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On a Dirichlet Series Connected to a Periodic Hurwitz Zeta-Function with Transcendental and Rational Parameter

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Abstract. In the paper, we construct an absolutely convergent Dirichlet series which in the mean is close to the periodic Hurwitz zeta-function, and has the universality property on the approximation of a wide class of analytic functions.

Keywords: Haar measure, periodic Hurwitz zeta-function, space of analytic functions, universality, weak convergence.

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1 Introduction

Let $\mathfrak{a} = \{a_m : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ be a periodic sequence of complex numbers with minimal period $q \in \mathbb{N}$, $0 < \alpha \leqslant 1$ fixed parameter, and $s = \sigma + it$ a complex variable. The periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{a})$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s, \alpha; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s}.$$

If $a_m \equiv 1$, then we have the classical Hurwitz zeta-function $\zeta(s,\alpha)$ which has the meromorphic continuation to the whole complex plane with unique simple pole at the point $s = 1$ with residue 1. The periodicity of the sequence $\mathfrak{a}$ implies, for $\sigma > 1$, the equality

$$\zeta(s,\alpha;\mathfrak{a}) = \frac{1}{q^s}\sum_{l=0}^{q-1} a_l \zeta\left(s, \frac{l+\alpha}{q}\right), \tag{1.1}$$

therefore, the function $\zeta(s,\alpha;\mathfrak{a})$ also can be meromorphically continued to the whole complex plane with a simple pole at the point $s = 1$ with residue

$$\widehat{a} \stackrel{def}{=} \frac{1}{q}\sum_{l=0}^{q-1} a_l.$$

If $\widehat{a} = 0$, then $\zeta(s,\alpha;\mathfrak{a})$ is an entire function. Clearly, $\zeta(s,1;\{1\}) = \zeta(s)$ is the Riemann zeta-function. Thus, the periodic Hurwitz zeta-function is a generalization of the classical Hurwitz and Riemann zeta-functions. Analytical properties of $\zeta(s,\alpha;\mathfrak{a})$ are governed by the sequence $\mathfrak{a}$, and, in particular, by arithmetic of the parameter α.

The function $\zeta(s,\alpha;\mathfrak{a})$, for some classes of the parameter α, as other zeta-functions, is universal, i. e., its shifts $\zeta(s+i\tau,\alpha;\mathfrak{a})$, $\tau \in \mathbb{R}$, approximate all analytic functions defined in the strip $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$. Note that the phenomenon of universality for zeta-functions was discovered by Voronin, in [6] he proved the universality of the Riemann zeta-function. Later, the universality of some other zeta-functions was obtained. By the Linnik-Ibragimov conjecture, all functions in some half-plane given by Dirichlet series, analytically continuable to the left of the absolute convergence half-plane and satisfying some natural growth conditions are universal in the Voronin sense. However, till now the universality of some zeta-functions are not known. The universality of the function $\zeta(s,\alpha;\mathfrak{a})$ with transcendental α was considered in [3], and with rational α in [4]. Denote by $\mathcal{K}$ the class of compact subsets of the strip D, and by $H(K)$ with $K \in \mathcal{K}$ the class of continuous functions on K that are analytic in the interior of K. Let meas$A$ stand for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then the universality of the function $\zeta(s,\alpha;\mathfrak{a})$ is described in the following theorem.

Theorem 1. *Suppose that the number α is transcendental, or $\alpha = a/b$, $a,b \in \mathbb{N}$, $a < b$, $(a,b) = 1$, $a/b \neq 1/2$ and $(lb+a, bq) = 1$ for all $l = 0,1,\dots,q-1$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau,\alpha;\mathfrak{a}) - f(s)| < \varepsilon\right\} > 0.$$

It is easily seen that the transcendence of α can be replaced by the linear independence over the field of rational numbers $\mathbb{Q}$ of the set

$$L(\alpha) = \{\log(m+\alpha) : m \in \mathbb{N}_0\}.$$

The universality of the function $\zeta(s,\alpha;\mathfrak{a})$ with algebraic irrational α is an open problem.

In the strip D, the function $\zeta(s,\alpha;\mathfrak{a})$ is defined by using (1.1) and analytic continuation of the Hurwitz zeta-function. Therefore, it is not easy to derive an information on the function $f(s)$ from the inequality

$$\sup_{s\in K}|\zeta(s+i\tau,\alpha;\mathfrak{a})-f(s)|<\varepsilon.$$

It is more convenient to use an absolutely convergent Dirichlet series in place of $\zeta(s,\alpha;\mathfrak{a})$. This paper is devoted to a realization of the mentioned idea.

Let $\theta>1/2$ be a fixed number, and, for $m\in\mathbb{N}_0$ and $u>0$,

$$v_u(m,\alpha)=\exp\left\{-\left((m+\alpha)/u\right)^{\theta}\right\}.$$

Since $|a_m|\leqslant C$, $m\in\mathbb{N}_0$, with some $C<\infty$, the series

$$\zeta_u(s,\alpha;\mathfrak{a})=\sum_{m=0}^{\infty}\frac{a_m v_u(m,\alpha)}{(m+\alpha)^s}$$

is absolutely convergent for $\sigma>\sigma_0$ with arbitrary finite σ_0.

Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of the space $\mathbb{X}$, by $H(D)$ the space of analytic on D functions endowed with the topology of uniform convergence on compacta, and let $\gamma=\{s\in\mathbb{C}:|s|=1\}$. Define the set $\Omega=\prod_{m\in\mathbb{N}_0}\gamma_m$, where $\gamma_m=\gamma$ for all $\mathbb{N}_0$. With the product topology and pointwise multiplication, the torus Ω is a compact topological Abelian group. Therefore, on $(\Omega,\mathcal{B}(\Omega))$, the probability Haar measure m_H exists, and we obtain a probability space $(\Omega,\mathcal{B}(\Omega),m_H)$. Denote by ω_m the mth component of an element $\omega\in\Omega$, $m\in\mathbb{N}_0$, and, on the probability space $(\Omega,\mathcal{B}(\Omega),m_H)$, define the $H(D)$-valued random element

$$\zeta(s,\alpha,\omega;\mathfrak{a})=\sum_{m=0}^{\infty}\frac{a_m\omega(m)}{(m+\alpha)^s}.$$

The latter series, for almost all $\omega\in\Omega$ is uniformly convergent on compact subsets of the strip D. Let $P_{\zeta,\alpha,\mathfrak{a}}$ be the distribution of the random element $\zeta(s,\alpha,\omega;\mathfrak{a})$, i. e.,

$$P_{\zeta,\alpha,\mathfrak{a}}(A)=m_H\{\omega\in\Omega:\zeta(s,\alpha,\omega;\mathfrak{a})\in A\},\quad A\in\mathcal{B}(H(D)).$$

Theorem 2. *Suppose that the number α is transcendental, and $u_T\to\infty$, $u_T\ll T^2$ as $T\to\infty$. Let $K\in\mathcal{K}$ and $f(s)\in H(K)$. Then the limit*

$$\lim_{T\to\infty}\frac{1}{T}\mathrm{meas}\left\{\tau\in[0,T]:\sup_{s\in K}|\zeta_{u_T}(s+i\tau,\alpha;\mathfrak{a})-f(s)|<\varepsilon\right\}$$
$$=m_H\left\{\omega\in\Omega:\sup_{s\in K}|\zeta(s,\alpha,\omega;\mathfrak{a})-f(s)|<\varepsilon\right\}>0$$

exists for all but at most countably many $\varepsilon>0$.

For the case of rational α, define one more infinite-dimensional torus $\Omega_1 = \prod_{p\in\mathbb{P}} \gamma_p$, where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$, and $\mathbb{P}$ is the set of all prime numbers. Analogically to Ω, we have the probability space $(\Omega_1, \mathcal{B}(\Omega_1), m_{1H})$, where m_{1H} is the probability Haar measure on $(\Omega_1, \mathcal{B}(\Omega_1))$. Denote by $\omega_1(p)$ the p-th component of an element $\omega_1 \in \Omega_1$, $p \in \mathbb{P}$, and extend $\omega_1(p)$ to the set $\mathbb{N}$ by the formula

$$\omega_1(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \omega^l(p), \quad m \in \mathbb{N}.$$

Now, let $\alpha = a/b$, $a, b \in \mathbb{N}$, $a < b$, $(a, b) = 1$. Denote by χ Dirichlet characters, and by $L(s,\chi)$ the corresponding Dirichlet L-functions

$$L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}, \quad \sigma > 1.$$

The functions $L(s,\chi)$ have meromorphic continuation to the whole complex plane. Let $\varphi(m)$ be the totient Euler function. On the probability space $(\Omega_1, \mathcal{B}(\Omega_1), m_{1H})$, define the $H(D)$-valued random element

$$\zeta\left(s, \frac{a}{b}, \omega_1; \mathfrak{a}\right) = \frac{b^s \omega_1(b)}{\varphi(bq)} \sum_{l=0}^{q-1} a_l \sum_{\chi \bmod bq} \overline{\chi}(a+bl) L(s, \omega_1, \chi),$$

where

$$L(s, \omega_1, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)\omega_1(m)}{m^s}$$

and $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$.

Theorem 3. *Suppose that $a, b \in \mathbb{N}$, $a < b$, $(a, b) = 1$, $a/b \neq 1/2$ and $(lb + a, q) = 1$ for all $l = 0, 1, \ldots, q-1$, and $u_T \to \infty$, $u_T \ll T^2$ as $T \to \infty$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the limit*

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{T} \text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} \left|\zeta_{u_T}\left(s + i\tau, \frac{a}{b}; \mathfrak{a}\right) - f(s)\right| < \varepsilon\right\} \\
&\quad = m_{1H}\left\{\omega_1 \in \Omega_1 : \sup_{s\in K} \left|\zeta\left(s, \frac{a}{b}, \omega_1; \mathfrak{a}\right) - f(s)\right| < \varepsilon\right\} > 0
\end{aligned}$$

exists for all but at most countably many $\varepsilon > 0$.

We will derive Theorems 2 and 3 from Theorem 1 by using the approximation of $\zeta(s, \alpha; \mathfrak{a})$ by $\zeta_{u_T}(s, \alpha; \mathfrak{a})$ in the mean.

2 Some estimates

In this section, we prove the following equality.

Lemma 1. *Let K be a compact subset of the strip D, and $u_T \to \infty$ and $u_T \ll T^2$ as $T \to \infty$. Then*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K} |\zeta(s + i\tau, \alpha; \mathfrak{a}) - \zeta_{u_T}(s + i\tau, \alpha; \mathfrak{a})| \, d\tau = 0.$$

Proof. From the Mellin formula

$$\frac{1}{2\pi i}\int_{c-\infty}^{c+\infty} \Gamma(s)d^{-s}\mathrm{d}s = e^{-d}, \quad c,d>0,$$

where $\Gamma(s)$ is the Euler gamma-function, we have

$$v_{u_T}(m,\alpha) = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty} \frac{1}{\theta}\Gamma\left(\frac{s}{\theta}\right)\left(\frac{m+\alpha}{u_T}\right)^{-s}\mathrm{d}s.$$

Hence, denoting $l_{u_T}(s) = \frac{s}{\theta}\Gamma\left(\frac{s}{\theta}\right)u_T^s$, we obtain, for $s\in K$,

$$\begin{aligned}\zeta_{u_T}(s,\alpha;\mathfrak{a}) =& \frac{1}{2\pi i}\sum_{m=0}^{\infty}\frac{a_m}{(m+\alpha)^s}\int_{\theta-i\infty}^{\theta+i\infty}\frac{z}{\theta}\Gamma\left(\frac{z}{\theta}\right)\left(\frac{m+\alpha}{u_T}\right)^{-z}\frac{\mathrm{d}z}{z}\\ =& \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\frac{l_{u_T}(z)}{z}\sum_{m=0}^{\infty}\frac{a_m}{(m+\alpha)^{s+z}}\mathrm{d}z\\ =& \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\zeta(s+z,\alpha;\mathfrak{a})l_{u_T}(z)\frac{\mathrm{d}z}{z}.\end{aligned} \tag{2.1}$$

There exists $\varepsilon>0$ such that $1/2+2\varepsilon \leqslant \sigma \leqslant 1-\varepsilon$ for $s=\sigma+it\in K$. We take $\theta=\frac{1}{2}+\varepsilon$ and $\theta_1=\frac{1}{2}+\varepsilon-\sigma$. Then $\theta_1<0$ for all $s\in K$. Therefore, the representation (2.1) and the residue theorem give

$$\zeta_{u_T}(s,\alpha;\mathfrak{a})-\zeta(s,\alpha;\mathfrak{a}) = \frac{1}{2\pi i}\int_{\theta_1-i\infty}^{\theta_1+i\infty}\zeta(s+z,\alpha;\mathfrak{a})\frac{l_{u_T}(z)}{z}\mathrm{d}z + \frac{\widehat{a}l_{u_T}(1-s)}{1-s}.$$

Thus, for all $s\in K$,

$$\begin{aligned}&\zeta_{u_T}(s+i\tau,\alpha;\mathfrak{a})-\zeta(s+i\tau,\alpha;\mathfrak{a}) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\zeta\left(\frac{1}{2}+\varepsilon+it+i\tau+iv,\alpha;\mathfrak{a}\right)\\ &\quad\times\frac{l_{u_T}(1/2+\varepsilon-\sigma+iv)}{1/2+\varepsilon-\sigma+iv}\mathrm{d}v+\frac{\widehat{a}l_{u_T}(1-s-i\tau)}{1-s-i\tau}\\ &=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\zeta\left(\frac{1}{2}+\varepsilon+i\tau+iv,\alpha;\mathfrak{a}\right)\frac{l_{u_T}(1/2+\varepsilon-s+iv)}{1/2+\varepsilon-s+iv}\mathrm{d}v\\ &\quad+\frac{\widehat{a}l_{u_T}(1-s-i\tau)}{1-s-i\tau} \ll \int_{-\infty}^{\infty}\left|\zeta\left(\frac{1}{2}+\varepsilon+i\tau+iv,\alpha;\mathfrak{a}\right)\right|\\ &\quad\sup_{s\in K}\left|\frac{l_{u_T}(1/2+\varepsilon-s+iv)}{1/2+\varepsilon-s+iv}\right|\mathrm{d}v+\widehat{a}\sup_{s\in K}\left|\frac{l_{u_T}(1-s-i\tau)}{1-s-i\tau}\right|.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{1}{T}\int_0^T\sup_{s\in K}\left|\zeta(s+i\tau,\alpha;\mathfrak{a})-\zeta_{u_T}(s+i\tau,\alpha;\mathfrak{a})\right|\mathrm{d}\tau\\ &\quad\ll\int_{-\infty}^{\infty}\left(\frac{1}{T}\int_0^T\left|\zeta\left(\frac{1}{2}+\varepsilon+i\tau+iv,\alpha;\mathfrak{a}\right)\right|^2\mathrm{d}\tau\right)^{1/2}\\ &\times\sup_{s\in K}\left|\frac{l_{u_T}(1/2+\varepsilon-s+iv)}{1/2+\varepsilon-s+iv}\right|\mathrm{d}v+\widehat{a}\frac{1}{T}\int_0^T\sup_{s\in K}\left|\frac{l_{u_T}(1-s-i\tau)}{1-s-i\tau}\right|\mathrm{d}\tau\stackrel{def}{=}I_1+I_2.\end{aligned} \tag{2.2}$$

It is well known that, for $1/2 < \sigma < 1$,

$$\int_0^T |\zeta(\sigma+it,\alpha)|^2\,\mathrm{d}t \ll_{\sigma,\alpha} T.$$

This and (1.1) show that, for $1/2 < \sigma < 1$,

$$\int_0^T |\zeta(\sigma+it,\alpha;\mathfrak{a})|^2\,\mathrm{d}t \ll_{\sigma,\alpha,\mathfrak{a}} T.$$

Hence, for the same σ and $v \in \mathbb{R}$,

$$\int_0^T |\zeta(\sigma+it+iv,\alpha;\mathfrak{a})^2\,\mathrm{d}\tau \ll_{\sigma,\alpha,\mathfrak{a}} T(1+|v|), \quad T \geqslant 1. \tag{2.3}$$

For the function $\Gamma(s)$, the estimate

$$\Gamma(\sigma+it) \ll \exp\{-c_1|t|\}$$

uniformly in $\sigma_1 \leqslant \sigma \leqslant \sigma_2$ with arbitrary $\sigma_1 < \sigma_2$ is valid. Therefore, for all $s \in K$,

$$\begin{aligned}\frac{l_{u_T}(1/2+\varepsilon-s+iv)}{1/2+\varepsilon-s+iv} &\ll u_T^{1/2+\varepsilon-\sigma}\left|\Gamma\left(\frac{1}{\theta}\left(\frac{1}{2}+\varepsilon-s+iv\right)\right)\right| \\ &\ll u_T^{-\varepsilon}\exp\left\{\frac{c_1}{\theta}|v-t|\right\} \ll_{\varepsilon,K} u_T^{-\varepsilon}\exp\{-c_2|v|\}, \quad c_2>0.\end{aligned}$$

Therefore, in view of (2.3),

$$I_1 \ll_{\varepsilon,\alpha,\mathfrak{a},K} u_T^{-\varepsilon}\int_{-\infty}^{\infty}(1+|v|)^{1/2}\exp\{-c_2|v|\}\,\mathrm{d}v \ll_{\varepsilon,\alpha,\mathfrak{a},K} u_T^{-\varepsilon}. \tag{2.4}$$

Similarly, we find that, for all $s \in K$,

$$\frac{l_{u_T}(1-s-i\tau)}{1-s-i\tau} \ll_\varepsilon u_T^{1-\sigma}\exp\{-c_1|t+\tau|\} \ll_{\varepsilon,\mathfrak{a},K} u_T^{1/2-2\varepsilon}\exp\{-c_3|\tau|\}, \quad c_3>0.$$

Thus,

$$I_2 \ll_{\varepsilon,\mathfrak{a},K} u_T^{1/2-2\varepsilon}\frac{1}{T}\int_0^T \exp\{-c_3\tau\}\,\mathrm{d}\tau \ll_{\varepsilon,\mathfrak{a},K} u_T^{1/2-2\varepsilon}/T.$$

This, (2.2) and (2.4) prove the lemma. $\square$

3 Limit theorems

Let P_n, $n \in \mathbb{N}$, and P be probability measures on $(\mathbb{X},\mathcal{B}(\mathbb{X}))$. Recall that P_n converges weakly to P as $n \to \infty$ if, for every real bounded continuous function g on $\mathbb{X}$,

$$\lim_{n\to\infty}\int_{\mathbb{X}} g\,\mathrm{d}P_n = \int_{\mathbb{X}} g\,\mathrm{d}P.$$

There are equivalents of weak convergence in terms of some classes of sets. We will use the following, see, for example, [1].

Lemma 2. *P_n converges weakly to P as $n \to \infty$ if and only if, for every closed set $F \subset \mathbb{X}$,*

$$\limsup_{n\to\infty} P_n(F) \leqslant P(F).$$

For $A \in \mathcal{B}(H(D))$, define

$$P_{T,\alpha,\mathfrak{a}}(A) = \frac{1}{T}\text{meas}\{\tau \in [0,T] : \zeta(s+i\tau,\alpha;\mathfrak{a}) \in A\}.$$

Lemma 3. *Suppose that α is a transcendental number. Then $P_{T,\alpha,\mathfrak{a}}$ converges weakly to*

$$m_H\{\omega \in \Omega : \zeta(s,\omega,\alpha;\mathfrak{a}) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

as $T \to \infty$.

The lemma was obtained in [2]. The transcendence of α can be replaced by the linear independence over $\mathbb{Q}$ for the set $L(\alpha)$.

Let $V > 0$ be an arbitrary fixed number, $D_V = \{s \in \mathbb{C} : 1/2 < \sigma < 1,\ |t| < V\}$, and $H(D_V)$ the space of analytic on D_V functions. Denote by $P^1_{T,\alpha,\mathfrak{a}}$ the analogue of $P_{T,\alpha,\mathfrak{a}}$ for the space $H(D_V)$.

Lemma 4. *Suppose that $\alpha = a/b$, $a,b \in \mathbb{N}$, $a < b$, $(a,b) = 1$, $a/b \neq 1/2$, and $(lb+a, bq) = 1$ for all $l = 0,1,\dots,q-1$. Then $P^1_{T,\alpha,\mathfrak{a}}$ converges weakly to*

$$m_{1H}\left\{\omega_1 \in \Omega_1 : \zeta\left(s, \frac{a}{b}, \omega_1; \mathfrak{a}\right) \in A\right\} \stackrel{def}{=} P^1_{\zeta,\alpha,\mathfrak{a}}(A), \quad A \in \mathcal{B}(H(D_V)),$$

as $T \to \infty$.

Proof of the lemma is given in [4], Theorem 3.

Now we will prove the analogues of Lemmas 3 and 4 for the function $\zeta_{u_T}(s,\alpha;\mathfrak{a})$. For $A \in \mathcal{B}(H(D))$, define

$$Q_{T,\alpha,\mathfrak{a}}(A) = \frac{1}{T}\text{meas}\{\tau \in [0,T] : \zeta_{u_T}(s+i\tau,\alpha;\mathfrak{a}) \in A\}.$$

Lemma 5. *Suppose that α is a transcendental number, and $u_T \to \infty$ and $u_T \ll T^2$ as $T \to \infty$. Then $Q_{T,\alpha,\mathfrak{a}}$ converges weakly to $P_{\zeta,\alpha,\mathfrak{a}}$ as $T \to \infty$.*

Proof. For $g_1, g_2 \in H(D)$, define

$$\rho(g_1,g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s\in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s\in K_l} |g_1(s) - g_2(s)|},$$

where $\{K_l : l \in \mathbb{N}\} \subset D$ is a sequence of compact embedded subsets such that $D = \bigcup_{l=1}^{\infty} K_l$, and if $K \subset D$ is a compact set, then K lies in some K_l. Then ρ is a metric in the space $H(D)$ inducing its topology of uniform convergence on compacta.

Let ξ_T be the random variable defined on a certain probability space $(\widehat{\Omega}, \mathcal{A}, \mu)$ and uniformly distributed in the interval $[0, T]$. Define the $H(D)$-valued random elements

$$X_{T,\alpha,\mathfrak{a}} = X_{T,\alpha,\mathfrak{a}}(s) = \zeta(s + i\xi_T, \alpha; \mathfrak{a}),$$
$$Y_{T,\alpha,\mathfrak{a}} = Y_{T,\alpha,\mathfrak{a}}(s) = \zeta_{u_T}(s + i\xi_T, \alpha; \mathfrak{a}).$$

Let F be an arbitrary closed set of the space $H(D)$ and $\varepsilon > 0$. Then the set

$$F_\varepsilon = \{g : \rho(g, F) \leqslant \varepsilon\}$$

is closed as well. Therefore, by Lemmas 3 and 2,

$$\limsup_{T\to\infty} P_{T,\alpha,\mathfrak{a}}(F_\varepsilon) \leqslant P_{\zeta,\alpha,\mathfrak{a}}(F_\varepsilon).$$

Using the inclusion

$$\{Y_{T,\alpha,\mathfrak{a}} \in F\} \subset \{X_{T,\alpha,\mathfrak{a}} \in F_\varepsilon\} \cup \{\rho(X_{T,\alpha,\mathfrak{a}}, Y_{T,\alpha,\mathfrak{a}}) \geqslant \varepsilon\},$$

we find

$$\mu\{Y_{T,\alpha,\mathfrak{a}} \in F\} \leqslant \mu\{X_{T,\alpha,\mathfrak{a}} \in F_\varepsilon\} + \mu\{\rho(X_{T,\alpha,\mathfrak{a}}, Y_{T,\alpha,\mathfrak{a}}) \geqslant \varepsilon\}. \tag{3.1}$$

Since the density of the random variable ξ_T is $1/T$ on $[0, T]$, and 0 elsewhere, for every measurable function $h : \widehat{\Omega} \to \mathbb{X}$, we have

$$\mu\{h(\xi_T) \in A\} = \frac{1}{T} \int\limits_{\substack{0\\ \tau \in h^{-1}A}}^{T} \mathrm{d}\tau, \quad A \in \mathcal{B}(\mathbb{X}).$$

Therefore, by the definitions of $X_{T,\alpha,\mathfrak{a}}$ and $Y_{T,\alpha,\mathfrak{a}}$,

$$\mu\{X_{T,\alpha,\mathfrak{a}} \in F_\varepsilon\} = \frac{1}{T}\mathrm{meas}\{\tau \in [0, T] : \zeta(s + i\tau, \alpha; \mathfrak{a}) \in F_\varepsilon\} = P_{T,\alpha;\mathfrak{a}}(F_\varepsilon),$$

and similarly

$$\mu\{Y_{T,\alpha;\mathfrak{a}} \in F\} = Q_{T,\alpha,\mathfrak{a}}(F).$$

Moreover, in view of Lemma 1 and the definition of the metric ρ, for every $\varepsilon > 0$,

$$\begin{aligned}
&\mu\{\rho(X_{T,\alpha,\mathfrak{a}}, Y_{T,\alpha,\mathfrak{a}}) \geqslant \varepsilon\}\\
&\quad = \frac{1}{T}\mathrm{meas}\{\tau \in [0, T] : \rho(\zeta(s + i\tau, \alpha, \mathfrak{a}), \zeta_{u_T}(s + i\tau, \alpha, \mathfrak{a})) \geqslant \varepsilon\}\\
&\quad \leqslant \frac{1}{T\varepsilon} \int_0^T \rho(\zeta(s + i\tau, \alpha, \mathfrak{a}), \zeta_{u_T}(s + i\tau, \alpha, \mathfrak{a}))\, \mathrm{d}\tau = 0.
\end{aligned}$$

These three equalities and (3.1) give

$$\liminf_{T\to\infty} Q_{T,\alpha,\mathfrak{a}}(F) \leqslant P_{\zeta,\alpha,\mathfrak{a}}(F_\varepsilon).$$

Letting $\varepsilon \to +0$ together with Lemma 2 proves the lemma. $\square$

For $A \in \mathcal{B}(H(D_V))$, let $Q^1_{T,\alpha,\mathfrak{a}}(A)$ be an analogue of $Q_{T,\alpha,\mathfrak{a}}$ for the space $H(D_V)$. Using Lemma 4 and repeating a proof of Lemma 5 lead to the following assertion.

Lemma 6. *Suppose that the hypotheses of Lemma 4 on the parameter α and the sequence $\mathfrak{a}$ are satisfied, and $u_T \to \infty$ and $u_T \ll T^2$ as $T \to \infty$. Then $Q^1_{T,\alpha,\mathfrak{a}}$ converges weakly to $P^1_{\zeta,\alpha,\mathfrak{a}}$ as $T \to \infty$.*

Lemmas 5 and 6 imply the weak convergence for the corresponding measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. To see this, recall a preservation property of weak convergence. Let P be a probability measure on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$, and $h : \mathbb{X} \to \mathbb{X}_1$ be a $(\mathcal{B}(\mathbb{X}), \mathcal{B}(\mathbb{X}_1))$-measurable mapping. Then the measure P defines the unique probability measure Ph^{-1} on $(\mathbb{X}_1, \mathcal{B}(\mathbb{X}_1))$ by the formula

$$Ph^{-1}(A) = P(h^{-1}A), \quad A \in \mathcal{B}(\mathbb{X}_1).$$

Lemma 7. *Suppose that P_n, $n \in \mathbb{N}$, and P are probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$, $h : \mathbb{X} \to \mathbb{X}_1$ a continuous mapping, and P_n converges weakly to P as $n \to \infty$. Then also $P_n h^{-1}$ converges weakly to Ph^{-1} as $n \to \infty$.*

Proof of the lemma can be found in [1].

For $A \in \mathcal{B}(\mathbb{R})$, define

$$\widehat{Q}_{T,\alpha,\mathfrak{a}}(A) = \frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \sup_{s\in K} |\zeta_{u_T}(s+i\tau, \alpha; \mathfrak{a}) - f(s)| \in A\Big\},$$

where K and $f(s)$ are from Theorems 2 and 3.

Lemma 8. *Suppose that α is a transcendental number, $u_T \to \infty$ and $u_T \ll T^2$ as $T \to \infty$, and $K \subset D$ is a compact set. Then $\widehat{Q}_{T,\alpha,\mathfrak{a}}$ converges weakly to*

$$m_H\Big\{\omega \in \Omega : \sup_{s\in K} |\zeta(s, \alpha, \omega; \mathfrak{a}) - f(s)| \in A\Big\}, \quad A \in \mathcal{B}(\mathbb{R}),$$

as $T \to \infty$.

Proof. The mapping $h : H(D) \to \mathbb{R}$ given by $h(g) = \sup_{s\in K} |g(s) - f(s)|$ is continuous. Therefore, the lemma follows from Lemmas 5 and 7. □

Similarly, Lemmas 6 and 7 imply the following statement.

Lemma 9. *Let $K \subset D$ be a compact set, $u_T \to \infty$ and $u_T \ll T^2$ as $T \to \infty$. Then, under hypotheses of Lemma 4 for the parameter α and sequence $\mathfrak{a}$, $\widehat{Q}_{T,\alpha,\mathfrak{a}}$ converges weakly to*

$$m_{1H}\left\{\omega_1 \in \Omega_1 : \sup_{s\in K} \left|\zeta\left(s, \frac{a}{b}, \omega_1; \mathfrak{a}\right) - f(s)\right| \in A\right\}, \quad A \in \mathcal{B}(\mathbb{R}),$$

as $T \to \infty$.

4 Proof of universality

Let $G_n(x)$, $n \in \mathbb{N}$ and $G(x)$ be the distribution functions. We recall that G_n converges weakly to G if

$$\lim_{n\to\infty} G_n(x) = G(x)$$

for every continuity point x of $G(x)$. Moreover, every distribution function has no more than a countable set of discontinuity points.

It is well known that the weak convergence of probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is equivalent to that of the corresponding distribution functions.

Proof. *(Proof of Theorem 2).* In [3], it is obtained that the support of the measure $P_{\zeta,\alpha,\mathfrak{a}}$ is the space $H(D)$. Therefore,

$$P_{\zeta,\alpha,\mathfrak{a}}(\mathcal{G}_\varepsilon) > 0, \tag{4.1}$$

where

$$\mathcal{G}_\varepsilon = \left\{ g \in H(D) : \sup_{s\in K} |g(s) - p(s)| < \frac{\varepsilon}{2} \right\},$$

and $p(s)$ is a polynomial. In view of the Mergelyan theorem on the approximation of analytic functions by polynomials [5], there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}.$$

This shows that

$$\mathcal{G}_\varepsilon \subset \left\{ g \in H(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon \right\} \stackrel{def}{=} \widehat{\mathcal{G}}_\varepsilon.$$

Thus, in view of (4.1), we have $P_{\zeta,\alpha,\mathfrak{a}}(\widehat{\mathcal{G}}_\varepsilon) > 0$, i. e.,

$$m_H \left\{ \omega \in \Omega : \sup_{s\in K} |\zeta(s,\alpha,\omega;\mathfrak{a}) - f(s)| < \varepsilon \right\} > 0.$$

Define the distribution function

$$G_{T,\alpha,\mathfrak{a}}(\varepsilon) = \frac{1}{T}\text{meas} \left\{ \tau \in [0,T] : \sup_{s\in K} |\zeta_{u_T}(s+i\tau,\alpha;\mathfrak{a}) - f(s)| < \varepsilon \right\}.$$

Then Lemma 8 implies that

$$\lim_{T\to\infty} G_{T,\alpha,\mathfrak{a}}(\varepsilon) = m_H \left\{ \omega \in \Omega : \sup_{s\in K} |\zeta(s,\alpha,\omega;\mathfrak{a}) - f(s)| < \varepsilon \right\}$$

for all continuity points ε of the latter limit distribution function, i. e., for all but at most countably many $\varepsilon > 0$. □

Proof. *(Proof of Theorem 3).* Let V be such that $K \subset D_V$. The support of the measure $P^1_{\zeta,\alpha,\mathfrak{a}}$ is the set $H(D_V)$ [4]. Therefore, the analogue of the set $\mathcal{G}_\varepsilon$, the set

$$\mathcal{G}_{\varepsilon,V} = \left\{ g \in H(D_V) : \sup_{s\in K} |g(s) - p(s)| < \frac{\varepsilon}{2} \right\},$$

is an open neighbourhood of an element of the support of the measure $P^1_{\zeta,\alpha,\mathfrak{a}}$. Thus $P^1_{\zeta,\alpha,\mathfrak{a}}(\mathcal{G}_{\varepsilon,V}) > 0$, and the further proof runs in the same way as that of Theorem 2 by using Lemma 9. □

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