corollary 3 are satisfied. Then for IVP with (2.4) we have the following asymptotic expansions

$$\begin{aligned}
\omega_s(t) &= \cos(xt) + \big(Q(t) - tQ_1(x)\big)\sin(xt)x^{-1} + \mathcal{O}\big(x^{-2}\big),\\
\omega'_s(t) &= -\,x\sin(xt) + \big(Q(t) - tQ_1(x)\big)\cos(xt) + \mathcal{O}\big(x^{-1}\big).
\end{aligned}$$

Example 1. If $q \in C^2[0,1]$, then we can calculate the first functions p_j^l and $\bar{p}_j^l$:

$$\begin{aligned}
&p_1^0 = -1,\ \ p_2^0 = Q(t),\ \ p_3^0 = -\tfrac12(Q(t))^2 + \tfrac14 q(t) + \tfrac14 q(0),\\
&p_1^1 = t,\ \ p_2^1 = 1 - tQ(t),\ \ p_1^2 = -t^2, \qquad (2.5)\\
&\bar{p}_0^0 = 1,\ \ \bar{p}_1^0 = -Q(t),\ \ \bar{p}_2^0 = \tfrac12(Q(t))^2 + \tfrac14 q(t) - \tfrac14 q(0),\\
&\bar{p}_0^1 = -t,\ \ \bar{p}_1^1 = tQ(t),\ \ \bar{p}_0^2 = t^2, \qquad (2.6)
\end{aligned}$$

and expresions

$$\begin{aligned}
R_1 &= -\sin(xt),\quad R_2 = \big(Q(t) - tQ_1(x)\big)\cos(xt),\\
R_3 &= -\big(p_3^0(t) - p_2^1(t)Q_1(x) + \tfrac12 p_1^2(t)(Q_1(x))^2\big)\sin(xt) - p_1^1(t)Q_2(x)\cos(xt),\\
\bar{R}_0 &= -\cos(xt),\quad \bar{R}_1 = -\big(Q(t) - tQ_1(x)\big)\sin(xt),\\
\bar{R}_2 &= -\big(\bar{p}_2^0(t) + \bar{p}_1^1(t)Q_1(x) + \tfrac12\bar{p}_0^2(t)(Q_1(x))^2\big)\cos(xt) - \bar{p}_0^1(t)Q_2(x)\sin(xt).
\end{aligned}$$

So, we have explicit expressions for R_3 and $\tilde{R}_2$ in the case $r = 2$, too, and

$$\omega_s(t) = R_1(t,x)x^{-1} + R_2(t,x)x^{-2} + R_3(t,x)x^{-3} + \mathcal{O}\big(x^{-4}\big), \qquad (2.7)$$

$$\omega'_s(t) = \bar{R}_0(t,x) + \bar{R}_1(t,x)x^{-1} + \bar{R}_2(t,x)x^{-2} + \mathcal{O}\big(x^{-3}\big). \qquad (2.8)$$

3 Properties of a spectrum in the case $q \equiv 0$

In this section, we present the results of articles [22,23] about eigenvalues of SLP (1.1)–(1.3) in the case $q(t) \equiv 0$. The spectrum of this problem has countably many eigenvalues. Negative eigenvalues exist for $\gamma > 0$ only. A unique negative eigenvalue exists: in Cases 1, 3 for $\gamma > 1/\xi$; in Case 2 for $\gamma > 1$. Also, $\lambda = 0$ is eigenvalue: in Cases 1, 3 for $\gamma = 1/\xi$; in Case 2 for $\gamma = 1$.

Let us define a *Constant Eigenvalue* (CE) as the eigenvalue λ that does not depend on the parameter $\gamma \in \mathbb{R}$ for fixed ξ. Constant eigenvalues exist only for rational numbers $\xi = m/n \in (0,1)$, $m,n \in \mathbb{N}$, $\gcd(m,n) = 1$, and those eigenvalues $\lambda_k = \pi^2 q_k^2$, $k \in \mathbb{N}$, are given by: $q_k = n(k-1/2)$ for $m \in \mathbb{N}_{\text{even}}$, $n \in \mathbb{N}_{\text{odd}}$ in Case 1, $q_k = n(k-1/2)$ for $m,n \in \mathbb{N}_{\text{odd}}$ in Case 2, $q_k = nk$ in Case 3. All CE are simple.

All nonconstant (that depend on the parameter $\gamma \in \mathbb{R}$) eigenvalues $\lambda = s^2$, $s \in \mathbb{C}_s$, are γ-points of the Characteristic Function (CF) $\gamma : \mathbb{C}_s \to \mathbb{R}$ [36] $\gamma(s) = Z(s)/P_\xi(s)$, where

$$Z(s) = \cos s, \quad P_\xi(s) = \frac{\sin(\xi s)}{s}, \tag{3.1$_1$}$$

$$Z(s) = \cos s, \quad P_\xi(s) = \cos(\xi s), \tag{3.1$_2$}$$

$$Z(s) = \frac{\sin s}{s}, \quad P_\xi(s) = \frac{\sin(\xi s)}{s}. \tag{3.1$_3$}$$

For fixed $\gamma \in \mathbb{R}$ the roots of this meromorphic function describe nonconstant eigenvalues. All zeroes of the functions $Z(s)$ and $P_\xi(s)$ are real, simple and belong to sets:

$$\hat{\mathcal{Z}} = \{z_l = \pi(l-1/2),\ l \in \mathbb{N}\}, \quad \overline{\mathcal{Z}}_\xi = \{p_k = \pi k/\xi,\ k \in \mathbb{N}\}, \tag{3.2$_1$}$$

$$\hat{\mathcal{Z}} = \{z_l = \pi(l-1/2),\ l \in \mathbb{N}\}, \quad \overline{\mathcal{Z}}_\xi = \{p_k = \pi(k-1/2)/\xi,\ k \in \mathbb{N}\}, \tag{3.2$_2$}$$

$$\hat{\mathcal{Z}} = \{z_l = \pi l,\ l \in \mathbb{N}\}, \quad \overline{\mathcal{Z}}_\xi = \{p_k = \pi k/\xi,\ k \in \mathbb{N}\}. \tag{3.2$_3$}$$

We denote the set of all CE as $\mathcal{C}_\xi = \hat{\mathcal{Z}} \cap \overline{\mathcal{Z}}_\xi$. All poles of CF are of the first order and belong to $\mathcal{P}_\xi = \overline{\mathcal{Z}}_\xi \setminus \hat{\mathcal{Z}} = \overline{\mathcal{Z}}_\xi \setminus \mathcal{C}_\xi$ [5]. Nonconstant eigenvalues can be complex [4, 5, 30, 37].

In the case $q \equiv 0$ the characteristic equation for SLP (1.1)–(1.3) is

$$-\cos s + \gamma s^{-1} \sin(\xi s) = 0, \tag{3.3$_1$}$$

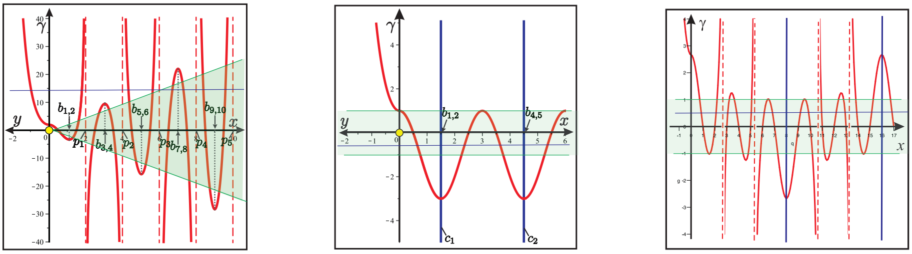
$$-\cos s + \gamma \cos(\xi s) = 0, \tag{3.3$_2$}$$

$$-s^{-1} \sin s + \gamma s^{-1} \sin(\xi s) = 0. \tag{3.3$_3$}$$

Lemma 4. *Let $\lambda_k = (s_k)^2$, $k \in \mathbb{N}$, be eigenvalues of the problem* (1.1)–(1.2), (1.3$_1$) *in the case $q \equiv 0$. Then exists $K \in \mathbb{N}$ such that for fixed $\gamma \in \mathbb{R}$ all eigenvalues λ_k, $k \geq K$, are positive, simple and $s_k = x_k \in I_{k-1}$ for all $k \geq K$.*

Proof. If x is multiple positive eigenvalue, then two equalities

$$\gamma \sin(\xi x) = x \cos x, \quad \xi\gamma \cos(\xi x) = x \sin x + \cos x$$



(a) $u'(1) = \gamma u(\xi)$ (b) $u'(1) = \gamma u'(\xi), |\gamma| < 1$ (c) $u(1) = \gamma u(\xi)$, $|\gamma| < 1$

Figure 2. Real CF (Dirichlet BC) and positive eigenvalues.

are valid. From this system we get

$$\xi^2\gamma^2 = x^2\big(1 - (1-\xi^2)\cos^2 x\big) + x\,\sin(2x) + \cos^2 x.$$

If $x \geq 9\xi^{-2}/4 \geq 1$, then we estimate

$$|\gamma|^2 \geq x^2 + \xi^{-2}\big(x\,\sin(2x) + \cos^2 x\big) \geq x^2 - \xi^{-2}(1+x) \geq x^2/9 \geq (x/\pi)^2.$$

So, all eigenvalues in the angle $|\gamma| < x/\pi$ for $x \geq 9\xi^{-2}/4$ are positive and simple. CE points are the first-order poles of CF. Eigenvalues corresponding to these points are positive and simple. Since CF has zeros at points $\pi(k-1/2)$, $k \in \mathbb{N}$, we have $|x_k - \pi(k-1/2)| < \pi$ in this domain. □

The graph of Real CF (see [36]) $\gamma\colon \mathbb{R}_s \to \mathbb{R}$ in Case 1 is presented in Figure 2(a). In the angle $|\gamma| < x/\pi$ we have positive simple eigenvalues only.

Lemma 5. (See [28, Lemma 4].) *Let* $|\gamma| < 1$, $0 < \xi < 1$, $\beta \geq 0$. *If* $\sin x - \gamma\xi^\beta \sin(\xi x) = 0$, *then there exists* $\kappa > 0$ *such that* $|\cos x| - |\gamma||\cos(\xi x)| \geq \kappa > 0$.

Lemma 6. (See [28, Lemma 5].) *Let* $|\gamma| < 1$, $0 < \xi < 1$, $\beta \geq 0$. *If* $\cos x - \gamma\xi^\beta \cos(\xi x) = 0$, *then there exists* $\tilde{\kappa} > 0$ *such that* $|\sin x| - |\gamma||\sin(\xi x)| \geq \tilde{\kappa} > 0$.

Remark 4. These two lemmas are valid for $\beta = \infty$ (in this case $\xi^\beta = 0$).

In Figure 2(b) and Figure 2(c) we see that for $|\gamma| < 1$ all eigenvalues are positive and simple. Now we present some properties of equation

$$f(s) := \cos s - \gamma\cos(\xi s) = 0, \tag{3.4$_2$}$$

$$f(s) := \sin s - \gamma\sin(\xi s) = 0, \tag{3.4$_3$}$$

$s \in \mathbb{C}_s$, $\xi \in [0,1]$.

Remark 5. The equation $\sin s - \gamma\sin(\xi s) = 0$, $|\gamma| < 1$, in Case 3 was investigated in [19, 28, 31].

Lemma 7. *If* $|\gamma| < 1$, *all roots of Equation* (3.4) *are real and simple.*

Proof. The lemma is valid for $\xi = 0$. Let $\xi \neq 0$, $y \neq 0$. If $s = x + \imath y$, then $\cos(x + \imath y) = \cos x \cosh y - \imath \sin x \sinh y$. So, Equation ($3.4_2$) is equivalent to the system

$$\cos x \cosh y = \gamma \cos(\xi x) \cosh(\xi y), \quad \sin x \sinh y = \gamma \sin(\xi x) \sinh(\xi y).$$

From these two equations we derive

$$1 > \gamma^2 = \frac{\cosh^2 y}{\cosh^2(\xi y)} \cos^2 x + \frac{\sinh^2 y}{\sinh^2(\xi y)} \sin^2 x \geq \sin^2 x + \cos^2 x = 1.$$

This contradiction shows that $s = x$, $y = 0$. If x is not a simple root, then

$$f(x) = \cos x - \gamma \cos(\xi x) = 0, \quad -f'(x) = \sin x - \xi\gamma \sin(\xi x) = 0,$$

and

$$1 = \sin^2 x + \cos^2 x = \gamma^2 \big(\cos^2(\xi x) + \xi^2 \sin^2(\xi x)\big) \leq \gamma^2 < 1.$$

This contradiction shows real roots are simple. The proof for Equation (3.4_3) is analogous. □

Lemma 8. *If* $|\gamma| < 1$*, then Equation* (3.4) *has infinitely many* (*countable*) *positive roots* x_k*,* $k \in \mathbb{N}$*, and* $x_k \in I_k$*.*

Proof. The lemma is valid for $\xi = 0$ or $\gamma = 0$. For definiteness, we take $0 < \gamma < 1$. Lemma 7 states that all roots are real. Then Equation (3.4_2) is equivalent to

$$x = (-1)^k \arcsin\big(\gamma \cos(\xi x)\big) + \pi(k - 1/2), \quad k \in \mathbb{N}. \tag{3.5}$$

Since

$$\Big|\Big((-1)^k \arcsin\big(\gamma \cos(\xi x)\big)\Big)'\Big| = \frac{\gamma\xi |\sin(\xi x)|}{\sqrt{1 - \gamma^2 \cos^2(\xi x)}} \leq \gamma\xi \leq \gamma < 1,$$

we have unique solution x_k of Equation (3.5) for every $k \in \mathbb{N}$ by the Banach Fixed-Point theorem, and

$$|x_k - \pi(k - 1/2)| = \big|\arcsin\big(\gamma \cos(\xi x)\big)\big| \leq \arcsin \gamma < \arcsin 1 = \pi/2,$$

i.e. $x_k \in (a_k, a_{k+1})$. The proof for Equation (3.4_3) is analogous. □

Additionaly we prove a few simple estimates for function f (see (3.4)).

Lemma 9. *The following inequality*

$$|f(s)| \geq \sinh |y| - |\gamma| \cosh(\xi y) \tag{3.6}$$

is valid.

Proof. We estimate

$$|\cos s-\gamma\cos(\xi s)|\ge\big||\cos s|-|\gamma\cos(\xi s)|\big|\ge|\cos s|-|\gamma|\,|\cos(\xi s)|,$$
$$|\sin s-\gamma\sin(\xi s)|\ge\big||\sin s|-|\gamma\sin(\xi s)|\big|\ge|\sin s|-|\gamma|\,|\sin(\xi s)|.$$

Then, using properties

$$\sinh|y|\le|\sin s|\le\cosh y,\quad \sinh|y|\le|\cos s|\le\cosh y$$

we get (3.6). □

We note that for $|\gamma|<1$ we have

$$\lim_{y\to+\infty}\big(\sinh y-|\gamma|\cosh(\xi y)\big)\mathrm{e}^{-y}=\tfrac12(1-|\gamma|\cdot\lfloor\xi\rfloor)\ge\tfrac12(1-|\gamma|)>0.$$

Corollary 4. If $|\gamma|<1$, then there exists $B>0$ such that

$$|f(s)|\ge\tfrac14(1-|\gamma|)\mathrm{e}^{|y|}\text{ for }|y|\ge B.$$

Lemma 10. *The following inequalities*

$$|\cos s-\gamma\cos(\xi s)|\ge(|\cos x|-|\gamma|\,|\cos(\xi x)|)\cosh y,\tag{3.7$_2$}$$
$$|\sin s-\gamma\sin(\xi s)|\ge(|\sin x|-|\gamma|\,|\sin(\xi x)|)\cosh y\tag{3.7$_3$}$$

are valid.

Proof. If $s=x+\imath y$, then $\operatorname{Re}\cos s=\cos x\,\cosh y$ and $\operatorname{Re}\sin s=\sin x\,\cosh y$. We estimate

$$\begin{aligned}|\cos s-\gamma\cos(\xi s)|&\ge|\text{Re }\cos s-\gamma\text{Re }\cos(\xi s)|=|\cos x\,\cosh y\\&-\gamma\cos(\xi x)\,\cosh(\xi y)|\ge|\cos x|\,\cosh y-|\gamma|\,|\cos(\xi x)|\,\cosh(\xi y)\\&\ge|\cos x|\,\cosh y-|\gamma|\,|\cos(\xi x)|\,\cosh y,\end{aligned}$$

because $\cosh(\xi y)\le\cosh y$ for $\xi\in[0,1]$. The proof of (3.7$_3$) is the similar. □

4 Characteristic equation for problem with two-point boundary condition

Substituting $\omega_s(t)$ into (1.3) we get the characteristic equation

$$h(s):=\omega_s'(1)-\gamma\omega_s(\xi)=0,\tag{4.1$_1$}$$
$$h(s):=\omega_s'(1)-\gamma\omega_s'(\xi)=0,\tag{4.1$_2$}$$
$$h(s):=\omega_s(1)-\gamma\omega_s(\xi)=0.\tag{4.1$_3$}$$

The set of eigenvalues of the BVP (1.1),(1.2), (1.3) coincides with the set $\{\lambda\colon\lambda=s^2,h(s)=0\}$.

We will use notation ρ: $\rho = 0$ in Cases 1, 2; $\rho = 1$ in Case 3 and introduce functions:

$$h_0^l(s) := -\bar{p}_0^l \cos(s - \tfrac{\pi}{2} l), \quad l \in \mathbb{N}_0, \tag{4.2}$$

in Case 1, and

$$h_j^l(s) := \gamma p_j^l(\xi) \cos\bigl(\xi s + \tfrac{\pi}{2}(j-l)\bigr) - \bar{p}_j^l(1) \cos\bigl(s + \tfrac{\pi}{2}(j-l)\bigr), \tag{4.3$_1$}$$
$$h_j^l(s) := \gamma \bar{p}_j^l(\xi) \cos\bigl(\xi s + \tfrac{\pi}{2}(j-l)\bigr) - \bar{p}_j^l(1) \cos\bigl(s + \tfrac{\pi}{2}(j-l)\bigr), \tag{4.3$_2$}$$
$$h_j^l(s) := \gamma p_j^l(\xi) \cos\bigl(\xi s + \tfrac{\pi}{2}(j-l)\bigr) - p_j^l(1) \cos\bigl(s + \tfrac{\pi}{2}(j-l)\bigr), \tag{4.3$_3$}$$

where $j = \overline{1,r}$ in Case 1, $j = \overline{0,r}$ in Case 2, $j = \overline{1,r+1}$ in Case 3, $l \in \mathbb{N}_0$.

In this article, we will need some expressions of these functions for $l = 0, 1, 2$. Using the formulas (2.5)–(2.6) for p_j^l and $\bar{p}_j^l$ we get

$$h_0^0(s) = -\cos s, \qquad h_1^0(s) = -Q(1)\sin s + \gamma \sin(\xi s), \tag{4.4$_1$}$$
$$h_0^0(s) = -\cos s + \gamma\cos(\xi s), \qquad h_1^0(s) = -Q(1)\sin s + \gamma Q(\xi)\sin(\xi s), \tag{4.4$_2$}$$
$$h_1^0(s) = -\sin s + \gamma\sin(\xi s), \qquad h_2^0(s) = Q(1)\cos s - \gamma Q(\xi)\cos(\xi s), \tag{4.4$_3$}$$

$$h_2^0(s) = -\gamma Q(\xi)\cos(\xi s) + \tfrac14\bigl(2\bigl(Q(1)\bigr)^2 + q(1) - q(0)\bigr)\cos s, \tag{4.5$_1$}$$
$$h_2^0(s) = -\tfrac14\gamma\bigl(2\bigl(Q(\xi)\bigr)^2 + q(\xi) - q(0)\bigr)\cos(\xi s) + \tfrac14\bigl(2\bigl(Q(1)\bigr)^2 + q(1) - q(0)\bigr)\cos s, \tag{4.5$_2$}$$
$$h_3^0(s) = \tfrac14\gamma\bigl(2\bigl(Q(\xi)\bigr)^2 - q(\xi) - q(0)\bigr)\sin(\xi s) - \tfrac14\bigl(2\bigl(Q(1)\bigr)^2 - q(1) - q(0)\bigr)\sin s, \tag{4.5$_3$}$$

$$h_0^1(s) = \sin s, \qquad h_0^2(s) = \cos s, \tag{4.6$_1$}$$
$$h_0^1(s) = \sin s - \gamma\xi\sin(\xi s), \qquad h_0^2(s) = \cos s - \gamma\xi^2\cos(\xi s), \tag{4.6$_2$}$$
$$h_1^1(s) = -\cos s + \gamma\xi\cos(\xi s), \qquad h_1^2(s) = \sin s - \gamma\xi^2\sin(\xi s), \tag{4.6$_3$}$$

$$h_1^1(s) = -Q(1)\cos s + \gamma\xi\cos(\xi s), \tag{4.7$_1$}$$
$$h_1^1(s) = -Q(1)\cos s + \gamma\xi Q(\xi)\cos(\xi s), \tag{4.7$_2$}$$
$$h_2^1(s) = \bigl(1 - Q(1)\bigr)\sin s - \gamma\xi\bigl(1 - \xi Q(\xi)\bigr)\sin(\xi s). \tag{4.7$_3$}$$

Lemma 11. *Let $s \in \mathbb{C}_s$ and $q \in C^r[0,1]$. Then for $|s| \geq q_0$ we have the asymptotic expansions*

$$h^{(l)}(s) = \sum_{j=\rho}^{r+\rho} h_j^l(s) s^{-j} + \mathcal{O}\bigl(s^{-(r+1+\rho)} \mathrm{e}^{(r+2)|y|}\bigr), \quad l \in \mathbb{N}_0. \tag{4.8}$$

Proof. Function h is an analytic function of parameter $s \in \mathbb{C}_s$ and

$$h^{(l)}(s) = (\omega_s')_s^{(l)}(1,s) - \gamma(\omega_s)_s^{(l)}(\xi,s), \tag{4.9$_1$}$$

$$h^{(l)}(s) = (\omega_s')_s^{(l)}(1,s) - \gamma(\omega_s')_s^{(l)}(\xi,s), \tag{4.9$_2$}$$

$$h^{(l)}(s) = (\omega_s)_s^{(l)}(1,s) - \gamma(\omega_s)_s^{(l)}(\xi,s), \tag{4.9$_3$}$$

$l \in \mathbb{N}_0$. Substituting (2.2) into (4.9) we get

$$\begin{aligned} h^{(l)}(s) = {}& \gamma \sum_{j=1}^{r} p_j^l(\xi)\cos(\xi s + \tfrac{\pi}{2}(j-l))s^{-j} \\ & - \sum_{j=0}^{r} \bar{p}_j^l(1)\cos(s + \tfrac{\pi}{2}(j-l))s^{-j} + \mathcal{O}(s^{-(r+1)}\mathrm{e}^{(r+2)|y|}), \end{aligned} \tag{4.10$_1$}$$

$$\begin{aligned} h^{(l)}(s) = {}& \gamma \sum_{j=0}^{r} \bar{p}_j^l(\xi)\cos(\xi s + \tfrac{\pi}{2}(j-l))s^{-j} \\ & - \sum_{j=0}^{r} \bar{p}_j^l(1)\cos(s + \tfrac{\pi}{2}(j-l))s^{-j} + \mathcal{O}(s^{-(r+1)}\mathrm{e}^{(r+2)|y|}), \end{aligned} \tag{4.10$_2$}$$

$$\begin{aligned} h^{(l)}(s) = {}& \gamma \sum_{j=1}^{r+1} p_j^l(\xi)\cos(\xi s + \tfrac{\pi}{2}(j-l))s^{-j} \\ & - \sum_{j=1}^{r+1} p_j^l(1)\cos(s + \tfrac{\pi}{2}(j-l))s^{-j} + \mathcal{O}(s^{-(r+2)}\mathrm{e}^{(r+2)|y|}). \end{aligned} \tag{4.10$_3$}$$

We look for terms at s^{-j} and get expressions (4.2)–(4.3). □

In the case $r = 0$ the last term is $\mathcal{O}(s^{-1}\mathrm{e}^{|y|})$ (Cases 1, 2) or $\mathcal{O}(s^{-2}\mathrm{e}^{|y|})$ (Case 3) (see Lemma 2) and we have

$$h^{(l)}(s) = \ h_\rho^l(s)s^{-\rho} + \mathcal{O}(s^{-1-\rho}\mathrm{e}^{|y|}), \quad l \in \mathbb{N}_0, \tag{4.11}$$

where

$$h_0^l(s) = -\,(-1)^l\cos(s - \tfrac{\pi}{2}l), \tag{4.12$_1$}$$

$$h_0^l(s) = -\,(-1)^l\cos(s - \tfrac{\pi}{2}l) + \gamma(-1)^l\xi^l\cos(\xi s - \tfrac{\pi}{2}l), \tag{4.12$_2$}$$

$$h_1^l(s) = -\,(-1)^l\sin(s - \tfrac{\pi}{2}l) + \gamma(-1)^l\xi^l\sin(\xi s - \tfrac{\pi}{2}l). \tag{4.12$_3$}$$

If $q \in C^1[0,1]$, then we have

$$h^{(l)}(s) = \ h_\rho^l(s)s^{-\rho} + h_{1+\rho}^l(s)s^{-1-\rho} + \mathcal{O}\big(s^{-2-\rho}\mathrm{e}^{3|y|}\big), \quad l \in \mathbb{N}_0, \tag{4.13}$$

where

$$h_1^l(s) = -\,(-1)^lQ(1)\sin(s - \tfrac{\pi}{2}l) + \gamma(-1)^l\xi^l\sin(\xi s - \tfrac{\pi}{2}l), \tag{4.14$_1$}$$

$$h_1^l(s) = -\,(-1)^lQ(1)\sin(s - \tfrac{\pi}{2}l) + \gamma(-1)^l\xi^lQ(\xi)\sin(\xi s - \tfrac{\pi}{2}l), \tag{4.14$_2$}$$

$$\begin{aligned} h_2^l(s) = {}& -\,(-1)^l\big(l - Q(1)\big)\cos(s - \tfrac{\pi}{2}l) \\ & + \gamma(-1)^l\xi^{l-1}\big(l - \xi Q(\xi)\big)\cos(\xi s - \tfrac{\pi}{2}l). \end{aligned} \tag{4.14$_3$}$$

Remark 6. Asymptotic expansions (4.8), (4.11) and (4.13) are valid for SLP with Neumann BC with $\rho = -1$ in Cases 1, 2; $\rho = 0$ in Case 3. Expressions for the functions $h_j^l(s)$ were found in [35].

Analytic functions $H := h(s)s^{\rho}$, $M := h(s)s^{1+\rho}$ have the same nonzero roots as function h and

$$H^{(l)}(s) = H_0^l(s) + \mathcal{O}\big(s^{-1}\mathrm{e}^{|y|}\big), \quad M^{(l)}(s) = M_{-1}^l(s)\cdot s + \mathcal{O}\big(\mathrm{e}^{|y|}\big), \qquad (4.15)$$

where $M_{-1}^l(s) = H_0^l(s) = h_\rho^l(s)$, $l \in \mathbb{N}_0$.

Function $H_0^0(s) = \bar{H}_0^0(s) + \hat{H}_0^0(s)$ (see (4.4)), where

$$H_0^0(s) = -\cos s, \qquad \bar{H}_0^0(s) = -\cos s, \qquad \hat{H}_0^0(s) = 0, \qquad (4.16_1)$$

$$H_0^0(s) = -\cos s + \gamma\cos(\xi s), \quad \bar{H}_0^0(s) = -\cos s, \quad \hat{H}_0^0(s) = \gamma\cos(\xi s), \quad (4.16_2)$$

$$H_0^0(s) = -\sin s + \gamma\sin(\xi s), \quad \bar{H}_0^0(s) = -\sin s, \quad \hat{H}_0^0(s) = \gamma\sin(\xi s). \quad (4.16_3)$$

Lemma 12. *Assume that $|\gamma| < 1$ in Cases 2, 3. Function H_0^0 has only simple nonnegative roots x_k: $k \in \mathbb{N}$ in Cases 1, 2; $k \in \mathbb{N}_0$ in Case 3. The root $x_0 = 0$. Positive roots $x_k \in I_k$, $k \in \mathbb{N}$. More precisely, $x_k = \pi(k - 1/2)$ in Case 1.*

Proof. In Case 1 the proof is obvious. In Cases 2, 3 the proof follows from Lemma 7 and Lemma 8. □

If $s \in \mathbb{R}_s^-$, i.e., $s = \imath y$, $y > 0$, $q \in C[0,1]$, then (see (2.3))

$$\tilde{H}(y) := -h(\imath y) = \mathrm{e}^y/2 + \mathcal{O}\big(y^{-1}\mathrm{e}^y\big), \qquad (4.17_{1,2})$$

$$\tilde{H}(y) := -h(\imath y) = y^{-1}\mathrm{e}^y/2 + \mathcal{O}\big(y^{-2}\mathrm{e}^y\big). \qquad (4.17_3)$$

Let to consider positive eigenvalues, $q \in C^r[0,1]$. In this case, (4.10) is valid with $s = x > 0$ ($y = 0$) and functions M_j^l, $j = \overline{-1, r-1}$, $l \in \mathbb{N}_0$, are bounded and from (4.11) we have $M^{(l)}(x) = \mathcal{O}(x)$, $l \in \mathbb{N}_0$ or

$$h^{(l)}(x) = \mathcal{O}(x^{-\rho}), \quad l \in \mathbb{N}_0. \qquad (4.18)$$

We investigate equation $M(x+\delta) = 0$, $\delta \in \mathbb{R}$, with additional condition

$$|h_\rho^1(x)| \geq \varkappa > 0. \qquad (4.19)$$

We note that this condition is equivalent to $|M_{-1}^1(x)| \geq \varkappa > 0$. For real $s = x > 0$ from (4.17) we have

$$M(x) = M_{-1}^0(x)\cdot x + \mathcal{O}(1). \qquad (4.20)$$

Lemma 13. *If x is such that $M_{-1}^0(x) = 0$ and $\delta = o(1)$, then $\delta = \mathcal{O}\big(x^{-1}\big)$.*

Proof. If $x+\delta$ is the root of function M, then from (4.20) we have equality $(x+\delta)M_{-1}^0(x+\delta) + \mathcal{O}(1) = 0$ or $M_{-1}^0(x+\delta) = \mathcal{O}\big((x+\delta)^{-1}\big) = \mathcal{O}\big(x^{-1}\big)$. Since

$$M_{-1}^0(x+\delta) = M_{-1}^0(x) + M_{-1}^1(x)\delta + \delta^2\int_0^1\int_0^1 M_{-1}^2(x+\xi\tau\delta)\xi\, d\xi\, d\tau$$

we get

$$\big(M^1_{-1}(x)+\mathcal{O}(\delta)\big)\delta=\mathcal{O}\big(x^{-1}\big).$$

If $|M^1_{-1}(x)|\geq\varkappa>0$ and $\delta=o(1)$, then from this formula we have $\delta=\mathcal{O}\big(x^{-1}\big)$. Lemma is proved. □

Let's denote the function

$$Q_1(x)=-h^0_{1+\rho}(x)\big(h^1_\rho(x)\big)^{-1}. \qquad (4.21)$$

If functions $Q_1,\ldots,Q_{k-1}$ are defined, then we can find functions

$$z_l(x)=\sum_{\substack{n_1+\ldots+n_{k-1}=i,\ j\geq0,\\ j+n_1+2n_2+\ldots+(k-1)n_{k-1}=l}}-h^{i+1}_{j+\rho}(x)(h^1_\rho(x))^{-1}\frac{Q_1^{n_1}(x)\cdots Q_{k-1}^{n_{k-1}}(x)}{(i+1)n_1!\cdots n_{k-1}!}, \qquad (4.22)$$

$l=\overline{1,k-1}$ and function

$$Q_k(x)=\sum_{\substack{n_1+\ldots+n_{k-1}=l,\ j>0,\\ j+n_1+2n_2+\ldots+(k-1)n_{k-1}=k}}-h^0_{j+\rho}(x)(h^1_\rho(x))^{-1}\frac{l!z_1^{n_1}(x)\cdots z_{k-1}^{n_{k-1}}(x)}{n_1!\cdots n_{k-1}!}. \qquad (4.23)$$

If $q\in C^r[0,1]$, then in such way we can find all $Q_j(x)$, $j=\overline{1,r}$, and they are bounded functions.

Lemma 14. *If $q\in C^r[0,1]$, $h^0_\rho(x)=0$ and $\delta=o(1)$, then we have asymptotic expansion*

$$\delta=\sum_{j=1}^{r}Q_j(x)x^{-j}+\mathcal{O}\big(x^{-(r+1)}\big). \qquad (4.24)$$

Proof. Formula (4.24) is valid for $r=0$. So, $\delta=\mathcal{O}\big(x^{-1}\big)$. If $r=1$, then substituting (4.13) and (4.18) expressions into equality

$$0=h(x+\delta)=h(x)+h'(x)\delta+h''(x+\theta\delta)\delta^2/2,\quad \theta\in[0,1],$$

we have $h^1_\rho(x)x^{-\rho}\delta=-h^0_{1+\rho}(x)x^{-1-\rho}+\mathcal{O}\big(x^{-2-\rho}\big)$, i.e., $\delta=Q_1(x)x^{-1}+\mathcal{O}\big(x^{-2}\big)$, where Q_1 is defined by (4.21).

Finally, suppose that $\delta=\sum_{j=1}^{k-1}Q_j(x)x^{-j}+\mathcal{O}\big(x^{-k}\big)$, $k=\overline{2,r}$. Substituting (4.8) in the case $y=0$ and (4.18) expressions into equality

$$0=h(x+\delta)=h(x)+\delta\sum_{i=0}^{k-1}h^{(i+1)}(x)\frac{\delta^i}{(i+1)!}+\frac{h^{(k+1)}(x+\theta\delta)}{(k+1)!}\delta^{k+1},\ \theta\in[0,1],$$

we derive recursive formulas (4.22) and (4.23). The full proof of general formula (4.24) one can find in [34,35]. □

Remark 7. We can use the functions $Q_j(x)$, $j=\overline{1,r}$ in Corollary 1.

Corollary 5. If $q \in C^1[0,1]$, then

$$Q_1(x) = \frac{Q(1)\sin x - \gamma \sin(\xi x)}{\sin x}, \tag{4.25$_1$}$$

$$Q_1(x) = \frac{Q(1)\sin x - \gamma Q(\xi)\sin(\xi x)}{\sin x - \gamma\xi \sin(\xi x)}, \tag{4.25$_2$}$$

$$Q_1(x) = \frac{Q(1)\cos x - \gamma Q(\xi)\cos(\xi x)}{\cos x - \gamma\xi \cos(\xi x)}. \tag{4.25$_3$}$$

Remark 8. Expression for Q_1 in Case 3 was proved in [28].

Remark 9. For SLP with the Neumann BC [35] we have the following

$$Q_1(x) = \frac{Q(1)\cos x - \gamma \cos(\xi x)}{\cos x}, \tag{4.26$_1$}$$

$$Q_1(x) = \frac{Q(1)\cos x - \gamma Q(\xi)\cos(\xi x)}{\cos x - \gamma\xi \cos(\xi x)}, \tag{4.26$_2$}$$

$$Q_1(x) = \frac{Q(1)\sin x - \gamma Q(\xi)\sin(\xi x)}{\sin x - \gamma\xi \sin(\xi x)}. \tag{4.26$_3$}$$

Corollary 6. If $q \in C^2[0,1]$, then

$$\begin{aligned} Q_2(x) &= -\, h^0_{2+\rho}(x)\big(h^1_\rho(x)\big)^{-1} - h^0_{1+\rho}(x)\big(h^1_\rho(x)\big)^{-1} z_1(x), \\ z_1(x) &= -\, h^1_{1+\rho}(x)\big(h^1_\rho(x)\big)^{-1} - \tfrac{1}{2} h^2_\rho(x)\big(h^1_\rho(x)\big)^{-1} Q_1(x) \end{aligned} \tag{4.27}$$

where functions $h^0_{1+\rho}$, $h^0_{2+\rho}$, h^1_ρ, h^2_ρ, $h^1_{1+\rho}$ are calculated in (4.4)–(4.7).

This explicit formula (4.27) for function $Q_2(x)$ allows to calculate the term at x_k^{-2} in the asymptotic expansion of real eigenvalues when $q \in C^2[0,1]$. In Case 3 such asymptotic expansion was announced in [28, see Remark 7] and only formula (4.26$_3$) was derived. Formula (2.7) allows to find asymptotic expansion for corresponding eigenfunctions.

The formula (4.23) allows find the asymptotic expansion (4.24) for all $r \in \mathbb{N}_0$, because all Q_k can be calculated using $h^l_j(x)$, and for $h^l_j(x)$ we have explicit formulas (4.2)–(4.3), where functions $p^l_j(t)$ and $\bar{p}^l_j(t)$ can be calculated by recursive formulas (2.1) and recursive formulas in Lemma 1. Unfortunately, if $r > 1$, then the formulas are very complicated (see, for example, (4.27), (4.5), (2.5)–(2.6) in the case $r = 2$), and all results for $q > 2$ are useful only in a theoretical sense, proving that such asymptotic expansions exist.

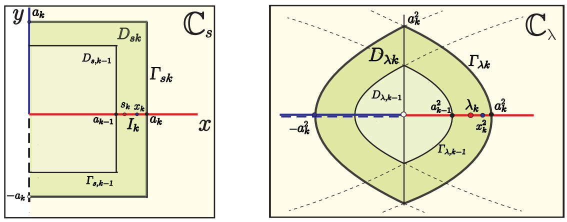
5 Spectral asymptotics for eigenvalues and eigenfunctions for problem with two-point boundary condition

In this section, we investigate eigenvalues for SLP (1.1)–(1.3).

Lemma 15. *Real eigenvalues of the SLP* (1.1)–(1.3) *are bounded from below.*

Proof. From (4.17) we have $\lim_{y\to+\infty}\tilde{H}(y)=+\infty$. Then there exists a $y_0>0$ such that $\tilde{H}(y)>0$ for $y>y_0$. Therefore, $h(\imath y)\neq 0$ for $y>y_0$. Accordingly, $-y_0^2\leq\lambda$ for negative λ. □

Corollary 7. The number of negative eigenvalues of problem (1.1)–(1.3) is finite *(*maybe zero*)*.



(a) D_{sk} (b) $D_{\lambda k}$

Figure 3. Domain D_{sk} and $D_{\lambda k}$.

In this section, we assume that $|\gamma|<1$ in Cases 2, 3. Let us denote domain $D_k=\{s\in\mathbb{C}\colon |x|\leq a_k,\ |y|\leq a_k\}$, $D_{sk}=\mathbb{C}_s\cap D_k$, $k\in\mathbb{N}$ ($k>1$ in Cases 1, 2), and a contour $\mathit{\Gamma}_{sk}=\mathbb{C}_s\cap\partial D_k$ (see Figure 3(a)).

Remark 10. The corresponding contour $\mathit{\Gamma}_{\lambda k}$ in the plane $\mathbb{C}_\lambda=\mathbb{C}$ will be the boundary of the domain $D_{\lambda k}$ (see Figure 3(b)). The contour $\mathit{\Gamma}_{\lambda k}$ belongs to two parabolas (see also Figure 1).

Lemma 16. *The function $H\colon\mathbb{R}_s^+\to\mathbb{R}$ has at least one positive root in the interval I_k for large k.*

Proof. For positive x we have formula (see (4.15))

$$H(x)=\bar{H}_0^0(x)+\hat{H}_0^0(x)+\mathcal{O}(x^{-1}).$$

In Case 2 we have $\bar{H}_0^0(x)=-\cos x$, $\hat{H}_0^0(x)=\gamma\cos(\xi x)$ and $a_k=\pi k$, $k\in\mathbb{N}$. So, in this case $\bar{H}_0^0(a_{2l})=-1$, $\bar{H}_0^0(a_{2l-1})=1$. Whereas $|\hat{H}_0^0(x)+\mathcal{O}\big(x^{-1}\big)|=|\gamma\cos(\xi x)+\mathcal{O}\big(x^{-1}\big)|<1$ for large x, therefore $H(a_{2l-1})>0$, $H(a_{2l})<0$. Then from Intermediate Value Theorem at least one root of the function $H(x)$ lies in each interval $I_k=\big(a_k,\,a_{k+1}\big)$, $K<k\in\mathbb{N}$ for large K. In Cases 1, 3 the proof is similar. □

Corollary 8. The SLP (1.1)–(1.3) have infinitely many *(*countable*)* positive eigenvalues.

Lemma 17. *There exists $q_1>0$ such that all eigenvalues of problem* (1.1)–(1.3) *in the domain $\{s\in\mathbb{C}_s\colon\ |s|>q_1\}$ are positive and, more precisely, there exists only one positive root of function $H(s)$ in each interval I_k for sufficiently large k.*

Proof. We consider formula (4.15) for $l = 0$:

$$H(s) = H_0^0(s) + \mathcal{O}\big(s^{-1}\mathrm{e}^{|y|}\big), \tag{5.1}$$

where

$$H_0^0(s) = -\cos s, \tag{5.2$_1$}$$

$$H_0^0(s) = -\cos s + \gamma\cos(\xi s), \tag{5.2$_2$}$$

$$H_0^0(s) = -\sin s + \gamma\sin(\xi s). \tag{5.2$_3$}$$

We claim that $|H_0^0(s)| \geq A\mathrm{e}^{|y|}$ for $s \in \Gamma_{sk}$ for sufficiently large k.

First of all in Case 1 the proof of this inequality is the same as in Case 2 with $\gamma = 0$. On the vertical part of contour Γ_{sk} we have $s = a_k + \imath y$, $y \in [-a_k, a_k]$, $k \in \mathbb{N}$. In Case 3 $\cos a_k = 0$ and from Lemma 6 (with $\beta = \infty$) it follows that $|\sin a_k| - |\gamma|\,|\sin(\xi a_k)| \geq \tilde{\kappa} > 0$. Then using the inequality (3.7$_3$) in Lemma 10 we estimate

$$|H_0^0(s)| = |-\sin s + \gamma\sin(\xi s)| \geq \big(|\sin a_k| - |\gamma|\,|\sin(\xi a_k)|\big)\cosh y \geq \tilde{\kappa}\mathrm{e}^{|y|}/2.$$

In Case 2 we use Lemma 5 (with $\kappa > 0$) and the inequality (3.7$_2$) in Lemma 10:

$$|H_0^0(s)| = |-\cos s + \gamma\cos(\xi s)| \geq \big(|\cos a_k| - |\gamma|\,|\cos(\xi a_k)|\big)\cosh y \geq \kappa\mathrm{e}^{|y|}/2.$$

On the remaining part of contour $y = \pm a_k$, $0 \leq x \leq a_k$, from Corollary 4 it follows that there exists $B > 0$ and $|H_0^0(s)| \geq \frac{1}{4}(1 - |\gamma|)\mathrm{e}^{|y|}$ for $|y| \geq B$. Finally, $|H_0^0(s)| \geq A\mathrm{e}^{|y|}$ for $s \in \Gamma_{sk}$ for sufficiently large k, where $A = \min\{\tilde{\kappa}/2, \kappa/2, (1 - |\gamma|)/4\}$.

Let's consider formula (5.1). We estimate

$$\big|\mathcal{O}\big(s^{-1}\mathrm{e}^{|y|}\big)\big| \leq c_1|s|^{-1}\mathrm{e}^{|y|} < A\mathrm{e}^{|y|} \leq |H_0^0(s)|$$

on the contours Γ_{sk} for sufficiently large k. Therefore, by Rouché theorem for domain $D_{\lambda k}$ with contour $\Gamma_{\lambda k}$ it follows that the number of zeros of $H(s) = H_0^0(s) + \mathcal{O}\big(s^{-1}\mathrm{e}^{|y|}\big)$ and $H_0^0(s)$ are the same inside Γ_{sk} for sufficiently large k.

There is exactly one root of the function $H_0^0(s)$ in the domain between contours Γ_{sk} and $\Gamma_{s,k+1}$ and it belongs to I_k (see Lemma 12). The function H has root in I_k for sufficiently large k (see Lemma 16). So, the single root of H in this domain is positive. □

Corollary 9. The function h has one positive root in I_k for large k.

So, there exist $q_1 > 0$ such that roots of functions $H(s)$ and $H_0^0(s)$ are positive for $|s| > q_1$. If a root $s \in I_k$, then we enumerate these roots as s_k for function $H(s)$ and x_k for function $H_0^0(s)$. We note, that $x_k = (k - 1/2)\pi$, $k \in \mathbb{N}$ in Case 1. We have $s_k \sim x_k \sim \pi k$ (as $k \to \infty$). Then $H(s_k) = H_0^0(s_k) + \mathcal{O}\big(k^{-1}\big) = 0$ and $\lim_{k\to\infty} H_0^0(s_k) = 0$. The function H_0^0 is analytic and has one root in I_k. Additionally, $|H_0^1(x_k)| \geq \kappa > 0$ (see Lemma 6 in Cases 2, Lemma 5 in Case 3 and equality $|H_0^1(x_k)| = |\sin x_k| = 1$ in Case 1). Therefore, $s_k \to x_k$ as $k \to \infty$ or

$$s_k = x_k + o(1) \quad (\text{as } k \to \infty). \tag{5.3}$$

Now we will investigate the distribution of these positive eigenvalues of problem (1.1)–(1.3), and we leave out the note about sufficiently large k. Now we consider only real positive $s > 0$.

Let us denote $\delta_k = s_k - x_k$. We have that $\delta_k = o(1)$.

Theorem 1. *Let $q \in C[0,1]$. For eigenvalues $\lambda_k = s_k^2$ and eigenfunctions u_k of problem* (1.1)–(1.3)*, we have the asymptotic expansions*

$$s_k = x_k + \mathcal{O}\big(k^{-1}\big), \qquad u_k(t) = -\sin(x_k t)x_k^{-1} + \mathcal{O}\big(k^{-2}\big) \tag{5.4}$$

for sufficiently large k.

Proof. We have $\delta_k = o(1)$ and $h_\rho^0(x_k) = H_0^0(x_k) = M_{-1}^0(x_k) = 0$, $|h_\rho^1(x_k)| = |H_0^1(x_k)| \geq \varkappa > 0$ $\big($see (4.19)$\big)$. So, all conditions of Lemma 13 are satisfied, and it follows $\delta_k = \mathcal{O}\big(x_k^{-1}\big) = \mathcal{O}\big(k^{-1}\big)$. Then we apply Corollary 2 and get

$$u_k = \omega_{s_k}(t) = -\sin(x_k t)x_k^{-1} + \mathcal{O}\big(x_k^{-2}\big) = -\sin(x_k t)x_k^{-1} + \mathcal{O}\big(k^{-2}\big).$$

□

Remark 11. Normalized eigenfunctions are

$$v_k(t) = \sqrt{2}\sin(x_k t) + \mathcal{O}\big(k^{-1}\big).$$

Theorem 2. *Let $q \in C^r[0,1]$. For eigenvalues $\lambda_k = s_k^2$ and eigenfunctions u_k of problem* (1.1)–(1.3)*, we have the asymptotic expansions*

$$s_k = x_k + \sum_{j=1}^{r} Q_j(x_k)x_k^{-j} + \mathcal{O}\big(k^{-(r+1)}\big), \tag{5.5}$$

$$u_k(t) = \sum_{j=1}^{r+1} R_j(t, x_k)x_k^{-j} + \mathcal{O}\big(k^{-(r+2)}\big) \tag{5.6}$$

for sufficiently large k.

Proof. We have $\delta_k = \mathcal{O}\big(k^{-1}\big)$ (see Theorem 1). So, all conditions of Lemma 14 are valid, and it follows

$$\delta_k = \sum_{j=1}^{r} Q_j(x_k)x_k^{-j} + \mathcal{O}\big(k^{-(r+1)}\big).$$

Then we apply Corollary 1 and get

$$u_k = \omega_{s_k}(t) = \sum_{j=1}^{r+1} R_j(t, x_k)x_k^{-j} + \mathcal{O}\big(k^{-(r+2)}\big).$$

□

Corollary 10. If $q \in C^1[0,1]$, then we have the asymptotic expansions

$$s_k = x_k + Q_1(x_k)x_k^{-1} + \mathcal{O}(k^{-2}),$$
$$u_k(t) = R_1(t,x_k)x_k^{-1} + R_2(t,x_k)x_k^{-2} + \mathcal{O}(k^{-3})$$

for sufficiently large k, where $Q_1(x)$ is defined by (4.25), $R_1(t,x) = -\sin(xt)$, $R_2(t,x) = \big(Q(t) - tQ_1(x)\big)\cos(xt)$.

Remark 12. Normalized eigenfunctions are

$$v_k(t) = \sqrt{2}\sin(x_k t) + \sqrt{2}\big(\tfrac{1}{4}\sin(2x_k)\sin(x_k t) - R_2(t,x_k)\big)x_k^{-1} + \mathcal{O}(k^{-2}). \quad (5.7)$$

Corollary 11. If $q \in C^2[0,1]$, then we have the asymptotic expansions

$$s_k = x_k + Q_1(x_k)x_k^{-1} + Q_2(x_k)x_k^{-1} + \mathcal{O}(k^{-3}), \quad (5.8)$$
$$u_k(t) = R_1(t,x_k)x_k^{-1} + R_2(t,x_k)x_k^{-2} + R_3(t,x_k)x_k^{-2} + \mathcal{O}(k^{-4})$$

for sufficiently large k, where $Q_1(x)$, $Q_2(x)$ are defined by (4.25),(4.27), $R_1(t,x)$, $R_2(t,x)$ and $R_3(t,x)$ are calculated in Example 1.

Remark 13. In [28] the asymptotic expansion (in Case 3) was published without proof. Now we calculate explicit formulas for this case.

Remark 14. For SLP with the Neumann BC the asymptotic expansion (5.5) in Theorem 2 remains correct, but formulas for functions Q_j, $j = \overline{1,r}$, differ from the Dirichlet case (see Remark 9). Formulas (4.21)–(4.23) for $Q_k(x)$ are the same, but parameter $\rho = -1$ (Cases 1,2) and $\rho = 0$ (Case 3) and functions h_j^l differ from the Dirichlet case. Instead the asymptotic expansion (5.6) we have

$$u_k(t) = \sum_{j=0}^{r} R_j(t,x_k)x_k^{-j} + \mathcal{O}(k^{-(r+1)}),$$

where $R_0(t,x) = \cos(xt)$, $R_1(t,x) = \big(Q(t) - tQ_1(x)\big)\cos(xt)$ (for more results with the Neumann BC see [35]).

6 Conclusions

In this paper, we investigated the asymptotic properties of the spectrum and eigenfunctions for a SLP with Dirichlet and nonlocal two-point boundary conditions. We obtained asymptotic expansions for eigenvalues and normalized eigenfunctions. Previous works in the literature of the Sturm–Liouville theory are especially for the cases $q \in C[0,1]$ or $q \in C^1[0,1]$. Contrary to the these previous works, the authors investigate the asymptotics of characteristic functions, eigenvalues and eigenfunctions for $q \in C^r[0,1]$, where r is arbitrary positive integer. This paper continues a series of similar studies for problems with the integral boundary condition [34], or the Neumann condition instead of the Dirichlet condition [35]. The results obtained in this work can be extended to differential equations with retarded argument [29].

References

[1] K. Aydemir and O.Sh. Mukhtarov. Asymptotic distribution of eigenvalues and eigenfunctions for a multi-point discontinuous Sturm–Liouville problem. *Electron. J. Differential Equations*, **2016**(131):1–14, 2016. Available from Internet: `https://ejde.math.txstate.edu/Volumes/2016/131/aydemir.pdf`.

[2] E. Başkaya. Periodic and semi-periodic eigenvalues of Hill's equation with symmetric double well potential. *TWMS J. App. and Eng. Math.*, **10**(2):346–352, 2020.

[3] A. Bayramov, S. Uslu and S. Ç̣alıṣkan. Computation of eigenvalues and eigenfunctions of a discontinuous boundary value problem with retarded argument. *Appl. Math. Comput.*, **191**(2):592–600, 2007. https://doi.org/10.1016/j.amc.2007.02.118.

[4] K. Bingelė, A. Bankauskienė and A. Štikonas. Spectrum curves for a discrete Sturm–Liouville problem with one integral boundary condition. *Nonlinear Anal. Model. Control*, **24**(5):755–774, 2019. https://doi.org/10.15388/NA.2019.5.5.

[5] K. Bingelė, A. Bankauskienė and A. Štikonas. Investigation of spectrum curves for a Sturm–Liouville problem with two-point nonlocal boundary conditions. *Math. Model. Anal.*, **25**(1):53–70, 2020. https://doi.org/10.3846/mma.2020.10787.

[6] A. Boumenir. Eigenvalues of periodic Sturm–Liouville problems by the Shannon–Whittaker sampling theorem. *Math. Comput.*, **68**(227):1057–1066, 1999. https://doi.org/10.1090/S0025-5718-99-01053-4.

[7] J. Cai, K. Li and Z. Zheng. A singular Sturm–Liouville problem with limit circle endpoint and boundary conditions rationally dependent on the eigenparameter. *Mediterr. J. Math.*, **19**(184):1–15, 2022. https://doi.org/10.1007/s00009-022-02109-z.

[8] R. Čiegis, A. Štikonas, O. Štikonienė and O. Subač. Stationary problems with nonlocal boundary conditions. *Math. Model. Anal.*, **6**(2):178–191, 2001. https://doi.org/10.3846/13926292.2001.9637157.

[9] H. Coşkun and E. Başkaya. Asymptotics of eigenvalues of regular Sturm–Liouville problems with eigenvalue parameter in the boundary condition for integrable potential. *Math. Scand.*, **107**(2):209–223, 2010.

[10] M. Eastham. *The Spectral Theory of Periodic Differential Equations*. Scottish Academic Press, Edinburgh, 1973.

[11] C.T. Fulton. Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions. *Proc. Edinb. Math. Soc. A*, **77**(3–4):293–308, 1977. https://doi.org/10.1017/S030821050002521X.

[12] N.J. Guliyev. Essentially isospectral transformations and their applications. *Ann. Mat. Pura Appl.*, **199**(4):1621–1648, 2020. https://doi.org/10.1007/s10231-019-00934-w.

[13] N.J. Guliyev. On two-spectra inverse problems. *Proc. Amer. Math. Soc.*, **148**(10):4491–4502, 2020. https://doi.org/10.1090/proc/15155.

[14] G. Infante. Eigenvalues of some nonlocal boundary-value problems. *Proc. Edinb. Math. Soc. (Series 2)*, **46**:75–86, 2003. https://doi.org/10.1017/S0013091501001079.

[15] G. Infante. Positive solutions of systems of perturbed Hammerstein integral equations with arbitrary order dependence. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci*, **379**(2191):20190376(1–10), 2021. https://doi.org/10.1098/rsta.2019.0376.

[16] G. Infante, P. Pietramala and F.A.F. Tojo. Non-trivial solutions of local and non-local Neumann boundary-value problems. *Proc. Roy. Soc. Edinburgh Sect. A*, **146**(2):337–369, 2016. https://doi.org/10.1017/S0308210515000499.

[17] A.G. Kostyuchenko and I.S. Sargsjan. *Distribution of eigenvalues.* Selfadjoint ordinary differential operators. Nauka, Moscow, 1979. (in Russian)

[18] B.M. Levitan and I.S. Sargsjan. *Sturm–Liouville and Dirac operators.* Kluwer, Dordrecht, 1991.

[19] V. Mityushev and P.M Adler. Darcy flow around a two-dimensional permeable lens. *J. Phys. A: Math. Gen.*, **39**(14):3545–3560, 2006. https://doi.org/10.1088/0305-4470/39/14/004.

[20] O.Sh. Mukhtarov, H. Olğar and K. Aydemir. Resolvent operator and spectrum of new type boundary value problems. *Filomat*, **29**(7):1671–1680, 2015. https://doi.org/10.2298/FIL1507671M.

[21] S.B. Norkin. *Differential equations of the second order with retarded argument,* Translations of Mathematical Monographs, volume 31. AMS, Providence, RI, 1972.

[22] S. Pečiulytė, O. Štikonienė and A. Štikonas. Investigation of negative critical points of the characteristic function for problems with nonlocal boundary conditions. *Nonlinear Anal. Model. Control*, **13**(4):467–490, 2008. https://doi.org/10.15388/NA.2008.13.4.14552.

[23] S. Pečiulytė and A. Štikonas. On positive eigenfunctions of Sturm–Liouville problem with nonlocal two-point boundary condition. *Math. Model. Anal.*, **12**(2):215–226, 2007. https://doi.org/10.3846/1392-6292.2007.12.215-226.

[24] M. Pikula, E. Čatrnja and I. Kalčo. Spectral problems for operators with deviating arguments. *Hacettepe J. Math. Stat.*, **47**(5):1172–1183, 2018. https://doi.org/10.15672/HJMS.2017.494.

[25] S. Roman and A. Štikonas. Third-order linear differential equation with three additional conditions and formula for Green's function. *Lith. Math. J.*, **50**(4):426–446, 2010. https://doi.org/10.1007/s10986-010-9097-x.

[26] S. Roman and A. Štikonas. Green's function for discrete second-order problems with nonlocal boundary conditions. *Bound. Value Probl.*, **2011**(Article ID 767024):1–23, 2011. https://doi.org/10.1155/2011/767024.

[27] E. Şen and A. Bayramov. Calculation of eigenvalues and eigenfunctions of a discontinuous boundary value problem with retarded argument which contains a spectral parameter in the boundary condition. *Math. Comput. Modelling*, **54**(11–12):3090–3097, 2011. https://doi.org/10.1016/j.mcm.2011.07.039.

[28] E. Şen and A. Štikonas. Asymptotic distribution of eigenvalues and eigenfunctions of a nonlocal boundary value problem. *Math. Model. Anal.*, **26**(2):253–266, 2021. https://doi.org/10.3846/mma.2021.13056.

[29] E. Şen and A. Štikonas. Computation of eigenvalues and eigenfunctions of a nonlocal boundary value problem with retarded argument. *Complex Var. Elliptic Equ.*, **67**(7):1662–1676, 2022. https://doi.org/10.1080/17476933.2021.1890054.

[30] A. Skučaitė and A. Štikonas. Spectrum curves for Sturm–Liouville Problem with Integral Boundary Condition. *Math. Model. Anal.*, **20**(6):802–818, 2015. https://doi.org/10.3846/13926292.2015.1116470.

[31] A. Štikonas. The Sturm–Liouville problem with a nonlocal boundary condition. *Lith. Math. J*,**47**(3):336–351,2007. https://doi.org/10.1007/s10986-007-0023-9.

[32] A. Štikonas. Investigation of characteristic curve for Sturm–Liouville problem with nonlocal boundary conditions on torus. *Math. Model. Anal.*, **16**(1):1–22, 2011. https://doi.org/10.3846/13926292.2011.552260.

[33] A. Štikonas. A survey on stationary problems, Green's functions and spectrum of Sturm–Liouville problem with nonlocal boundary conditions. *Nonlinear Anal. Model. Control*, **19**(3):301–334, 2014. https://doi.org/10.15388/NA.2014.3.1.

[34] A. Štikonas and E. Şen. Asymptotic analysis of Sturm–Liouville problem with nonlocal integral-type boundary condition. *Nonlinear Anal. Model. Control*, **26**(5):969–991, 2021. https://doi.org/10.15388/namc.2021.26.24299.

[35] A. Štikonas and E. Şen. Asymptotic analysis of Sturm–Liouville problem with Neumann and nonlocal two-point boundary conditions. *Lith. Math. J*, **62**(4):519–541, 2022. https://doi.org/10.1007/s10986-022-09577-6.

[36] A. Štikonas and O. Štikonienė. Characteristic functions for Sturm–Liouville problems with nonlocal boundary conditions. *Math. Model. Anal.*, **14**(2):229–246, 2009. https://doi.org/10.3846/1392-6292.2009.14.229-246.

[37] F. Sun, K. Li and J. Cai. Bounds on the non-real eigenvalues of nonlocal indefinite Sturm–Liouville problems with coupled boundary conditions. *Complex Anal. Oper. Theory*, **16**(30):1–12, 2022. https://doi.org/10.1007/s11785-022-01202-1.

[38] E.C. Titchmarsh. *Eigenfunction expansions associated with second-order differential equations.* Clarendon Press, Oxford, 1946.