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Study on Temporal-Fuzzy Fractional p-KdV Equation with Non-Singular Mittag Leffler Kernel

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Abstract. This work discusses the solution of temporal-fuzzy fractional non-linear p-KdV equations employing a singular kernel and a non-singular Mittag Leffler kernel. A novel q-homotopy analysis approach with a generalised transform is proposed to study the fuzzy time-fractional model with two distinct fractional operators, and the behaviour of the solution is studied in both crisp and uncertain cases. Consequently, the efficiency and accuracy of the proposed method have been obtained by comparing the obtained numerical results with the available results under the assumption of crisp case for $\alpha = 1$ that validate the obtained results. Finally, the efficiency of the proposed fractional orders is checked with distinct fractional operators.

Keywords: Atangana-Baleanu operator, Liouville-Caputo operator, fuzzy double parametric approach, fuzzy fractional differential equation, q-homotopy analysis Shehu transform method.

AMS Subject Classification: 35R11;35R13.

1 Introduction

Fractional calculus was introduced in the advancement of classical calculus by extending approach of order from integer to non-integer for differentiation and integration operators. The concept of generalizing fractional operators is introduced simultaneously on behalf of classical ones. To have high accuracy in real-world problems, fractional calculus is stealing eye of almost all researchers. Now, when it comes to fractional derivative operators, there are already available derivative operators with singular kernel [12,19,20,21] such as Riemann-Liouville(RL), Liouville-Caputo(LC), etc. Everytime new definition of fractional operators and integrals is given, which is just the modification to existing LC operator in the direction to add some extra properties which are not available earlier, and this is needed for development of theory of fractional calculus. When we consider list of modification to already existing fractional derivative operator LC, first one added Caputo Fabrizio(CF) operator [9], then Atangana-Baleanu derivative [6] in both senses one in Riemann(ABR) and one in Caputo sense(ABC), and so on. CF, given by M. Caputo and M. Fabrizio was the first fractional derivative with non-singular kernel. Atangana-Baleanu fractional derivative was proposed with Mittag-Leffler function [32] as the non-local and non-singular kernel. LC is known as non-local operator and it is assumed by many scientists that LC doesn't follow chain rule, in spite of this ABC operator satisfies chain-rule.

Solving a fractional order partial differential equation (FPDE) involves a lot of computations; when we come to non-linear ones, it gets more challenging to crack them and get their solutions. We have many numerical methods which provide numerical solutions for non-linear FPDE. Also by using analytic methods, we solve non-linear FPDE. Numerical and analytical approaches differ mainly because the analytical technique gives a continuous graph to the solution while the numerical method provides the solution at discrete points. Non-linear FPDEs are challenging to solve, and hence they need better computational software and systems.

Liao [15, 16] proposed a new analytic technique, namely Homotopy analytical technique, in 1992. This technique has been used widely over various non-linear problems in physical science and engineering [13, 18, 33] after Liao published a book [14] in 2004. When we approach towards fractional order PDEs it becomes really tough to solve them with different type of fractional derivatives. Advancement to HAM is provided by firstly reducing a fractional order PDE with the help of some transformation and after applying some method thus becomes more convenient for researchers to solve problems. We have many transformations like Laplace, $\mathcal{J}$, Sumudu, etc. and with the help of HAM we get the advanced methods called as Laplace Transform HAM, Sumudu Transform HAM and so on. And with the help of Shehu transform and q- homotopy method we have q- Homotopy Analysis Shehu Transform Method (q-HAShTM) [25,26,27]. Here we have used two convergence parameters n and h for the series solution to control the convergence of series.

Fuzzy set was introduced in 1965 [34], and later on, almost every branch of science started to consider uncertainty in their problems which was not there earlier. Fuzzy function's differentiation was explained by Puri and Ralescu [22]. Later on, many researchers in various domains used fuzzy functions and parameters [4, 5, 24] to solve their problems. Recently overview over computational and fuzzy mathematics [10] is provided by Chakraverty and Perera. In this real-world nothing is certain, everywhere uncertainty is there. To overcome this uncertainty we started to study fuzzy theory over mathematical problems.

Considering real-world problems, we know that PDEs are only sometimes the best fit. Several investigators in fuzzy environment have broadened the concept of derivatives to include fuzzy fractional differential equations(FDEs) as a base of modelling of complex system which allowed to write differential equations in a fuzzy framework [23]. Currently in mathematical modelling there are complex systems with insufficient data and fuzzy PDEs have the capacity to explain such systems, this elevated fuzzy PDEs importance in attracting young researchers towards this field [2]. Fuzzy Set Theory [35, 36] is used in various mathematical fields such as fixed-point theory, topology, bifurcation, fractional calculus, integral inequality, operations research, image processing, control theory, artificial intelligence, and consumer electronics. Third-order fuzzy dispersive PDEs using the Liouville Caputo, Caputo-Fabrizio, and Atangana-Baleanu fractional operator frameworks was explored by Ahmad et al. [1]. To solve such fuzzy FDEs, many analytical and numerical approaches have been adopted but homotopy analysis method (HAM) is providing superior results. For abating the complexity while reproducing the solutions of fuzzy fractional problems, different type of transforms are used with HAM. For solving fuzzy fractional differential equations, double parametric approach in q-HAShTM have been used in [29, 30], which was introduced by Verma and Meher [28]. The main advantage of q-HAShTM is not just reducing the problem's complexity, but it gives a free hand to choose the values of two controlling parameters, namely, h and n for faster convergence of the solutions. The q-HAShTM technique is simpler to implement and saves time, making it more efficient than prior methods.

Korteweg-de Vries (KdV) equation has diverse application in various domains which includes acoustic waves in crystal lattices, weakly non-linear restoring forces in shallow water waves and many more. KdV equation plays vital role [11] in ocean and blood pressure pulses while exemplifying internal gravity waves. Here we considered special case of KdV equation as p-KdV. While studying the long wave arise in shallow water this equation plays an important role. This research paper contains Potential-KdV [31] as follows:

$$\vartheta_t(\zeta,t) + \alpha(\vartheta_\zeta(\zeta,t))^2 + \beta\vartheta_{\zeta\zeta\zeta}(\zeta,t) = 0, \quad 0 \le t, \tag{1.1}$$

where α, & β are real constants, ζ denotes spatial variables, t denotes temporal variables & $\vartheta(\zeta,t)$ denotes wave profile. First term in Equation (1.1) is evolution term, second term $\vartheta_\zeta(\zeta,t)$ and third term $\vartheta_{\zeta\zeta\zeta}(\zeta,t)$ symbolizes non-linearity and 3rd order dispersion term, respectively.

The central theme of this work is to discuss the solution of temporal-fuzzy fractional non-linear p-KdV equations by employing singular and non-singular kernels having fractional operators, i.e., Liouville-Caputo (LC) and Atangana-Baleanu in Caputo sense (ABC). A novel q-homotopy analysis approach with a generalised transform is proposed here to solve the fuzzy time-fractional model in both crisp and uncertain cases and study the proposed model's solution with distinct fractional operators. Finally, the results are validated for the crisp case to check the efficiency of the proposed fractional operators.

2 Preliminaries

DEFINITION 1. [21] Let ν in $H^1[a,b]$, $0<\mu<1$, $t>0$, the Liouville-Caputo (LC) fractional derivative of order μ is

$$^{LC}D_t^{\mu}\nu(t)=\frac{1}{\Gamma(1-\mu)}\int_0^t\frac{1}{(t-\tau)^{\mu}}\nu'(\tau)\,d\tau.$$

DEFINITION 2. [6] Let ν in $H^1[a,b]$, $0<\mu<1$, $t>0$, the Atangana-Baleanu fractional derivative in Caputo sense(ABC) of order μ is

$$^{ABC}D_t^{\mu}\nu(t)=\frac{B(\mu)}{1-\mu}\int_0^t E_{\mu}\Big(\frac{-\mu(t-\tau)}{1-\mu}\Big)\nu'(\tau)\,d\tau,$$

where E_{μ} is a Mittag-Leffler function [9] and $B(\mu)$ is a normalization function with $B(0)=1$ and $B(1)=1$.

DEFINITION 3. The Shehu transform [17] of the function $\nu(t)\in A$ with

$$A=\big\{\nu(t)\ :\ \exists\ K,\kappa_1,\kappa_2>0,|\nu(t)|<K\exp\big(|t|/\kappa_i\big),\ \text{if}\ t\in(-1)^i\times[0,\infty)\big\},$$

is given by

$$V(s,u)=\mathcal{S}[\nu(t)]=\int_0^{\infty}\exp\big(-st/u\big)\nu(t)\,dt,$$

where s, and u are positive numbers.

Some of results of Shehu transform are as follows:
(a) $\mathcal{S}(b)=bu/s$, where b is constant.
(b) $\mathcal{S}(t)=u/s$.
(c) $\mathcal{S}(t^b)=\Gamma(b+1)(u/s)^{b+1}$, $b>-1$.
(d) It holds linearity: $\mathcal{S}[b\nu_1(t)\pm d\nu_2(t)]=b\mathcal{S}[\nu_1(t)]\pm d\mathcal{S}[\nu_2(t)]$.

DEFINITION 4. If $V(s,u)=\mathcal{S}[\nu(t)]$, then we have its inverse Shehu transform [17] as:

$$\nu(t)=\mathcal{S}^{-1}[V(s,u)]=\lim_{\Delta\to\infty}\int_{\gamma-i\Delta}^{\gamma+i\Delta}\frac{1}{u}\exp\Big(\frac{st}{u}\Big)V(s,u)\,ds.$$

DEFINITION 5. Shehu transform of Liouville-Caputo fractional derivative of order μ $(0<\mu\le 1)$ is given by [7]:

$$\mathcal{S}(^{LC}D_t^{\mu}\nu(t))=\big(s/u\big)^{\mu}\mathcal{S}[\nu(t)]-\big(s/u\big)^{\mu-1}\nu(0).$$

DEFINITION 6. Shehu transform of Atangana-Baleanu Caputo fractional derivative of order μ $(0<\mu\le 1)$ is given by [8]:

$$\mathcal{S}(^{ABC}D_t^{\mu}\nu(t))=\frac{B(\mu)}{1-\mu+\mu\big(u/s\big)^{\mu}}\Big(\mathcal{S}[\nu(t)]-\frac{u}{s}\nu(0)\Big).$$

DEFINITION 7. For any $f\in U$ universal set and considering membership value of f as $M(f)$ then a fuzzy set $\tilde{F}$ will be collection of ordered pairs $(f,M(f))$, and it can be written as [10,22,34]:

$$\tilde{F}=\{(f,M(f)):f\in U,M(f)\in(0,1)\}.$$

Definition 8. A normalized fuzzy set is defined on a real line which is convex is called fuzzy number [10, 22, 34] if it have piecewise continuous membership function that have atleast at one point its value as 1.

Definition 9. For any triangular fuzzy number(TFN) $\tilde{F} = [f_l, f_c, f_u]$, membership value of f is defined as [10, 22, 34]:

$$M(f) = \begin{cases} 0, & f \le f_l, \\ (f - f_l)/(f_c - f_l), & f_l \le f \le f_c, \\ (f_u - f)/(f_u - f_c), & f_c \le f \le f_u, \\ 0, & f > f_u \ , \end{cases}$$

where f_l, f_c and f_u are lower, center and upper fuzzy value, respectively.

Definition 10. Δ-cut is a crisp set which can be defined as $\tilde{F}_\Delta = \{f \in U, M(f) \ge \Delta\}$, and we can write it in another form as [10, 22, 34]:

$$\tilde{F} = [f_l, f_c, f_u] = [(f_c - f_l)\Delta + f_l, f_u - (f_u - f_c)\Delta], \quad \text{where } \Delta \in (0, 1).$$

Definition 11. We can write an interval $K = (\underline{K}, \bar{K})$ in double parametric form as [10]: $K = \tau(\bar{K} - \underline{K}) + \underline{K}$, where $\tau \in (0, 1)$.

3 General discussion of *q*-HAShTM with singular and Mittag Leffler kernel

To obtain better understanding of q-HAShTM, let us consider following non-linear time-fractional partial differential equation

$$^{\varpi}D_t^{\mu}\tilde{\vartheta}(\zeta, t) + \mathcal{L}(\tilde{\vartheta}(\zeta, t)) + \mathcal{N}(\tilde{\vartheta}(\zeta, t)) = \tilde{\mathcal{R}}(\zeta, t), \quad 0 \le t, \quad 0 < \mu \le 1, \tag{3.1}$$

where $^{\varpi}D_t^{\mu}\tilde{\vartheta}(\zeta, t)$ represents fractional derivative operator in ϖ sense with ϖ = LC and ABC fractional derivative operators, $\tilde{\mathcal{R}}(\zeta, t)$ is the source term and $\mathcal{L}(\tilde{\vartheta}(\zeta, t))$, $\mathcal{N}(\tilde{\vartheta}(\zeta, t))$ are linear and non-linear differential operators, respectively. Using Δ-cut technique to write above Equation (3.1) in interval form, we have

$$\begin{aligned}&[^{\varpi}D_t^{\mu}\bar{\vartheta}(\zeta, t, \Delta), {}^{\varpi}D_t^{\mu}\underline{\vartheta}(\zeta, t, \Delta)] + [\mathcal{L}(\bar{\vartheta}(\zeta, t, \Delta)), \mathcal{L}(\underline{\vartheta}(\zeta, t, \Delta))] \\ &\qquad + [\mathcal{N}(\bar{\vartheta}(\zeta, t, \Delta)), \mathcal{N}(\underline{\vartheta}(\zeta, t, \Delta))] = [\bar{\mathcal{R}}(\zeta, t, \Delta), \underline{\mathcal{R}}(\zeta, t, \Delta)].\end{aligned}$$

On writing above equation in another parametric form τ, we have

$$\begin{aligned}&[\tau\{^{\varpi}D_t^{\mu}\bar{\vartheta}(\zeta, t, \Delta) - {}^{\varpi}D_t^{\mu}\underline{\vartheta}(\zeta, t, \Delta)\} + {}^{\varpi}D_t^{\mu}\underline{\vartheta}(\zeta, t, \Delta)] + [\tau\{\mathcal{L}(\bar{\vartheta}(\zeta, t, \Delta)) \\ &\quad - \mathcal{L}(\underline{\vartheta}(\zeta, t, \Delta))\} + \mathcal{L}(\underline{\vartheta}(\zeta, t, \Delta))] + [\tau\{\mathcal{N}(\bar{\vartheta}(\zeta, t, \Delta)) - \mathcal{N}(\underline{\vartheta}(\zeta, t, \Delta))\} \\ &\quad + \mathcal{N}(\underline{\vartheta}(\zeta, t, \Delta))] = [\tau\{\bar{\mathcal{R}}(\zeta, t, \Delta) - \underline{\mathcal{R}}(\zeta, t, \Delta)\} + \underline{\mathcal{R}}(\zeta, t, \Delta)].\end{aligned} \tag{3.2}$$

Here, $\varDelta$ and τ are parameters with $\varDelta,\tau \in [0,1]$. As above equation representing the fuzzy PDEs double parametric form. Therefore, we can write

$$\tau\{{}^{\varpi}D_t^{\mu}\bar{\vartheta}(\zeta,t,\varDelta) - {}^{\varpi}D_t^{\mu}\underline{\vartheta}(\zeta,t,\varDelta)\} + {}^{\varpi}D_t^{\mu}\underline{\vartheta}(\zeta,t,\varDelta)] = {}^{\varpi}D_t^{\mu}\tilde{\vartheta}(\zeta,t,\varDelta,\tau), \tag{3.3}$$

$$\tau\{\mathcal{L}(\bar{\vartheta}(\zeta,t,\varDelta)) - \mathcal{L}(\underline{\vartheta}(\zeta,t,\varDelta))\} + \mathcal{L}(\underline{\vartheta}(\zeta,t,\varDelta) = \mathcal{L}(\tilde{\vartheta}(\zeta,t,\varDelta,\tau)), \tag{3.4}$$

$$\tau\{\mathcal{N}(\bar{\vartheta}(\zeta,t,\varDelta)) - \mathcal{N}(\underline{\vartheta}(\zeta,t,\varDelta))\} + \mathcal{N}(\underline{\vartheta}(\zeta,t,\varDelta)) = \mathcal{N}(\tilde{\vartheta}(\zeta,t,\varDelta,\tau)), \tag{3.5}$$

$$\tau\{\bar{\mathcal{R}}(\zeta,t,\varDelta) - \underline{\mathcal{R}}(\zeta,t,\varDelta)\} + \underline{\mathcal{R}}(\zeta,t,\varDelta) = \tilde{\mathcal{R}}(\zeta,t,\varDelta,\tau), \tag{3.6}$$

$$\tau\{\bar{\vartheta}(\zeta,t,\varDelta) - \underline{\vartheta}(\zeta,t,\varDelta)\} + \underline{\vartheta}(\zeta,t,\varDelta) = \tilde{\vartheta}(\zeta,t,\varDelta,\tau). \tag{3.7}$$

Hence we can write Equation (3.1) in double parametric form as

$$\begin{aligned}&{}^{\varpi}D_t^{\mu}\tilde{\vartheta}(\zeta,t,\varDelta,\tau) + \mathcal{L}(\tilde{\vartheta}(\zeta,t,\varDelta,\tau)) + \mathcal{N}(\tilde{\vartheta}(\zeta,t,\varDelta,\tau)) = \tilde{\mathcal{R}}(\zeta,t,\varDelta,\tau), 0 \le t, 0 < \mu \le 1,\\ &\tilde{\vartheta}(\zeta,0,\varDelta,\tau) = \tilde{\vartheta}_0(\zeta,\varDelta,\tau),\end{aligned} \tag{3.8}$$

where $\varpi =$ LC and ABC fractional derivative operators. The solution of the considered equation with help of q-HAShTM is explained as follows:

Firstly, taking Shehu Transform on both sides of Equation (3.8) and using initial conditions, we get

$$\begin{aligned}&\mathcal{S}[\tilde{\vartheta}(\zeta,t,\varDelta,\tau)] - \frac{u}{s}\tilde{\vartheta}(\zeta,0,\varDelta,\tau)\\ &\quad + \phi(.)\mathcal{S}\Big[\mathcal{L}(\tilde{\vartheta}(\zeta,t,\varDelta,\tau)) + \mathcal{N}(\tilde{\vartheta}(\zeta,t,\varDelta,\tau)) - \tilde{\mathcal{R}}(\zeta,t,\varDelta,\tau)\Big] = 0,\end{aligned} \tag{3.9}$$

for $\varpi = LC$, using Definition 5, we get $\phi(.) = (\frac{u}{s})^{\mu}$ and for $\varpi = ABC$, using Definition 6, we get $\phi(.) = (1-\mu+\mu(\frac{u}{s})^{\mu})/B(\mu)$.

The non-linear operator will be defined as

$$\begin{aligned}N[\tilde{\rho}(\zeta,t,\varDelta,\tau;q)] &= \mathcal{S}[\tilde{\rho}(\zeta,t,\varDelta,\tau;q)] - \frac{u}{s}\tilde{\rho}(\zeta,0,\varDelta,\tau;q) + \phi(.)\mathcal{S}\big[\mathcal{L}(\tilde{\rho}(\zeta,t,\varDelta,\tau;q))\\ &\quad + \mathcal{N}(\tilde{\rho}(\zeta,t,\varDelta,\tau;q)) - \tilde{\mathcal{R}}(\zeta,t,\varDelta,\tau)\big],\end{aligned}$$

where $\tilde{\rho}(\zeta,t,\varDelta,\tau;q)$ is a fuzzy-valued function of $\zeta,t,\varDelta,\tau,q$ with $q \in [0,\frac{1}{n}](n \ge 1)$ an embedding parameter.

Now, we use q-HAShTM, to construct homotopy as

$$(1-nq)\mathcal{S}[\tilde{\rho}(\zeta,t,\varDelta,\tau;q) - \tilde{\vartheta}_0(\zeta,t,\varDelta,\tau)] = hq\tilde{H}(\zeta,t,\varDelta,\tau)N(\tilde{\rho}(\zeta,t,\varDelta,\tau;q)),$$

where $\mathcal{S}$ is Shehu transform, $\tilde{H}(\zeta,t,\varDelta,\tau)$ is non-zero auxiliary function, $\tilde{\rho}(\zeta,t,\varDelta,\tau;q)$ is unknown function, $h(\neq 0)$ is auxiliary parameter, and this equation is called as 0^{th}- order deformation equation. Here we see that, for $q = 0$, we have $\tilde{\rho}(\zeta,t,\varDelta,\tau;0) = \tilde{\vartheta}_0(\zeta,t,\varDelta,\tau)$ and for $q = \frac{1}{n}$, we have $\tilde{\rho}(\zeta,t,\varDelta,\tau;\frac{1}{n}) = \tilde{\vartheta}(\zeta,t,\varDelta,\tau)$, i.e., as q varies from 0 to $\frac{1}{n}$, the solution $\tilde{\rho}(\zeta,t,\varDelta,\tau;q)$ from above equation varies from initial guess $(\tilde{\vartheta}_0(\zeta,t,\varDelta,\tau))$ to exact solution$(\tilde{\vartheta}(\zeta,t,\varDelta,\tau))$.

Now, with the help of Taylor's series expanding $\tilde{\rho}(\zeta,t,\varDelta,\tau;q)$ with respect to q, we have

$$\tilde{\rho}(\zeta,t,\varDelta,\tau;q) = \tilde{\vartheta}_0(\zeta,t,\varDelta,\tau) + \sum_{m=1}^{\infty}\tilde{\vartheta}_m(\zeta,t,\varDelta,\tau)q^m,$$

where $\tilde{\vartheta}_m(\zeta, t, \varDelta, \tau)$ stands for

$$\tilde{\vartheta}_m(\zeta, t, \varDelta, \tau) = \frac{1}{m!}\frac{\partial^m \tilde{\rho}(\zeta, t, \varDelta, \tau; q)}{\partial q^m}\bigg|_{q=0}.$$

At $q = 1/n$, the choice of initial assumption $\tilde{\vartheta}_0(\zeta, t, \varDelta, \tau)$, the auxiliary parameter h and n is proper then above series converges, and we obtain

$$\tilde{\vartheta}(\zeta, t, \varDelta, \tau) = \tilde{\vartheta}_0(\zeta, t, \varDelta, \tau) + \sum_{m=1}^{\infty}\tilde{\vartheta}_m(\zeta, t, \varDelta, \tau)\Big(\frac{1}{n}\Big)^m. \tag{3.10}$$

Equation (3.10) is original non-linear equation's one of the solution. As shown in Equation (3.10), the controlling equation can be obtained from 0^{th}- order deformation equation. Define vector,

$$\vec{\tilde{\vartheta}}_m(\zeta, t, \varDelta, \tau) = \{\tilde{\vartheta}_0(\zeta, t, \varDelta, \tau), \tilde{\vartheta}_1(\zeta, t, \varDelta, \tau), \tilde{\vartheta}_2(\zeta, t, \varDelta, \tau), ..., \tilde{\vartheta}_m(\zeta, t, \varDelta, \tau)\}.$$

Taking m-times differentiation of zero-order deformation equation concerning embedding parameter q, putting $q = 0$ and after dividing with $m!$, the deformation equation of m^{th}-order can be presented as follows

$$\mathcal{S}[\tilde{\vartheta}_m(\zeta, t, \varDelta, \tau) - \psi_m\tilde{\vartheta}_{m-1}(\zeta, t, \varDelta, \tau)] = h\tilde{H}(\zeta, t, \varDelta, \tau)R_m(\vec{\tilde{\vartheta}}_{m-1}(\zeta, t, \varDelta, \tau)), \tag{3.11}$$

where $R_m(\vec{\tilde{\vartheta}}_{m-1}(\zeta, t, \varDelta, \tau))$ stands for

$$\begin{aligned} R_m(\vec{\tilde{\vartheta}}_{m-1}(\zeta, t, \varDelta, \tau)) = &\frac{1}{(m-1)!}\frac{\partial^{m-1} N[\tilde{\rho}(\zeta, t, \varDelta, \tau; q)]}{\partial q^{m-1}}\bigg|_{q=0} \\ &= \mathcal{S}[\tilde{\vartheta}_{m-1}(\zeta, t, \varDelta, \tau)] - \frac{u}{s}\Big(1 - \frac{\psi_m}{n}\Big)\tilde{\vartheta}(\zeta, 0, \varDelta, \tau) \\ &+ \phi(.)\mathcal{S}[\mathcal{L}(\tilde{\vartheta}_{m-1}(\zeta, t, \varDelta, \tau) + \mathcal{N}(\tilde{\vartheta}_{m-1}(\zeta, t, \varDelta, \tau)) - \tilde{\mathcal{R}}(\zeta, t, \varDelta, \tau)], \end{aligned}$$

and

$$\psi_m = \begin{cases} 0, & m \leq 1, \\ n, & m > 1. \end{cases} \tag{3.12}$$

also $\phi(.)$ mentioned below Equation (3.9), is accordingly with the differential operators. Here, it is surely noted that when $m \geq 1$, $\tilde{\vartheta}_m(\zeta, t, \varDelta, \tau)$ is controlled by the linear m^{th}- order deformation equation.

Taking inverse Shehu transform of Equation (3.11), we get

$$\tilde{\vartheta}_m(\zeta, t, \varDelta, \tau) = \psi_m\tilde{\vartheta}_{m-1}(\zeta, t, \varDelta, \tau) + h\mathcal{S}^{-1}[\tilde{H}(\zeta, t, \varDelta, \tau)R_m(\vec{\tilde{\vartheta}}_{m-1}(\zeta, t, \varDelta, \tau))].$$

Now, by choosing suitable values of h and n, we get q-HAShTM series solution as

$$\tilde{\vartheta}(\zeta, t, \varDelta, \tau) = \lim_{N\to\infty}\sum_{m=0}^{N}\tilde{\vartheta}_m(\zeta, t, \varDelta, \tau)\Big(\frac{1}{n}\Big)^m.$$

4 Convergence and absolute error analysis with singular and Mittag Leffler kernel

Theorem 1. ***Theorem on uniqueness of solution (Non singular Mittag Leffler kernel)*** *For a Fuzzy FPDE equation* (3.8)*, when solved using q-HAShTM technique with double parametric approach will have unique solution, wherever* $0 < R < 1$*, where*

$$R = (n+h) + h(\rho+\lambda)\frac{(1-\mu+\mu T)}{B(\mu)}.$$

Proof. The derived solution for fuzzy Atangana-Baleanu Caputo FPDE equation (3.8) is as follows

$$\tilde{\vartheta}(\zeta,t,\Delta,\tau) = \sum_{m=0}^{\infty} \tilde{\vartheta}_m(\zeta,t,\Delta,\tau)\left(\frac{1}{n}\right)^m.$$

Let us assume if the there are two different solutions $\tilde{\vartheta}$ and $\tilde{\vartheta}*$ of Equation (3.8), then using precedent equations, we have

$$\begin{aligned}|\tilde{\vartheta} - \tilde{\vartheta} * | = \big|&(n+h)(\tilde{\vartheta}-\tilde{\vartheta}*) \\ &+ h\mathcal{S}^{-1}\big(\big((1-\mu+\mu(u/s)^{\mu})/B(\mu)\big)\mathcal{S}(\mathcal{L}(\tilde{\vartheta}-\tilde{\vartheta}*)+\mathcal{N}(\tilde{\vartheta}-\tilde{\vartheta}*))\big)\big|.\end{aligned}$$

Our Shehu transform will follow up upon using convolution theorem as

$$\begin{aligned}\left|\tilde{\vartheta}-\tilde{\vartheta}*\right| &\le (n+h)\left|\tilde{\vartheta}-\tilde{\vartheta}*\right| + \frac{(1-\mu)h}{B(\mu)}\left(\left|\mathcal{L}(\tilde{\vartheta}-\tilde{\vartheta}*)\right| + \left|\mathcal{N}(\tilde{\vartheta}-\tilde{\vartheta}*)\right|\right) \\ &\quad + \frac{h\mu}{B(\mu)}\int_0^t \left(\left|\mathcal{L}(\tilde{\vartheta}-\tilde{\vartheta}*)\right| + \left|\mathcal{N}(\tilde{\vartheta}-\tilde{\vartheta}*)\right|\right)\frac{(t-\delta)^{\mu}}{\Gamma(1+\mu)}d\delta \\ &\le (n+h)\left|\tilde{\vartheta}-\tilde{\vartheta}*\right| + \frac{(1-\mu)h}{B(\mu)}\left(\rho\left|\tilde{\vartheta}-\tilde{\vartheta}*\right|| + \lambda\left|\tilde{\vartheta}-\tilde{\vartheta}*\right|\right) \\ &\quad + \frac{h\mu}{B(\mu)}\int_0^t \left(\rho\left|\tilde{\vartheta}-\tilde{\vartheta}*\right| + \lambda\left|\tilde{\vartheta}-\tilde{\vartheta}*\right|\right)\frac{(t-\delta)^{\mu}}{\Gamma(1+\mu)}d\delta\end{aligned}$$

Last term will follow up upon using integral mean value theorem as

$$\begin{aligned}\left|\tilde{\vartheta}-\tilde{\vartheta}*\right| &\le (n+h)\left|\tilde{\vartheta}-\tilde{\vartheta}*\right| + \frac{(1-\mu)h}{B(\mu)}\left(\rho\left|\tilde{\vartheta}-\tilde{\vartheta}*\right| + \lambda\left|\tilde{\vartheta}-\tilde{\vartheta}*\right|\right) \\ &\quad + \frac{h\mu}{B(\mu)}\left(\rho\left|\tilde{\vartheta}-\tilde{\vartheta}*\right| + \lambda\left|\tilde{\vartheta}-\tilde{\vartheta}*\right|\right)T \le \left|\tilde{\vartheta}-\tilde{\vartheta}*\right|R.\end{aligned}$$

It provides $(1-R)|\tilde{\vartheta}-\tilde{\vartheta}*| \le 0$. Because $R \in (0,1)$; therefore, $|\tilde{\vartheta}-\tilde{\vartheta}*| = 0$, which leads to $\tilde{\vartheta} = \tilde{\vartheta}*$. Hence solution is unique. □

Theorem 2. ***Theorem on uniqueness of solution (Singular kernel)*** *For a Fuzzy FPDE equation* (3.8)*, when solved using q-HAShTM technique with double parametric approach will have unique solution, wherever* $0 < r < 1$*, where* $r = (n+h) + h(\rho+\lambda)T$*.*

Proof. Similar proof to Theorem 1. □

Theorem 3. ***Theorem on convergence analysis of the solution (non singular Mittag Leffler kernel)*** *Let us consider Y to be a Banach space with non-linearity mapping $G : Y \to Y$ in view of*

$$\big\|G(\tilde{\vartheta}) - G(\tilde{z})\big\| \leq R\big\|\tilde{\vartheta} - \tilde{z}\big\|.$$

Then G has a fixed point, by fixed point theory concept provided by Banach. In addition, with selection of arbitrarily $\tilde{\vartheta}_0, \tilde{z}_0 \in Y$, q-HAShTM generated sequence will converge to the fixed point of G and

$$\left\|\tilde{\vartheta}_q - \tilde{\vartheta}_p\right\| \leq \frac{R^p}{1-R}\left\|\tilde{\vartheta}_1 - \tilde{\vartheta}_0\right\|, \ \ \forall\, \tilde{\vartheta}, \tilde{z} \in Y.$$

Proof. Defining a Banach space $(C[I], \|\cdot\|)$, over $C[I]$, i.e., set of all of the continuous functions on I with norm condition as $\|g(t)\| = \max_{t\in I}|g(t)|$.
In the Banach space, we will express that the sequence $\{\tilde{\vartheta}_p\}$ is a Cauchy sequence.

$$\begin{aligned}
\big\|\tilde{\vartheta}_q - \tilde{\vartheta}_p\big\| &= \max_{t\in I}\big|\tilde{\vartheta}_q - \tilde{\vartheta}_p\big| = \max_{t\in I}\big|(n+h)(\tilde{\vartheta}_{q-1} - \tilde{\vartheta}_{p-1}) \\
&\quad + h\mathcal{S}^{-1}\big(((1-\mu+\mu(u/s)^{\mu})/B(\mu)\mathcal{S}\big(\mathcal{L}(\tilde{\vartheta}_{q-1} - \tilde{\vartheta}_{p-1}) + \mathcal{N}(\tilde{\vartheta}_{q-1} - \tilde{\vartheta}_{p-1})\big)\big)\big| \\
&= \max_{t\in I}\big[(n+h)\big|\tilde{\vartheta}_{q-1} - \tilde{\vartheta}_{p-1}\big| + h\mathcal{S}^{-1}\big(((1-\mu+\mu(u/s)^{\mu})/B(\mu)) \\
&\quad \times \mathcal{S}\big(\big\|\mathcal{L}(\tilde{\vartheta}_{q-1} - \tilde{\vartheta}_{p-1})\big\| + \big\|\mathcal{N}(\tilde{\vartheta}_{q-1} - \tilde{\vartheta}_{p-1})\big\|\big)\big)\big].
\end{aligned}$$

Our Shehu transform will follow up upon using convolution theorem as

$$\begin{aligned}
\left\|\tilde{\vartheta}_q - \tilde{\vartheta}_p\right\| \leq & \max_{t\in I}\Big[(n+h)\Big|\tilde{\vartheta}_{q-1} - \tilde{\vartheta}_{p-1}\Big| \\
&+ \frac{h(1-\mu)}{B(\mu)}\left(\Big|\mathcal{L}(\tilde{\vartheta}_{q-1} - \tilde{\vartheta}_{p-1})\Big| + \Big|\mathcal{N}(\tilde{\vartheta}_{q-1} - \tilde{\vartheta}_{p-1})\Big|\right) \\
&+ \frac{h\mu}{B(\mu)}\int_0^t\left(\Big|\mathcal{L}(\tilde{\vartheta}_{q-1} - \tilde{\vartheta}_{p-1})\Big| + \Big|\mathcal{N}(\tilde{\vartheta}_{q-1} - \tilde{\vartheta}_{p-1})\Big|\right)\frac{(t-\delta)^{\mu}}{\Gamma(1+\mu)}d\delta\Big] \\
\leq & \max_{t\in I}\Big[(n+h)\Big|\tilde{\vartheta}_{q-1} - \tilde{\vartheta}_{p-1}\Big| \\
&+ \frac{h(1-\mu)}{B(\mu)}\left(\rho\Big|\tilde{\vartheta}_{q-1} - \tilde{\vartheta}_{p-1}\Big| + \lambda\Big|\tilde{\vartheta}_{q-1} - \tilde{\vartheta}_{p-1}\Big|\right) \\
&+ \frac{h\mu}{B(\mu)}\int_0^t\left(\rho\Big|\tilde{\vartheta}_{q-1} - \tilde{\vartheta}_{p-1}\Big| + \lambda\Big|\tilde{\vartheta}_{q-1} - \tilde{\vartheta}_{p-1}\Big|\right)\frac{(t-\delta)^{\mu}}{\Gamma(1+\mu)}d\delta\Big].
\end{aligned}$$

Last term will follow up upon using integral mean value theorem as

$$\begin{aligned}
\left\|\tilde{\vartheta}_q - \tilde{\vartheta}_p\right\| \leq & \max_{t\in I}\Big[(n+h)\Big|\tilde{\vartheta}_{q-1} - \tilde{\vartheta}_{p-1}\Big| \\
&+ \frac{h(1-\mu)}{B(\mu)}\left(\rho\Big|\tilde{\vartheta}_{q-1} - \tilde{\vartheta}_{p-1}\Big| + \lambda\Big|\tilde{\vartheta}_{q-1} - \tilde{\vartheta}_{p-1}\Big|\right) \\
&+ \frac{h\mu}{B(\mu)}\left(\rho\Big|\tilde{\vartheta}_{q-1} - \tilde{\vartheta}_{p-1}\Big| + \lambda\Big|\tilde{\vartheta}_{q-1} - \tilde{\vartheta}_{p-1}\Big|\right)T\Big] \leq R\left\|\tilde{\vartheta}_{q-1} - \tilde{\vartheta}_{p-1}\right\|.
\end{aligned}$$

Let $q = p + 1$, then we have

$$\left\|\tilde{\vartheta}_{p+1} - \tilde{\vartheta}_p\right\| \leq R\left\|\tilde{\vartheta}_p - \tilde{\vartheta}_{p-1}\right\| \leq R^2\left\|\tilde{\vartheta}_{p-1} - \tilde{\vartheta}_{p-2}\right\| \leq \cdots \leq R^p\left\|\tilde{\vartheta}_1 - \tilde{\vartheta}_0\right\|.$$

Now, the triangular inequality leads to

$$\begin{aligned}
\left\|\tilde{\vartheta}_q - \tilde{\vartheta}_p\right\| &\leq \left\|\tilde{\vartheta}_{p+1} - \tilde{\vartheta}_p\right\| + \left\|\tilde{\vartheta}_{p+2} - \tilde{\vartheta}_{p+1}\right\| + \cdots + \left\|\tilde{\vartheta}_q - \tilde{\vartheta}_{q-1}\right\| \\
&\leq \left[R^p + R^{p+1} + \cdots + R^{q-1}\right]\left\|\tilde{\vartheta}_1 - \tilde{\vartheta}_0\right\| \\
&\leq R^p\left[1 + R + R^2 + \cdots + R^{(q-1-p)}\right]\left\|\tilde{\vartheta}_1 - \tilde{\vartheta}_0\right\| \\
&\leq R^p\left[\left(1 - R^{(q-1-p)}\right)/(1-R)\right]\left\|\tilde{\vartheta}_1 - \tilde{\vartheta}_0\right\|.
\end{aligned}$$

Since $0 < R < 1$, which means $1 - R^{(q-1-p)} < 1$, it implies

$$\left\|\tilde{\vartheta}_q - \tilde{\vartheta}_p\right\| \leq \frac{R^p}{1-R}\left\|\tilde{\vartheta}_1 - \tilde{\vartheta}_0\right\|.$$

But $\left\|\tilde{\vartheta}_1 - \tilde{\vartheta}_0\right\| < \infty$, on increasing value of q and taking $q \to \infty$, we get $\left\|\tilde{\vartheta}_q - \tilde{\vartheta}_p\right\| \to 0$. Therefore, it can be said that the sequence $\{\tilde{\vartheta}_p\}$ is Cauchy in $C[I]$ and hence, the sequence is convergent. □

Theorem 4. ***Theorem on convergence analysis of the solution (singular kernel).*** *Let us consider Y to be a Banach space with non-linearity mapping $G : Y \to Y$ in view of*

$$\left\|G(\tilde{\vartheta}) - G(\tilde{z})\right\| \leq r\left\|\tilde{\vartheta} - \tilde{z}\right\|.$$

Then G has a fixed point, by fixed point theory concept provided by Banach. In addition, with selection of arbitrarily $\tilde{\vartheta}_0, \tilde{z}_0 \in Y$, q-HAShTM generated sequence will converge to the fixed point of G and

$$\left\|\tilde{\vartheta}_q - \tilde{\vartheta}_p\right\| \leq \frac{r^p}{1-r}\left\|\tilde{\vartheta}_1 - \tilde{\vartheta}_0\right\|, \ \forall\ \tilde{\vartheta}, \tilde{z} \in Y.$$

Proof. Similar proof to Theorem 3. □

Theorem 5. ***Absolute error analysis.*** *Let the approximate solution of $\tilde{\vartheta}(\zeta, t, \Delta, \tau)$ to be $\sum_{p=0}^{q}\tilde{\vartheta}_p(\zeta, t, \Delta, \tau)\left(\frac{1}{n}\right)^p$. Let us assume, there exists a real number $\mathbb{J} \in (0, 1)$ and $\mathbb{J} = \frac{s}{n}$, where $s \in (0, 1)$ such that $\left\|\tilde{\vartheta}_{p+1}(\zeta, t, \Delta, \tau)\right\| \leq s\left\|\tilde{\vartheta}_p(\zeta, t, \Delta, \tau)\right\|$, for all p. Then, the maximum absolute error is given by*

$$\left\|\tilde{\vartheta}(\zeta, t, \Delta, \tau) - \sum_{p=0}^{q}\tilde{\vartheta}_p(\zeta, t, \Delta, \tau)\left(\frac{1}{n}\right)^p\right\| \leq \frac{\mathbb{J}^{q+1}}{1-\mathbb{J}}\left\|\tilde{\vartheta}_0(\zeta, t, \Delta, \tau)\right\|,$$

where $s = r$ and R for LC and ABC approach, respectively.

Proof. In q-HAShTM series solution, we cannot obtain successive terms in infinite number, so we can write

$$\begin{aligned}
&\left\|\tilde{\vartheta}(\zeta,t,\Delta,\tau)-\sum_{p=0}^{q}\tilde{\vartheta}_p(\zeta,t,\Delta,\tau)\left(\frac{1}{n}\right)^p\right\| = \left\|\sum_{p=q+1}^{\infty}\tilde{\vartheta}_p(\zeta,t,\Delta,\tau)\left(\frac{1}{n}\right)^p\right\| \\
&\leq \sum_{p=q+1}^{\infty}\left\|\tilde{\vartheta}_p(\zeta,t,\Delta,\tau)\right\|\left(\frac{1}{n}\right)^p \leq \sum_{p=q+1}^{\infty} s^p\left\|\tilde{\vartheta}_0(\zeta,t,\Delta,\tau)\right\|\frac{1}{n^p} \\
&\leq \sum_{p=q+1}^{\infty}\mathbb{J}^p\left\|\tilde{\vartheta}_0(\zeta,t,\Delta,\tau)\right\| \leq \mathbb{J}^{q+1}(1+\mathbb{J}+\mathbb{J}^2+\cdots)\left\|\tilde{\vartheta}_0(\zeta,t,\Delta,\tau)\right\| \\
&\leq \frac{\mathbb{J}^{q+1}}{1-\mathbb{J}}\left\|\tilde{\vartheta}_0(\zeta,t,\Delta,\tau)\right\|.
\end{aligned}$$

Hence proved. □

5 Solution of fuzzy-fractional *p*-KdV equation

Consider the following fuzzy non-linear time fractional p-KdV equation in view of Equation (1.1), based on Liouville-Caputo time fractional and Atangana-Baleanu Caputo time fractional derivative operator of order μ, with ϖ as fractional derivative operator (i.e., ϖ = LC and ABC)

$$^{\varpi}D_t^{\mu}\tilde{\vartheta}(\zeta,t)+\alpha(\tilde{\vartheta}_{\zeta}(\zeta,t))^2+\beta\tilde{\vartheta}_{\zeta\zeta\zeta}(\zeta,t)=0,\quad 0\leq t,\quad 0<\mu\leq 1, \tag{5.1}$$

subject to the fuzzy initial condition as

$$\tilde{\vartheta}(\zeta,0)=\tilde{B}(\omega\tanh(\lambda\zeta)), \tag{5.2}$$

where $\lambda=\frac{\sqrt{\nu}}{2\sqrt{\beta}}$, $\omega=\frac{6\beta\lambda}{\alpha}$ and $\tilde{B}$=[0.8,1,1.2] is a triangular fuzzy number(TFN). TFN $\tilde{B}$ can also be expressed in Δ - cut form as $[\underline{B},\bar{B}]=[0.8+(0.2)\Delta,\ 1.2-(0.2)\Delta]$.

In view of Equations (3.2)–(3.7), above Equations (5.1) and (5.2) can be written in double parametric (DP) form as follows:

$$^{\varpi}D_t^{\mu}\tilde{\vartheta}(\zeta,t,\Delta,\tau)+\alpha(\tilde{\vartheta}_{\zeta}(\zeta,t,\Delta,\tau))^2+\beta\tilde{\vartheta}_{\zeta\zeta\zeta}(\zeta,t,\Delta,\tau)=0,\quad 0\leq t,\quad 0<\mu\leq 1, \tag{5.3}$$

writing DP form of TFN as

$$\tilde{E}(\Delta,\tau)=\tau\{\bar{B}-\underline{B}\}+\underline{B}=\tau\{0.4(1-\Delta)\}+0.2\Delta+0.8$$

and we can write DP form for fuzzy initial condition as

$$\tilde{\vartheta}(\zeta,0,\Delta,\tau)=(\tilde{E}(\Delta,\tau))(\omega\tanh(\lambda\zeta)).$$

For solving Equation (5.1) using q-HAShTM, let the initial approximation be

$$\tilde{\vartheta}_0(\zeta,t,\Delta,\tau)=\big(\tilde{E}(\Delta,\tau)\big)\omega\tanh(\lambda\zeta). \tag{5.4}$$

Using Shehu transform and Equation (5.4) we transform the Equation (5.3) as follows:

$$\mathcal{S}[\tilde{\vartheta}(\zeta,t,\Delta,\tau)] - \frac{u}{s}\Big(\big(\tilde{E}(\Delta,\tau)\big)\omega\tanh(\lambda\zeta)\Big) + \Big(\phi(.)\Big)\mathcal{S}\Big[\alpha\tilde{\vartheta}_{\zeta}^{2} + \beta\tilde{\vartheta}_{\zeta\zeta\zeta}\Big] = 0,$$

where

$$\phi(.) = \begin{cases} (u/s)^{\mu}, & \varpi = \text{LC}, \\ (1-\mu+\mu(u/s)^{\mu})/B(\mu), & \varpi = \text{ABC}. \end{cases}$$

The non-linear operator can be defined as

$$\begin{aligned} N[\rho(\zeta,t,\Delta,\tau;q)] =& \mathcal{S}[\rho(\zeta,t,\Delta,\tau;q)] - \frac{u}{s}\Big(\big(\tilde{E}(\Delta,\tau)\big)\omega\tanh(\lambda\zeta)\Big) \\ &+ \Big(\phi(.)\Big)\mathcal{S}\Big[\alpha\Big(\frac{\partial\rho(\zeta,t,\Delta,\tau;q)}{\partial\zeta}\Big)^{2} + \beta\frac{\partial^{3}\rho(\zeta,t,\Delta,\tau;q)}{\partial\zeta^{3}}\Big]. \end{aligned}$$

Now, we have m^{th} order deformation equation as per the method described in 3^{rd} section with $\tilde{H}(\zeta,t,\Delta,\tau) = 1$ as follows:

$$\mathcal{S}[\tilde{\vartheta}_{m}(\zeta,t,\Delta,\tau) - \psi_{m}\tilde{\vartheta}_{m-1}(\zeta,t,\Delta,\tau)] = h\mathcal{R}_{m}(\overrightarrow{\vartheta}_{m-1}), \tag{5.5}$$

where

$$\begin{aligned} \mathcal{R}_{m}(\overrightarrow{\vartheta}_{m-1}) =& \mathcal{S}[\tilde{\vartheta}_{m-1}] - \Big(1-\frac{\psi_{m}}{n}\Big)\Big(\frac{u}{s}\Big)\Big(\big(\tilde{E}(\Delta,\tau)\big)\omega\tanh(\lambda\zeta)\Big) \\ &+ \Big(\phi(.)\Big)\mathcal{S}\Big[\alpha\sum_{i=0}^{m-1}\frac{\partial\tilde{\vartheta}_{i}}{\partial\zeta}\frac{\partial\tilde{\vartheta}_{m-i-1}}{\partial\zeta} + \beta\frac{\partial^{3}\tilde{\vartheta}_{m-1}}{\partial\zeta^{3}}\Big], \end{aligned}$$

where ψ_m is given by Equation (3.12).

(A) Using Singular kernel approach, i.e., ϖ = LC

Upon using the initial assumption from Equation (5.4) and inverse Shehu transform formula over Equation (5.5), the first few terms of the solution can be expressed as following

$$\begin{aligned} \tilde{\vartheta}_{1}^{LC}(\zeta,t,\Delta,\tau) =& \frac{ht^{\mu}\Big(4(\cosh(\lambda\zeta))^{2}\beta\lambda + \alpha\omega\big(\tilde{E}(\Delta,\tau)\big) - 6\beta\lambda\Big)\omega\big(\tilde{E}(\Delta,\tau)\big)\lambda^{2}}{\Gamma(\mu+1)(\cosh(\lambda\zeta))^{4}}, \\ \tilde{\vartheta}_{2}^{LC}(\zeta,t,\Delta,\tau) =& \frac{(h+n)ht^{\mu}\Big(4(\cosh(\lambda\zeta))^{2}\beta\lambda + \alpha\omega\big(\tilde{E}(\Delta,\tau)\big) - 6\beta\lambda\Big)\omega\big(\tilde{E}(\Delta,\tau)\big)\lambda^{2}}{\Gamma(\mu+1)(\cosh(\lambda\zeta))^{4}} \\ &- \frac{8h^{2}\sinh(\lambda\zeta)\Big(A_{1}\Big)\omega\big(\tilde{E}(\Delta,\tau)\big)\lambda^{4}t^{2\mu}}{(\cosh(\lambda\zeta))^{7}\Gamma(2\mu+1)}, \ \cdots \end{aligned}$$

where

$$\begin{aligned} A_{1} =& 4\beta^{2}\lambda^{2}(\cosh(\lambda\zeta))^{4} + 10\alpha\beta\lambda\omega\big(\tilde{E}(\Delta,\tau)\big)(\cosh(\lambda\zeta))^{2} \\ &+ 60\beta^{2}\lambda^{2}(\cosh(\lambda\zeta))^{2} - \alpha^{2}\omega^{2}\big(\tilde{E}(\Delta,\tau)\big)^{2} + 21\alpha\beta\lambda\omega\big(\tilde{E}(\Delta,\tau)\big) - 90\beta^{2}\lambda^{2}. \end{aligned}$$

Thus, the approximated series solution can be expressed as follows:

$$\tilde{\vartheta}^{LC}(\zeta,t,\mathit{\Delta},\tau)=\tilde{\vartheta}_0^{LC}(\zeta,t,\mathit{\Delta},\tau)+\sum_{m=1}^{\infty}\tilde{\vartheta}_m^{LC}(\zeta,t,\mathit{\Delta},\tau)\Big(\frac{1}{n}\Big)^m.$$

(B) Using non-singular Mittag Leffler kernel approach, i.e., ϖ = ABC

Upon using the initial assumption from Equation (5.4) and inverse Shehu transform formula over Equation (5.5), the first few terms of the solution can be expressed as following

$$\tilde{\vartheta}_1^{ABC}(\zeta,t,\mathit{\Delta},\tau)=\frac{h\Big(A_2\Big)\omega\big(\tilde{E}(\mathit{\Delta},\tau)\big)\lambda^2}{B(\mu)\ (\cosh(\lambda\zeta))^4}\left(1-\mu+\frac{\mu t^\mu}{\Gamma(\mu+1)}\right),$$

$$\tilde{\vartheta}_2^{ABC}(\zeta,t,\mathit{\Delta},\tau)=\frac{(h+n)h\Big(A_2\Big)\omega\big(\tilde{E}(\mathit{\Delta},\tau)\big)\lambda^2}{B(\mu)\ (\cosh(\lambda\zeta))^4}\left(1-\mu+\frac{\mu t^\mu}{\Gamma(\mu+1)}\right)$$
$$\times\frac{8h^2\sinh(\lambda\zeta)\Big(A_3\Big)\omega\big(\tilde{E}(\mathit{\Delta},\tau)\big)\lambda^4}{B(\mu)^2\ (\cosh(\lambda\zeta))^7\big(\tilde{E}(\mathit{\Delta},\tau)\big)^{-1}}\left((1-\mu)^2+\frac{2(1-\mu)\mu t^\mu}{\Gamma(\mu+1)}+\frac{\mu^2t^{2\mu}}{\Gamma(2\mu+1)}\right),\ \cdots$$

where

$$\begin{aligned}A_2=&4(\cosh(\lambda\zeta))^2\beta\lambda+\alpha\omega\big(\tilde{E}(\mathit{\Delta},\tau)\big)-6\beta\lambda,\\A_3=&4\beta^2\lambda^2(\cosh(\lambda\zeta))^4+10\alpha\beta\lambda\omega(\cosh(\lambda\zeta))^2-\alpha^2\omega^2\big(\tilde{E}(\mathit{\Delta},\tau)\big)^2\\&+60\beta^2\lambda^2(\cosh(\lambda\zeta))^2+21\alpha\beta\lambda\omega\big(\tilde{E}(\mathit{\Delta},\tau)\big)-90\beta^2\lambda^2.\end{aligned}$$

Thus, the approximated series solution can be expressed as follows:

$$\tilde{\vartheta}^{ABC}(\zeta,t,\mathit{\Delta},\tau)=\tilde{\vartheta}_0^{ABC}(\zeta,t,\mathit{\Delta},\tau)+\sum_{m=1}^{\infty}\tilde{\vartheta}_m^{ABC}(\zeta,t,\mathit{\Delta},\tau)\Big(\frac{1}{n}\Big)^m.$$

6 Results and discussion

This section reports the numerical discussion of the obtained results of the nonlinear time fractional fuzzy p-KdV equation which is solved using q-HAShTM based on LC and ABC fractional derivative operators. Here, we consider the values of α, β, ν as $\alpha=1$, $\beta=1$, $\nu=0.5$, where as the values of the parameters for crisp case are taken as $(\mathit{\Delta},\tau)=(1,0)$ and for fuzzy case taken as $(\mathit{\Delta},\tau)=(0.9,0.1)$.

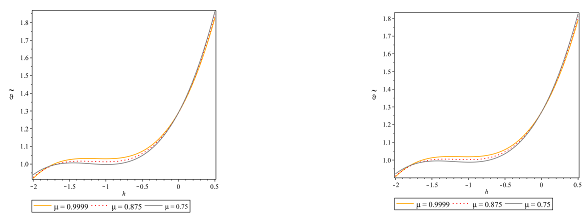
Table 1 discusses the absolute error of the term approximation at $\mu=1$, $n=1$, $h=-1$ for different values of ζ & t which shows that the comparison of the solution is valid for both fractional derivative operators LC and ABC as $\mu=1$. Furthermore, the comparison results concludes that as the order of the terms increases, the obtained solution approaches faster to the exact

Table 1. Comparison of the absolute errors of the term approximation of the obtained solution $\tilde{\vartheta}(\zeta, t, \varDelta, \tau)$ using q-HAShTM for crisp case with fractional operator in LC and ABC sense at $\mu = 1$, $h = -1$, $n = 1$ with different ζ and t.

ζ	t	$\lvert\tilde{\vartheta} - \tilde{\vartheta}^1\rvert$	$\lvert\tilde{\vartheta} - \tilde{\vartheta}^2\rvert$	$\lvert\tilde{\vartheta} - \tilde{\vartheta}^3\rvert$
	0.2	1.555×10^{-4}	3.077×10^{-5}	1.3901×10^{-7}
0.2	0.6	8.405×10^{-4}	8.358×10^{-4}	8.778×10^{-6}
	1	7.793×10^{-4}	3.877×10^{-3}	4.841×10^{-5}
	0.2	5.036×10^{-4}	2.637×10^{-5}	4.023×10^{-7}
0.6	0.6	4.038×10^{-3}	7.319×10^{-4}	3.075×10^{-5}
	1	9.782×10^{-3}	3.468×10^{-3}	2.220×10^{-4}
	0.2	7.779×10^{-4}	1.864×10^{-5}	5.453×10^{-7}
1	0.6	6.637×10^{-3}	5.318×10^{-4}	4.346×10^{-5}
	1	1.732×10^{-2}	2.590×10^{-3}	3.283×10^{-4}

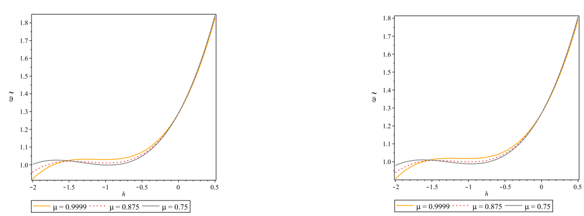
solution. Throughout this problem we considered 3 - term approximation, i.e., $\tilde{\vartheta}^3(\zeta, t, \varDelta, \tau)$ with both derivative operators.

Similarly, Tables 2 and 3 discuss the comparison results of the obtained solution with LC and ABC approach with fractional order $\mu = 0.9999$ with different ζ and t in crisp and fuzzy cases, respectively and compared the obtained solution(at $\mu = 0.9999$) and its error with the exact solution(at $\mu = 1$). Also, for the crisp case, the obtained results are compared with the reduced differential transform method (RDTM) [3] as shown in Table 2 and found to be in well agreement. It can be seen from the tables that the obtained errors are almost same in both LC and ABC approach.



(a) Crisp case (b) Uncertain case

Figure 1. h-curves with LC approach at $\zeta = 2, t = 1$.



(a) Crisp case (b) Uncertain case

Figure 2. h-curves with ABC approach at $\zeta = 2, t = 1$.

Figures 1 and 2 show the h-curves for the time-fractional fuzzy p-KdV equation with LC and ABC approach in both crisp and fuzzy cases, respectively. From figures it can be observed that they are alike in nature. Also, we can get

Table 2. Comparison of solutions with both the approaches and available solution by reduced differential transform method (RDTM) [3] in certain case at $h=-1$, $n=1$.

ζ	t	$\mu=0.9999$ $\tilde{\vartheta}(LC)$	$\tilde{\vartheta}(ABC)$	RDTM [3] at $\mu=1$	Exact Solution at $\mu=1$
0.5	0.3	0.2611484923	0.2611221697	0.2610737427	0.2611683123
	0.5	0.1869795450	0.1869603802	0.1865706094	0.1870132399
	0.7	0.1123269306	0.1123152430	0.1111680230	0.1123946498
	0.9	0.0373548960	0.0373509558	0.0348659835	0.0374960942
1	0.3	0.6189577798	0.6189327479	0.6189136425	0.6189774349
	0.5	0.5496359503	0.5496169594	0.5493732813	0.5496769348
	0.7	0.4789951912	0.4789826916	0.4782398922	0.4790954700
	0.9	0.4071433307	0.4071377404	0.4055134749	0.4073782341
5	0.3	1.988154235	1.988150597	1.988175910	1.988156133
	0.5	1.978727169	1.978723958	1.978819817	1.978727033
	0.7	1.968663609	1.968660914	1.968913080	1.968655080
	0.9	1.957929036	1.957926956	1.958455698	1.957900059
10	0.3	2.117317486	2.117317373	2.117318270	2.117317539
	0.5	2.117024605	2.117024504	2.117027992	2.117024547
	0.7	2.116710613	2.116710528	2.116719760	2.116710134
	0.9	2.116374241	2.116374176	2.116393571	2.116372734

ζ	t	Absolute Error $\tilde{\vartheta}(LC)$	$\tilde{\vartheta}(ABC)$	RDTM [3]
0.5	0.3	1.9820×10^{-5}	4.6142×10^{-5}	9.4569×10^{-5}
	0.5	3.3694×10^{-5}	5.2859×10^{-5}	4.4263×10^{-4}
	0.7	6.7719×10^{-5}	7.9406×10^{-5}	1.2266×10^{-3}
	0.9	1.4119×10^{-4}	1.4513×10^{-4}	2.6301×10^{-3}
1	0.3	1.9655×10^{-5}	4.4687×10^{-5}	6.3792×10^{-5}
	0.5	4.0984×10^{-5}	5.9975×10^{-5}	3.0365×10^{-4}
	0.7	1.0027×10^{-4}	1.1277×10^{-4}	8.5557×10^{-4}
	0.9	2.3490×10^{-4}	2.4049×10^{-4}	1.8647×10^{-3}
5	0.3	1.8992×10^{-6}	5.5359×10^{-6}	1.9777×10^{-5}
	0.5	1.3559×10^{-7}	3.0761×10^{-6}	9.2784×10^{-5}
	0.7	8.5295×10^{-6}	5.8332×10^{-6}	2.5800×10^{-4}
	0.9	2.8976×10^{-5}	2.6896×10^{-5}	5.5563×10^{-4}
10	0.3	5.2046×10^{-8}	1.6588×10^{-7}	7.3100×10^{-7}
	0.5	5.8803×10^{-8}	4.3375×10^{-8}	3.4450×10^{-6}
	0.7	4.8198×10^{-7}	3.9411×10^{-7}	9.6260×10^{-6}
	0.9	1.5131×10^{-6}	1.4425×10^{-6}	2.0837×10^{-5}

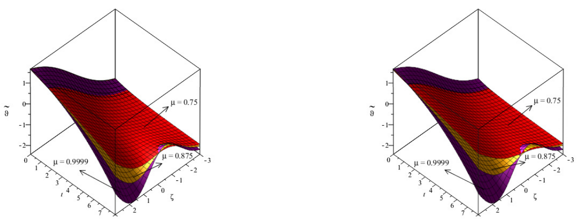
a convergence range of h with fixed $n=1$ as $h\in(-1.5,-0.5)$ for both LC and ABC approaches in crisp and fuzzy cases and here we have kept the value of h as $h=-1$ throughout the discussion.

Figures 3 and 4 discuss the graphical illustrations of the solutions with LC and ABC approach for different fractional order $\mu=0.75, 0.875, 0.9999$ in crisp and uncertain case, respectively.

Whereas Figures 5, 6 and 7 indicate lower bound (i.e., $\tau=0$) and upper bound (i.e., $\tau=1$) of the solution for LC and ABC approach with $\zeta=1$,

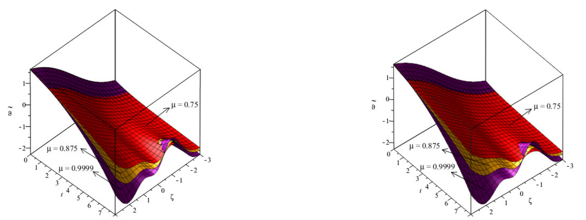
Table 3. Comparison of solutions with both the approaches in uncertain case at $h = -1$, $n = 1$.

ζ	t	$\mu = 0.9999$ $\tilde{\vartheta}(LC)$	$\tilde{\vartheta}(ABC)$	Exact solution at $\mu = 1$	Absolute error $\tilde{\vartheta}(LC)$	$\tilde{\vartheta}(ABC)$
0.5	0.3	0.2594792961	0.2594539651	0.2569896193	2.48967×10^{-3}	2.46434×10^{-3}
	0.5	0.1881548390	0.1881363312	0.1840210280	4.13381×10^{-3}	4.11530×10^{-3}
	0.7	0.1163109043	0.1162995074	0.1105963354	5.71456×10^{-3}	5.70317×10^{-3}
	0.9	0.0440820075	0.0440779683	0.0368961567	7.18585×10^{-3}	7.18111×10^{-3}
1	0.3	0.6112232954	0.6111992693	0.6090737958	2.14949×10^{-3}	2.12547×10^{-3}
	0.5	0.5445843360	0.5445662117	0.5408821037	3.70223×10^{-3}	3.68410×10^{-3}
	0.7	0.4767826029	0.4767708269	0.4714299424	5.35266×10^{-3}	5.34088×10^{-3}
	0.9	0.4079522499	0.4079472279	0.4008601823	7.09206×10^{-3}	7.08704×10^{-3}
5	0.3	1.956392784	1.956389236	1.956345635	4.71490×10^{-5}	4.36005×10^{-5}
	0.5	1.947172238	1.947169123	1.947067400	1.04837×10^{-4}	1.01721×10^{-4}
	0.7	1.937348621	1.937346027	1.937156598	1.92021×10^{-4}	1.89428×10^{-4}
	0.9	1.926892105	1.926890132	1.926573658	3.18446×10^{-4}	3.16474×10^{-4}



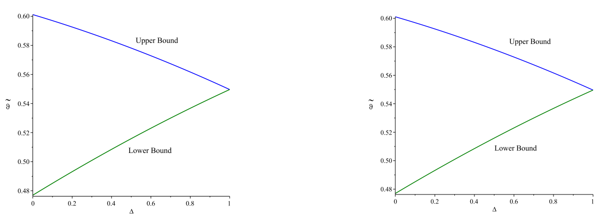
(a) LC approach (b) ABC approach

Figure 3. Comparison of solutions at different $\mu = 0.9999, 0.875, 0.75$ for crisp case.



(a) LC approach (b) ABC approach

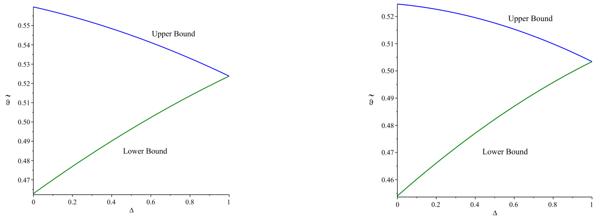
Figure 4. Comparison of solutions at different $\mu = 0.9999, 0.875, 0.75$ in uncertain case.



(a) LC approach (b) ABC approach

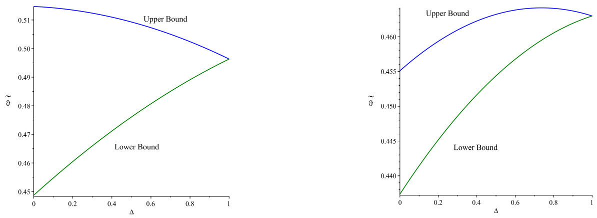
Figure 5. Lower bound (i.e., $\tau = 0$) and upper bound (i.e., $\tau = 1$) solutions with $\zeta = 1$, $t = 0.5$, $h = -1$ for fractional order $\mu = 0.9999$ with varying Δ.

$t = 0.5$ at different fractional orders $\mu = 0.9999, 0.875, 0.75$, respectively.



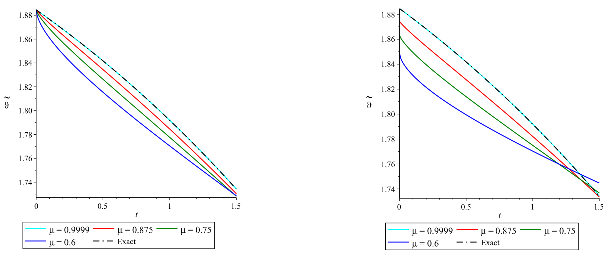
(a) LC approach (b) ABC approach

Figure 6. Lower bound (i.e., $\tau = 0$) and upper bound (i.e., $\tau = 1$) solutions with $\zeta = 1$, $t = 0.5$, $h = -1$ for fractional order $\mu = 0.875$ with varying Δ.



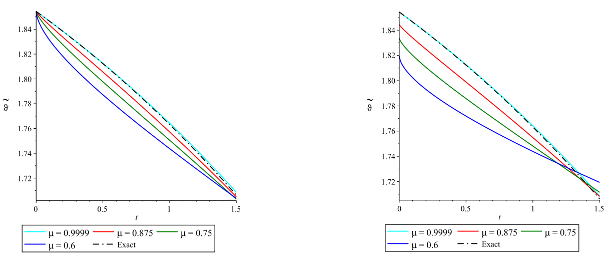
(a) LC approach (b) ABC approach

Figure 7. Lower bound (i.e., $\tau = 0$) and upper bound (i.e., $\tau = 1$) solutions with $\zeta = 1$, $t = 0.5$, $h = -1$ for fractional order $\mu = 0.75$ with varying Δ.



(a) LC approach (b) ABC approach

Figure 8. Comparison of solutions at $\zeta = 4$ with different $\mu = 0.9999$, 0.875, 0.75, 0.6 in crisp case.



(a) LC approach (b) ABC approach

Figure 9. Comparison of solutions at $\zeta = 4$ with different $\mu = 0.9999, 0.875, 0.75, 0.6$ in an uncertain case.

Finally, the Figures 8 and 9 indicate the graphically comparison between exact solution (at $\mu = 1$) and obtained solution with LC and ABC approach at $\zeta = 4$ with varying t at different fractional orders $\mu = 0.6, 0.75, 0.875, 0.9999$ for crisp and uncertain case, respectively.

7 Conclusions

Here, a non-linear temporal fuzzy fractional p-KdV equation is considered with a singular kernel and a non-singular Mittag Leffler kernel for two distinct fractional operators LC and ABC. The numerical results and the graphical illustration of the obtained results have been validated with both fractional operators to test the efficacy of the differential operators and validated the obtained results with the available results with the integer-order for crisp and fuzzy cases, respectively. It can be observed that both the fractional operators provide the same errors, and there are no significant changes in errors occurring by shifting the fractional operator from LC to ABC. It can also be seen from the discussion that the ABC operator holds a similar fractional nature as LC. In contrast, the LC operator works better in dealing with non-linear PDE than the ABC operator.

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