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Eigenvalues of Sturm-Liouville Problems with Eigenparameter Dependent Boundary and Interface Conditions

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Abstract. In this paper, a regular discontinuous Sturm-Liouville problem which contains eigenparameter in both boundary and interface conditions is investigated. Firstly, a new operator associated with the problem is constructed, and the self-adjointness of the operator in an appropriate Hilbert space is proved. Some properties of eigenvalues are discussed. Finally, the continuity of eigenvalues and eigenfunctions is investigated, and the differential expressions in the sense of ordinary or Fréchet of the eigenvalues concerning the data are given.

Keywords: Sturm-Liouville problems, interface conditions, continuity, differential expression.

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1 Introduction

For the classical Sturm-Liouville problems, the eigenparameter λ generally only appears in differential equation. However, for many practical problems in physics, engineering, and other fields, the corresponding mathematical models require that the eigenparameter λ not only appears in differential equation, but also in boundary conditions. For the classical Sturm-Liouville problems with eigenparameter dependent boundary conditions, there have been a sea of research results, (see, for example, [3, 7, 8, 9] and references cited therein).

Recently, there has been increasing interest in Sturm-Liouville problems with discontinuity and eigenparameter dependent boundary conditions, that is, a discontinuous point appears in the interval, namely, problems are considered in two disjoint intervals. To deal with such problems, some conditions are imposed on these points, which are also called transmission conditions, interface conditions etc [20, 30]. Such problems arise in many problems of physics and mechanics [14,24]. For example, heat and mass transfer problems. For such problems, many researchers study the asymptotic of eigenvalues and eigenfunctions, inverse problems, the completeness of eigenfunctions and resolvent and so on. Many important results have been obtained for this kind of problems (see [1, 2, 4, 13, 15, 16, 17, 18, 19, 20, 22, 23, 25, 27, 30]).

The dependence of eigenvalues for regular or singular Sturm-Liouville operators has been well investigated in recent years, see [26, 28, 29]. This kind of problems consists of a certain second-order differential expression with self-adjoint boundary conditions, and study the continuity and differentiability of eigenvalues with respect to the given parameters appear in the equation and boundary conditions. These results provided a fundamental to the numerical computation of spectrum, for example, the codes SLEUTH and SLEIGN2 constructed by Greenberg, Bailey et al. [6].

Kong and Zettl in [12] showed the continuity of eigenvalue of regular Sturm-Liouville problems and each eigenvalue is differentiable with respect to a given parameter. Such a problem gained various generalizations since then, for example, Sturm-Liouville operators with interface conditions, third-order and fourth-order differential operators, and general even order case, etc. [10, 13, 21, 30]. Particularly, in recent papers, we generalized these results to third-order differential operators with eigen-dependent boundary conditions [5], and Ao et al. considered the case of third-order differential operators with discontinuity [27].

Inspired by the above results, we consider the following differential equation

$$ly{:=}-(py')'+qy{=}\lambda\omega y,\ on\ I=[a,c)\cup(c,b],\ -\infty<a<c<b<\infty, \tag{1.1}$$
$$L_1y:=\lambda(\alpha_1'y(a)-\alpha_2'(py')(a))-(\alpha_1y(a)-\alpha_2(py')(a))=0, \tag{1.2}$$
$$L_2y:=\lambda(\beta_1'y(b)-\beta_2'(py')(b))-(\beta_2(py')(b)-\beta_1y(b))=0, \tag{1.3}$$
$$L_3y:=y(c+)-\gamma_1y(c-)=0, \tag{1.4}$$
$$L_4y:=\lambda\delta y(c-)-[(\gamma_2(py')(c-)-(py')(c+)]=0, \tag{1.5}$$

here λ is a complex spectral parameter;

$$\frac{1}{p},q,\omega{\in}L(I)\text{and have finite left and right limits at}\,c,\ p,\omega>0\text{a.e. on}\,I, \tag{1.6}$$
$$\alpha_i,\beta_i,\alpha_i',\beta_i',\gamma_i,\delta\in\mathbb{R},\ i=1,2\text{ and }\zeta:=\gamma_1\gamma_2>0,$$
$$\rho_1=\alpha_1'\alpha_2-\alpha_1\alpha_2'>0,\ \rho_2=\beta_1'\beta_2-\beta_1\beta_2'>0. \tag{1.7}$$

Then we consider a discontinuous Sturm-Liouville problem with eigenparameter contained in both boundary and interface conditions (1.1)–(1.5). Noting that the problem (1.1)–(1.5) can be gotten by using the method of separation of variables to various physical problems in some special cases. For example, some boundary-value problems arising in diffraction problems etc. (see [14,24]). Using operator theory and analysis technique, the problem (1.1)–(1.5) is transferred to a self-adjoint operator in a proper Hilbert space. We introduce some properties of eigenvalues and eigenfunctions of this operator. Moreover, we introduce the dependence of eigenvalues of the problem (1.1)–(1.5) with respect to the parameters given in the problem.

The outline of this paper is arranged as follows: in Section 2, a new Hilbert space related to the problem is constructed, and a new operator is defined in this space such that the eigenvalues of the problem are consistent with the eigenvalues of this operator. The fundamental solutions are constructed, and it is proved that the new operator is self-adjoint, and the simplicity of the eigenvalue is proved. The continuity of eigenvalues and eigenfunctions are proved in Section 3, followed by the differential expressions of eigenvalues about each parameter are given in Section 4.

2 Preliminaries and basic results

In this section, we define a new Hilbert space $H = H_1 \bigoplus \mathbb{C}^3$, where $H_1 = L^2[a,c) \bigoplus L^2(c,b]$, $\mathbb{C}^3$ denotes three-dimension complex vector space. And the inner product in H is defined as

$$\langle M, N\rangle = \gamma_1\gamma_2 \int_a^c m\overline{n}\omega dx + \int_c^b m\overline{n}\omega dx + \frac{\gamma_1\gamma_2}{\rho_1} m_1\overline{n}_1 + \frac{1}{\rho_2} m_2\overline{n}_2 + \frac{\gamma_1}{\delta} m_3\overline{n}_3,$$

for $M := (m(x), m_1, m_2, m_3)^T,\ N := (n(x), n_1, n_2, n_3)^T \in H$.
Then, in the Hilbert space H, we define an operator $\mathcal{T}$ with domain

$$\begin{aligned} D(\mathcal{T}) = \{Y = (y, y_1, y_2, y_3)^T | \quad & y, py' \in AC(I), \quad ly \in L^2(I), \\ & y_1 = \alpha_1' y(a) - \alpha_2'(py')(a),\ y_2 = \beta_1' y(b) - \beta_2'(py')(b),\ y_3 = \delta y(c-)\} \end{aligned}$$

and the rule

$$\mathcal{T}Y = \begin{pmatrix} \frac{1}{\omega} ly \\ \alpha_1 y(a) - \alpha_2(py')(a) \\ \beta_2(py')(b) - \beta_1 y(b) \\ \gamma_2(py')(c-) - (py')(c+) \end{pmatrix} \text{ for } Y = \begin{pmatrix} y \\ \alpha_1' y(a) - \alpha_2'(py')(a) \\ \beta_1' y(b) - \beta_2'(py')(b) \\ \delta y(c-) \end{pmatrix} \in D(\mathcal{T}).$$

For convenience, we shall use the following notations:

$$\begin{aligned} N_c(y) &= \gamma_2(py')(c-) - (py')(c+), & N_c'(y) &= \delta y(c-), \\ Q_a(y) &= \alpha_1 y(a) - \alpha_2(py')(a), & Q_a'(y) &= \alpha_1' y(a) - \alpha_2'(py')(a), \\ Q_b(y) &= \beta_1 y(b) - \beta_2(py')(b), & Q_b'(y) &= \beta_1' y(b) - \beta_2'(py')(b). \end{aligned}$$

So, the problem (1.1)–(1.5) can be transformed into the following form

$$\mathcal{T}Y = \lambda Y.$$

Then we have the following lemmas.

Lemma 1. *The eigenvalues of the problem (1.1)–(1.5) are consistent with the eigenvalues of the operator $\mathcal{T}$, and the eigenfunctions are the first component of the corresponding eigenfunctions of operator $\mathcal{T}$.*

Lemma 2. *$D(\mathcal{T})$ is dense in H.*

Proof. Let $M := (m(x), m_1, m_2, m_3)^T \in H$, $H \perp D(\mathcal{T})$ and C_0^∞ be a functional set such that

$$\psi(x) = \begin{cases} \psi_1(x), \ x \in [a, c), \\ \psi_2(x), \ x \in (c, b] \end{cases}$$

for $\psi_1(x) \in C_0^\infty[a, c)$, $\psi_2(x) \in C_0^\infty(c, b]$. Since $C_0^\infty \bigoplus 0 \bigoplus 0 \bigoplus 0 \subset D(\mathcal{T})(0 \in \mathbb{C})$, any $N = (n(x), 0, 0, 0)^T \in C_0^\infty \bigoplus 0 \bigoplus 0 \bigoplus 0$ is orthogonal to M, that is,

$$\langle M, N \rangle = \gamma_1\gamma_2 \int_a^c m\overline{n}\omega dx + \int_c^b m\overline{n}\omega dx = 0.$$

We can obtain that $m(x)$ is orthogonal to C_0^∞ in H_1, so $m(x) = 0$. So for all $T = (t(x), t_1, 0, 0)^T \in D(\mathcal{T})$, $\langle M, T \rangle = \frac{\gamma_1\gamma_2}{\rho_1} m_1 \overline{t}_1 = 0$. Thus, $m_1 = 0$ since t_1 can be chosen arbitrary. Further, for all $J = (j(x), j_1, j_2, 0)^T \in D(\mathcal{T})$, $\langle M, J \rangle = \frac{1}{\rho_2} m_2 \overline{j}_2 = 0$. Thus, $m_2 = 0$ since j_2 can be chosen arbitrary. Finally, for all $K = (k(x), k_1, k_2, k_3)^T \in D(\mathcal{T})$, $\langle M, K \rangle = \frac{\gamma_1}{\delta} m_3 \overline{k}_3 = 0$, thus we can obtain $m_3 = 0$. So, $m = (0, 0, 0, 0)^T$. □

Lemma 3. *Linear operator $\mathcal{T}$ is symmetric.*

Proof. Let $M, N \in D(\mathcal{T})$. Integration by parts we have

$$\begin{aligned} \langle \mathcal{T}M, N \rangle - \langle M, \mathcal{T}N \rangle &= \gamma_1\gamma_2 W(m, \overline{n}; c-) - \gamma_1\gamma_2 W(m, \overline{n}; a) + W(m, \overline{n}; b) \\ &- W(m, \overline{n}; c+) + \frac{\gamma_1\gamma_2}{\rho_1}[Q_a(m)Q_a'(\overline{n}) - Q_a'(m)Q_a(\overline{n})] + \frac{1}{\rho_2}\big[- Q_b(m)Q_b'(\overline{n}) \\ &+ Q_b'(m)Q_b(\overline{n})\big] + \frac{\gamma_1}{\delta}[N_c(m)N_c'(\overline{n}) - N_c(\overline{n})N_c'(m)], \end{aligned} \tag{2.1}$$

here we use $W(m, n; x)$ to denote the Wronskians $m(x)(pn')(x) - (pm')(x)n(x)$. From (1.4)–(1.5), we get

$$N_c(m)N_c'(\overline{n}) - N_c(\overline{n})N_c'(m) = \frac{\delta}{\gamma_1}[W(m, \overline{n}; c+) - \gamma_1\gamma_2 W(m, \overline{n}; c-)]. \tag{2.2}$$

In addition, it is easy to prove that

$$Q_a(m)Q_a'(\overline{n}) - Q_a'(m)Q_a(\overline{n}) = \rho_1 W(m, \overline{n}; a), \tag{2.3}$$

$$Q_b'(m)Q_b(\overline{n}) - Q_b(m)Q_b'(\overline{n}) = -\rho_2 W(m, \overline{n}; b). \tag{2.4}$$

Substituting (2.2)–(2.4) into (2.1) yields $\langle \mathcal{T}M, N \rangle = \langle M, \mathcal{T}N \rangle$. So $\mathcal{T}$ is symmetric. □

Theorem 1. *Linear operator $\mathcal{T}$ is a self-adjoint operator in H.*

Proof. Since $\mathcal{T}$ is symmetric, we just have to prove that if $\langle \mathcal{T}M, W\rangle = \langle M, T\rangle$ for all $M = (m(x), Q'_a(m), Q'_b(f), N'_c(m))^T \in D(\mathcal{T})$, then $W \in D(\mathcal{T})$, and $\mathcal{T}W = T$, where $W = (w(x), w_1, w_2, w_3)^T, T = (t(x), t_1, t_2, t_3)^T$, i.e.,

(1) $w(x), (pw')(x) \in AC(I)$ and $lw \in L^2(I)$;

(2) $w_1 = \alpha'_1 w(a) - \alpha'_2(pw')(a), w_2 = \beta'_1 w(b) - \beta'_2(pw')(b), w_3 = \delta w(c-)$;

(3) $L_3(w) = 0$;

(4) $t(x) = \frac{1}{\omega} lw(x)$;

(5) $t_1 = \alpha_1 w(a) - \alpha_2(pw')(a),\ t_2 = \beta_2(pw')(b) - \beta_1 w(b),\ t_3 = \gamma_2(pw')(c-) - (pw')(c+)$.

For $\forall M \in C_0^\infty \bigoplus 0 \bigoplus 0 \bigoplus 0 \subset D(\mathcal{T})$ such that

$$\gamma_1\gamma_2 \int_a^c (lm)\overline{w}\omega dx + \int_c^b (lm)\overline{w}\omega dx = \gamma_1\gamma_2 \int_a^c m\bar{t}\omega dx + \int_c^b m\bar{t}\omega dx,$$

that is, $\langle lm, w\rangle_1 = \langle m, t\rangle_1$. According to classical Sturm-Liouville theory, (1) and (4) hold. By (4), equation $\langle \mathcal{T}M, W\rangle = \langle M, T\rangle$, for all $M \in D(\mathcal{T})$, becomes

$$\frac{\gamma_1\gamma_2}{\rho_1}[Q'_a(m)\overline{t_1} - Q_a(m)\overline{w_1}] + \frac{1}{\rho_2}[Q'_b(m)\overline{t_2} + Q_b(m)\overline{w_2}] + \frac{\gamma_1}{\delta}\left[N'_c(m)\overline{t_3}\right.$$
$$\left. - N_c(m)\overline{w_3}\right] = \gamma_1\gamma_2[W(m, \overline{w}; c-) - W(m, \overline{w}; a)] + [W(m, \overline{w}; b) - W(m, \overline{w}; c+)].$$

By Naimark's Patching Lemma, there is an $M \in D(\mathcal{T})$ satisfying

$$m(b) = (pm')(b) = m(c \pm 0) = (pm')(c \pm 0) = 0, m(a) = \alpha'_2, (pm')(a) = \alpha'_1.$$

Thus, $w_1 = \alpha'_1 w(a) - \alpha'_2(pw')(a)$. Next, choose $M \in D(\mathcal{T})$ such that

$$m(a) = (pm')(a) = m(c \pm 0) = (pm')(c \pm 0) = 0, m(b) = \beta'_2, (pm)'(b) = \beta'_1.$$

Then, $w_2 = \beta'_1 w(b) - \beta'_2(pw)'(b)$. Finally, choose $M \in D(\mathcal{T})$ such that

$$m(a) = (pm')(a) = m(b) = (pm')(b) = m(c\pm) = (pm')(c+) = 0, (pm')(c-) = \alpha_1.$$

We have $w_3 = \delta w(c-)$, hence (2) is true. (5) can be proved in the same way. Further, let $M \in D(\mathcal{T})$ and satisfy

$$m(a) = (pm')(a) = m(b) = (pm')(b) = m(c\pm) = (pm')(c-) = 0, (pm')(c+) = \beta_1.$$

We have $w(c+) - \gamma_1 w(c-) = 0$. Consequently, the operator $\mathcal{T}$ is self-adjoint. □

Corollary 1. All eigenvalues of the problem (1.1)–(1.5) are real, and for two different eigenvalues, the corresponding eigenfunctions $m(x)$ and $n(x)$ are orthogonal in the following sense

$$\gamma_1\gamma_2 \int_a^c m\overline{n}\omega dx + \int_c^b m\overline{n}\omega dx + \frac{\gamma_1\gamma_2}{\rho_1} Q'_a(m)Q'_a(\overline{n})$$
$$+ \frac{1}{\rho_2} Q'_b(m)Q'_b(\overline{n}) + \frac{\gamma_1}{\delta} N'_c(m)N'_c(\overline{n}) = 0.$$

In what follows, we define two fundamental solutions of Equation (1.1)

$$\theta(x,\lambda)=\begin{cases}\theta_1(x,\lambda), & x\in[a,c),\\ \theta_2(x,\lambda), & x\in(c,b];\end{cases}\qquad \eta(x,\lambda)=\begin{cases}\eta_1(x,\lambda), & x\in[a,c),\\ \eta_2(x,\lambda), & x\in(c,b],\end{cases}$$

where $\theta_1(x,\lambda)$ is the solution of Equation (1.1) on the interval $[a,c)$, satisfying the initial conditions

$$\begin{pmatrix}\theta_1(a,\lambda)\\ (p\theta_1')(a,\lambda)\end{pmatrix}=\begin{pmatrix}\lambda\alpha_2'-\alpha_2\\ \lambda\alpha_1'-\alpha_1\end{pmatrix}. \tag{2.5}$$

We can define the solution $\theta_2(x,\lambda)$ of Equation (1.1) on the interval $(c,b]$ by the initial conditions

$$\begin{pmatrix}\theta_2(c+)\\ \theta_2'(c+)\end{pmatrix}=\begin{pmatrix}\gamma_1\theta_1(c-)\\ \gamma_2\theta_1'(c-)-\lambda\delta\theta_1(c-)\end{pmatrix}. \tag{2.6}$$

Similarly, define the solution $\eta_2(x,\lambda)$ and $\eta_1(x,\lambda)$ by the initial conditions

$$\begin{pmatrix}\eta_2(b)\\ (p\eta_2')(b)\end{pmatrix}=\begin{pmatrix}\lambda\eta_2'+\beta_2\\ \lambda\eta_1'+\beta_1\end{pmatrix}, \tag{2.7}$$

$$\begin{pmatrix}\eta_1(c-)\\ \eta_1'(c-)\end{pmatrix}=\begin{pmatrix}\frac{\eta_2(c+)}{\gamma_1}\\ \frac{\gamma_1\eta_2'(c+)+\lambda\delta\eta_2(c+)}{\gamma_1\gamma_2}\end{pmatrix}. \tag{2.8}$$

Let us consider the Wronskians

$$w_i(\lambda)=W_\lambda(\theta_i,\eta_i;x)=\theta_i(p\eta_i')-(p\theta_i')\eta_i,\,(i=1,2),$$

where w_1,w_2 are entire functions of λ on the interval $[a,c)$ and $(c,b]$.

Lemma 4. *For each $\lambda\in\mathbb{C}$, $\gamma_1\gamma_2w_1(\lambda)=w_2(\lambda)$.*

Proof. According to (2.5)–(2.8), by simply calculation we can get

$$W_\lambda(\theta_2,\eta_2;c+0)=\gamma_1\gamma_2W_\lambda(\theta_1,\eta_1;c-0),$$

so $\gamma_1\gamma_2w_1(\lambda)=w_2(\lambda)$ for each $\lambda\in\mathbb{C}$. □

Now, let $w(\lambda):=w_1(\lambda)=\frac{1}{\gamma_1\gamma_2}w_2(\lambda)$.

Theorem 2. *The eigenvalues of the problem (1.1)–(1.5) coincide with the zeros of $w(\lambda)$.*

Proof. Using similar methods proposed in [1], we can prove the assertion. □

Corollary 2. Suppose $\lambda=\lambda_0$ is an eigenvalue, then $\theta(x,\lambda_0)$ and $\eta(x,\lambda_0)$ are linearly independent.

Theorem 3. *The eigenvalues of the problem (1.1)–(1.5) are analytically simple.*

Proof. Let $\lambda = \sigma + it$. For simplicity, let $\theta = \theta(x, \lambda)$, $\theta_{1\lambda} = \frac{\partial \theta_1}{\partial \lambda}$, $(p\theta_1')_\lambda = \frac{\partial (p\theta_1')}{\partial \lambda}$. Differentiating the equation $l\eta = \lambda\omega\eta$ with respect to λ, we have

$$l\eta_\lambda = \omega\eta + \lambda\omega\eta_\lambda. \tag{2.9}$$

Using integration by parts, we have

$$\left\langle \frac{l\eta_\lambda}{\omega}, \theta \right\rangle_1 - \left\langle \eta_\lambda, \frac{l\theta}{\omega} \right\rangle_1 = \gamma_1\gamma_2(\eta_{1\lambda}(p\overline{\theta_1'}) - (p\eta_1')_\lambda\overline{\theta_1})|_a^c + (\eta_{2\lambda}(p\overline{\theta_2'}) - (p\eta_2')_\lambda\overline{\theta_2})|_c^b. \tag{2.10}$$

Substituting (2.9) and $l\theta = \lambda\omega\theta$ into the left side of (2.10), we have

$$\lambda\langle\eta_\lambda, \theta\rangle_1 + \langle\eta, \theta\rangle_1 - \langle\eta_\lambda, \lambda\theta\rangle_1 = \langle\eta, \theta\rangle_1 + 2it\langle\eta_\lambda, \theta\rangle_1.$$

Furthermore,

$$\begin{aligned}\gamma_1\gamma_2(\eta_{1\lambda}(p\overline{\theta_1'}) - (p\eta_1')_\lambda\overline{\theta_1})|_a^c + (\eta_{2\lambda}(p\overline{\theta_2'}) - (p\eta_2')_\lambda\overline{\theta_2})|_c^b = (\beta_2'(p\overline{\theta}_2')(b) \\ - \beta_1'\overline{\theta}_2(b)) - \gamma_1\gamma_2[(\lambda\alpha_1' - \alpha_1)\eta_{1\lambda}(a) - (\lambda\alpha_2' - \alpha_2)(p\eta_1')_\lambda(a)].\end{aligned}$$

Note that

$$w'(\lambda) = \alpha_2'(p\eta_1')(a) - \alpha_1'\eta_1(a) + (\lambda\alpha_2' - \alpha_2)(p\eta_1')_\lambda(a) - (\lambda\alpha_1' - \alpha_1)\eta_{1\lambda}(a),$$

so, Equation (2.10) becomes

$$\begin{aligned}\gamma_1\gamma_2 w'(\lambda) &= \langle\eta, \theta\rangle_1 + 2it\langle\eta_\lambda, \theta\rangle_1 - \beta_2'(p\overline{\theta}_2')(b) + \beta_1'\overline{\theta}_2(b) \\ &\quad + \gamma_1\gamma_2[\alpha_2'(p\eta_1')(a) - \alpha_1'\eta_1(a).\end{aligned} \tag{2.11}$$

Then, let ξ be any zero of $w(\lambda)$. As $w(\xi) = 0$, we have $\theta_1(x, \xi) = c_1\eta_1(x, \xi)$ ($c_1 \neq 0$), $\theta_2(x, \xi) = c_2\eta_2(x, \xi)$ ($c_2 \neq 0$), where c_1, $c_2 \in \mathbb{C}$. From

$$\theta_2(c+, \xi) = \gamma_1\theta_1(c-, \xi) = c_1\gamma_1\eta_1(c-, \xi) = c_1\eta_2(c+, \xi)$$

we have $c_1 = c_2 \neq 0$. Therefore, a short calculation (2.11) becomes

$$\gamma_1\gamma_2 w'(\xi) = \bar{c}_1(\gamma_1\gamma_2 \int_a^c |\eta_1(x, \xi)|^2\omega(x)dx + \int_c^b |\eta_2(x, \xi)|^2\omega(x)dx + \gamma_1\gamma_2\rho_1 c_0 + \rho_2).$$

Here, $\rho_1 > 0$, $\rho_2 > 0$, $\gamma_1\gamma_2 > 0$, $c_0 > 0$ and $\bar{c}_1 > 0$, hence $w'(\xi) \neq 0$. Therefore, the analytic multiplicity of ξ is simple. □

Corollary 3. The eigenvalues of problem (1.1)–(1.5) are bounded below and can be ordered to satisfy

$$-\infty < \lambda_0 < \lambda_1 < \lambda_2 < ..., \quad \lambda_n \to +\infty \quad as \quad n \to +\infty,$$

moreover, they are countably infinite and can cluster only at ∞.

Proof. The proof can be completed by using similar methods in [4], hence we omit it here. □

3 Continuity of eigenvalues and eigenfunctions

In this section, we prove continuity of the eigenvalues and normalized eigenfunctions for the problem (1.1)–(1.5).

Denote

$$A = \begin{pmatrix} \alpha_1' & \alpha_1 \\ \alpha_2' & \alpha_2 \end{pmatrix},\ B = \begin{pmatrix} \beta_1' & \beta_1 \\ \beta_2' & \beta_2 \end{pmatrix}.$$

Consider a Banach space

$$\mathbf{B} = L^1(I) \bigoplus L^1(I) \bigoplus L^1(I) \bigoplus M_{2\times 2}(\mathbb{R}) \bigoplus M_{2\times 2}(\mathbb{R}) \bigoplus \mathbb{R}^7$$

equipped with the norm

$$\begin{aligned} \|Z\| \;\; := \;\; & \int_a^c (\frac{1}{|p|} + |q| + |\omega|)dx + \int_c^b (\frac{1}{|p|} + |q| + |\omega|)dx + \|A\| + \|B\| \\ & + |\gamma_1| + |\gamma_2| + |\delta| + |a| + |b| + |c-| + |c+| \end{aligned}$$

for any $Z = (\frac{1}{p}, q, \omega, A, B, \gamma_1, \gamma_2, \delta, a, b, c-, c+) \in \mathbf{B}$.

Let $\Omega = \{Z \in \mathbf{B} : \text{(1.6)–(1.7) hold}\}$. When considering the variables in the parameter matrix of boundary conditions separately, we use the symbol $\Omega_1 = \{Z = (\frac{1}{p}, q, \omega, \alpha_1, \alpha_2, \alpha_1', \alpha_2', \beta_1, \beta_2, \beta_1', \beta_2', \gamma_1, \gamma_2, \delta, a, b, c-, c+) \in \mathbf{B}_1 :$ (1.6)–(1.7) hold$\}$, where

$$\mathbf{B}_1 := L^1(I) \bigoplus L^1(I) \bigoplus L^1(I) \bigoplus \mathbb{R}^{15}.$$

Then we get the continuous dependence of the eigenvalues on the parameters in the SL problems.

Theorem 4. *Let* $\widetilde{Z} = (\frac{1}{\tilde{p}}, \tilde{q}, \tilde{\omega}, \tilde{A}, \tilde{B}, \tilde{\gamma_1}, \tilde{\gamma_2}, \tilde{\delta}, \tilde{a}, \tilde{b}, \tilde{c-}, \tilde{c+})$ *and* $\lambda(Z)$ *be an eigenvalue of (1.1)–(1.5) with* Z*. Then,* λ *is continuous at* $\widetilde{Z}$*. That is, give any* $\varepsilon > 0$ *sufficiently small, there exists a* $\sigma > 0$ *such that* $|\lambda(\widetilde{Z}) - \lambda(Z)| < \varepsilon$ *if* $Z = (\frac{1}{p}, q, \omega, A, B, \gamma_1, \gamma_2, \delta, a, b, c-, c+)$ *satisfies*

$$\begin{aligned} \left\| Z - \widetilde{Z} \right\| = & \int_a^c \Big(\Big| \frac{1}{p} - \frac{1}{\tilde{p}} \Big| + |q - \tilde{q}| + |\omega - \tilde{\omega}| \Big) dx + \int_c^b \Big(\Big| \frac{1}{p} - \frac{1}{\tilde{p}} \Big| + |q - \tilde{q}| \\ & + |\omega - \tilde{\omega}|) dx + \left\| A - \tilde{A} \right\| + \left\| B - \tilde{B} \right\| + |\gamma_1 - \tilde{\gamma_1}| + |\gamma_2 - \tilde{\gamma_2}| \\ & + |\delta - \tilde{\delta}| + |a - \tilde{a}| + |b - \tilde{b}| + |(c-) - (\tilde{c-})| + |(c+) - (\tilde{c+})| < \sigma. \end{aligned}$$

Proof. By Theorem 2, λ is an eigenvalue of (1.1)–(1.5) if and only if $w(Z, \lambda(Z)) = 0$, for any $Z \in \Omega$. It is easy to get that $w(Z, \lambda)$ is an entire function of λ and is continuous in Z. By Corollary 3, we get that $\lambda(Z)$ is an isolated eigenvalue, then $w(Z, \lambda)$ is not a constant. By the well-known theorem on continuity of the roots of an equation, the statements follows. □

Definition 1. By a normalized eigenvector $M = (m, m_1, m_2, m_3)^T$ of the problem (1.1)–(1.5), we mean M satisfies

$$\langle M, M\rangle = \langle (m, m_1, m_2, m_3)^T, ((m, m_1, m_2, m_3)^T\rangle = \gamma_1\gamma_2 \int_a^c m\bar{m}w dx$$
$$+ \int_c^b m\bar{m}w dx + \frac{\gamma_1\gamma_2}{\rho_1} m_1\overline{m_1} + \frac{1}{\rho_2} m_2\overline{m_2} + \frac{\gamma_1}{\delta} m_3\overline{m_3} = 1.$$

Next, we give the continuity of the corresponding eigenvector.

Theorem 5. *Let $\lambda(\widetilde{Z})$ be an eigenvalue of problems (1.1)–(1.5) with $\widetilde{Z} \in \Omega$ and $(f, f_1, f_2, f_3) \in H$ be a normalized eigenvector for $\widetilde{Z}$. Then there exists a normalized eigenvector $(g, g_1, g_2, g_3) \in H$ for $\lambda(Z)$ with $Z \in \Omega$, which is specified in Theorem 4, such that when $Z \to \widetilde{Z} \in \Omega$, we have*

$$g(\cdot, Z) \to f(\cdot, \widetilde{Z}), \quad (pg')(\cdot, Z) \to (pf')(\cdot, \widetilde{Z})$$

and $g_1 \to f_1, \quad g_2 \to f_2, \quad g_3 \to f_3$ all uniformly on $[a, c) \cup (c, b]$.

Proof. As $\lambda(\widetilde{Z})$ is simple, there exists a neighborhood M of $\widetilde{Z}$ such that $\lambda(Z)$ is simple for any $Z \in \Omega$. For each $Z \in \Omega$, choose an eigenfunction $v = v(\cdot, Z)$ of $\lambda(Z)$ satisfying

$$\|V(c_0, Z)\| = |v(c_0, Z)| + |(pv')(c_0, Z)| = 1, v(x, Z) > 0$$

for some $c_0 \in [a, c) \cup (c, b]$ and x near c_0, where $V(\cdot, Z) = (v(\cdot, Z), (pv')(\cdot, Z))$. In the following we prove

$$V(c_0, Z) \to V(c_0, \widetilde{Z}), \quad Z \to \widetilde{Z}, \quad Z \in \Omega. \tag{3.1}$$

If (3.1) does not holds, then there exists a sequence $Z_k \to \widetilde{Z}$ such that

$$V(c_0, Z_k) \to Y, \quad Z_k \to \widetilde{Z}, \quad Z \in \Omega,$$

where Y and $V(c_0, \widetilde{Z})$ are linearly independent vectors. Let $W(x)$ be the vector solutions of (1.1) with $Z = \widetilde{Z}, \lambda = \lambda(\widetilde{Z})$ and the initial condition $W(c_0) = Y$. Therefore, $V(x, Z_k) \to W(x)$ uniformly on $[a, c) \cup (c, b]$. In particular,

$$V(a, Z_k) \to W(a), V(b, Z_k) \to W(b), V(c-, Z_k) \to W(c-), V(c+, Z_k) \to W(c+).$$

Since $Y(\cdot, Z_k)$ satisfies the conditions

$$A_{\lambda(Z_k)} Y(a, Z_k) + B_{\lambda(Z_k)} Y(b, Z_k) = 0, \quad Y(c+, Z_k) = C_{\lambda(Z_k)} Y(c-, Z_k).$$

Taking the limit $k \to \infty$, we have

$$A_{\lambda(\widetilde{Z})} W(a) + B_{\lambda(\widetilde{Z})} W(b) = 0, \quad W(c+) = C_{\lambda(\widetilde{Z})} W(c-).$$

Therefore, $W(x)$ is a vector eigenfunction for $Z = \widetilde{Z}, \lambda = \lambda(\widetilde{Z})$, which contradicts that $\lambda(\widetilde{Z})$ is simple. Thus, (3.1) holds.

Therefore, we have

$$v_1(Z) \to v_1(\widetilde{Z}), \quad v_2(Z) \to v_2(\widetilde{Z}), \quad v_3(Z) \to v_3(\widetilde{Z}), \quad as \quad Z \to \widetilde{Z}.$$

Let

$$\begin{aligned}
(g, g_1, g_2, g_3)^T &= \frac{(v(x,Z), v_1(Z), v_2(Z), v_3(Z))^T}{\|(v(x,Z), v_1(Z), v_2(Z), v_3(Z))^T\|},\\
(f, f_1, f_2, f_3)^T &= \frac{(v(x,\widetilde{Z}), v_1(\widetilde{Z}), v_2(\widetilde{Z}), v_3(\widetilde{Z}))^T}{\|(v(x,\widetilde{Z}), v_1(\widetilde{Z}), u_2(\widetilde{Z}), v_3(\widetilde{Z}))^T\|},\\
pg' &= (pv')(x,Z)/\|(v(x,Z), y_1(Z), v_2(Z), v_3(Z))^T\|,\\
pf' &= (pv')(x,\widetilde{Z})/\|(v(x,\widetilde{Z}), v_1(\widetilde{Z}), v_2(\widetilde{Z}), v_3(\widetilde{Z}))^T\|.
\end{aligned}$$

Then, Theorem 5 holds. □

4 Differential expression of eigenvalues

In this section we show that the eigenvalues are differentiable functions of all the parameters of the problem.

DEFINITION 2. [12] A map T from a Banach space X into another Banach space Y is differentiable at a point $x \in X$ if there exists a bounded linear operator $dT_x : X \to Y$ such that for $h \in X$

$$|T(x+h) - T(x) - dT_x(h)| = o(h), as\, h \to 0.$$

Theorem 6. *Let $Z = (K, M, \gamma_1, \gamma_2, \delta, \frac{1}{p}, q, \omega) \in \Omega$ with $\lambda = \lambda(Z)$ be an eigenvalue of operator $\mathcal{T}$ connected with Z, and let (u, u_1, u_2, u_3) be a normalized eigenvector for $\lambda(Z)$. Then λ is differential with respect to all the parameters in Z, and more precisely, the derivative formulas of λ are given as follows:*

(1) Fix all the parameters of Z except the boundary condition (1.2) parameter matrix

$$K = \begin{pmatrix} \alpha_1' & \alpha_1 \\ \alpha_2' & \alpha_2 \end{pmatrix},$$

and let $\lambda(K) := \lambda(Z)$. Then,

$$d\lambda_K(L) = \gamma_1\gamma_2(u(a), -(pu)'(a))[(E - K(K+L)^{-1})] \begin{pmatrix} (p\overline{u'})(a) \\ \overline{u}(a) \end{pmatrix}$$

for all H satisfying $\det(K+L) = \det K = \rho_1$.

(2) Fix all the parameters of Z except the boundary condition (1.3) parameter matrix

$$M = \begin{pmatrix} \beta_1' & \beta_1 \\ \beta_2' & \beta_2 \end{pmatrix},$$

and let $\lambda(K) := \lambda(Z)$. Then,

$$d\lambda_M(L) = (u(b), -(pu)'(b))[-E + M(M+L)^{-1})] \begin{pmatrix} (p\overline{u'})(b) \\ \overline{u}(b) \end{pmatrix}$$

for all H satisfying $\det(M+L) = \det M = \rho_2$.

(3) Fix all the parameters of Z except p and let $\lambda(\frac{1}{p}) := \lambda(Z)$. Then, λ is Frechet differentiable and

$$d\lambda_{\frac{1}{p}}(h) = -\gamma_1\gamma_2 \int_a^c |pu'|^2 h dx - \int_c^b |pu'|^2 h dx,\ \ h \in L(I).$$

(4) Fix all the parameters of Z except q and let $\lambda(q) := \lambda(Z)$. Then, λ is Frechet differentiable and

$$d\lambda_q(h) = \gamma_1\gamma_2 \int_a^c |u|^2 h dx + \int_c^b |u|^2 h dx,\ \ h \in L(I).$$

(5) Fix all the parameters of Z except ω and let $\lambda(\omega) := \lambda(Z)$. Then, λ is Frechet differentiable and

$$d\lambda_\omega(h) = -\lambda(Z)[\gamma_1\gamma_2 \int_a^c |u|^2 h dx + \int_c^b |u|^2 h dx],\ \ h \in L(I).$$

Proof. Fix all but one of the parameters in Z and let $\lambda(\widetilde{Z})$ be the eigenvalue satisfying Theorem 6 when $\|Z - \widetilde{Z}\| \le \varepsilon$ for sufficiently small $\varepsilon > 0$. For the above five cases, we replace $\lambda(\widetilde{Z})$ by

$$\lambda(K+L), \lambda(M+L), \lambda(\frac{1}{p}+h), \lambda(q+h), \lambda(\omega+h).$$

Let (v, v_1, v_2, v_3) be the corresponding normalized eigenvector.

(1) By (1.1) we have

$$-(pu')' + qu = \lambda(K)\omega u, \tag{4.1}$$
$$-(p\overline{v}')' + q\overline{v} = \lambda(K+L)\omega\overline{v}. \tag{4.2}$$

It follows from (4.1) and (4.2) that

$$[\lambda(K+L) - \lambda(K)]u\overline{v}\omega = -[p\overline{v}')'u - (pu')'\overline{v}].$$

Integrating from a to c and c to b, then we have

$$\begin{aligned}
& [\lambda(K+L) - \lambda(K)](\gamma_1\gamma_2 \int_a^c u\overline{v}\omega dx + \int_c^b u\overline{v}\omega dx) \\
= \ & [u(a)(p\overline{v}')(a) - \overline{v}(a)(pu')(a)] - [u(c-)(p\overline{v}')(c-) - \overline{v}(c-)(pu')(c-)] \\
& + [u(c-)(p\overline{v}')(c+) - \overline{v}(c+)(pu')(c+)] - [u(b)(p\overline{v}')(b) - \overline{v}(b)(pu')(b)] \\
= \ & -\gamma_1\gamma_2[u,v]_a^{c-} - [u,v]_{c+}^b.
\end{aligned} \tag{4.3}$$

Let $K+L = \begin{pmatrix} \tilde{\alpha_1'} & \tilde{\alpha_1} \\ \tilde{\alpha_2'} & \tilde{\alpha_2} \end{pmatrix}$. Then, it follows from (1.2) that

$$\begin{aligned}
\lambda(K+L)[\tilde{\alpha_1'}\overline{v}(a) - \tilde{\alpha_2'}(p\overline{v}')(a)] &= \tilde{\alpha}_1\overline{v}(a) - \tilde{\alpha}_2(p\overline{v}')(a), \\
\lambda(K)[\alpha_1' u(a) - \alpha_2'(pu')(a)] &= \alpha_1 u(a) - \alpha_2(pu')(a).
\end{aligned}$$

Therefore,

$$[\lambda(K+L)-\lambda(K)]u_1\overline{v_1}\frac{\gamma_1\gamma_2}{\rho_1}=\frac{\gamma_1\gamma_2}{\rho_1}\big[(\alpha_1'u(a)-\alpha_2'(pu')(a)) \\ \times(\tilde{\alpha_1}\overline{v}(a)-\tilde{\alpha_2}(p\overline{v}')(a))-(\alpha_1u(a)-\alpha_2(pu')(a))(\tilde{\alpha_1'}\overline{v}(a)-\tilde{\alpha_2'}(p\overline{v}')(a))\big]. \quad (4.4)$$

It follows from (1.3) that

$$\begin{aligned}\lambda(K+L)[\beta_1'\overline{v}(b)-\beta_2'(p\overline{v}')(b)]=&\beta_2(p\overline{v}')(b)-\beta_1\overline{v}(b),\\ \lambda(K)[\beta_1'u(b)-\beta_2'(pu')(b)]=&\beta_2(pu')(b)-\beta_1u(b).\end{aligned}$$

Therefore,

$$\begin{aligned}&[\lambda(K+L)-\lambda(K)]u_2\overline{v_2}\frac{1}{\rho_2}=\frac{1}{\rho_2}[(\beta_1'u(b)-\beta_2'(pu')(b))(\beta_2(p\overline{v}')(b)-\beta_1\overline{v}(b))\\ &\quad-(\beta_2(pu')(b)-\beta_1u(b))(\beta_1'\overline{v}(b)-\beta_2'(p\overline{v}')(b))]\\ &=\frac{1}{\rho_2}[(\beta_1'\beta_2-\beta_1\beta_2')u(b)p\overline{v}'(b)-(\beta_1'\beta_2-\beta_1\beta_2')(pu')(b)\overline{v}(b)]=[u,v](b). \quad (4.5)\end{aligned}$$

It follows from (1.4) that

$$\begin{aligned}&\lambda(K+L)\delta\overline{v}(c-)=\gamma_2(p\overline{v}')(c-)-(p\overline{v}')(c+),\\ &\lambda(K)\delta u(c-)=\gamma_2(pu')(c-)-(pu')(c+).\end{aligned}$$

Therefore,

$$\begin{aligned}&[\lambda(K+L)-\lambda(K)]u_3\overline{v_3}\frac{\gamma_1}{\delta}=\frac{\gamma_1}{\delta}[(\gamma_2(p\overline{v}')(c-)-(p\overline{v}')(c+))\delta u(c-)\\ &\quad-(\gamma_2(pu')(c-)-(pu')(c+))\delta\overline{v}(c-)]=\gamma_1\gamma_2[u,v](c-)-[u,v](c+). \quad (4.6)\end{aligned}$$

By (4.3)–(4.6) we have the following

$$\begin{aligned}&[\lambda(K+L)-\lambda(K)]\Big(\gamma_1\gamma_2\int_a^c u\overline{v}\omega dx+\int_c^b u\overline{v}\omega dx+\frac{\gamma_1\gamma_2}{\rho_1}u_1\overline{v_1}+\frac{1}{\rho_2}u_2\overline{v_2}\\ &+\frac{\gamma_1}{\delta}\Big)u_3\overline{v_3}=-\gamma_1\gamma_2[u,v](c-)+[u,v](c+)+\gamma_1\gamma_2[u,v](a)-[u,v](b)\\ &+(u(a),-(pu')(a))\begin{pmatrix}\alpha_1'\\ \alpha_2'\end{pmatrix}\frac{\gamma_1\gamma_2}{\rho_1}(-\tilde{\alpha_2},\tilde{\alpha_1})\begin{pmatrix}(p\overline{v}')(a)\\ \overline{v}(a)\end{pmatrix}\\ &-(u(a),-(pu')(a))\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}\frac{\gamma_1\gamma_2}{\rho_1}(-\tilde{\alpha_2}',\tilde{\alpha_1}')\begin{pmatrix}(p\overline{v}')(a)\\ \overline{v}(a)\end{pmatrix}\\ &+[u,v](b)+[u,v](c-)-[u,v](c+)\\ &=\gamma_1\gamma_2(u(a),-(pu')(a))\begin{pmatrix}\alpha_1'\\ \alpha_2'\end{pmatrix}[(E-K(K+L)^{-1})]\begin{pmatrix}(pu')(a)\\ u(a)\end{pmatrix}.\end{aligned}$$

Let $L\to 0$, the desired result can be obtained by Theorem 3. Similarly, we can get that (2) is also true.

(3) For $h \in L^1(I)$, let $\frac{1}{p} + h = \frac{1}{\tilde{p}}$. Integrating by parts, we have

$$\begin{aligned}
&[\lambda(\frac{1}{p}+h)-\lambda(\frac{1}{p})](\gamma_1\gamma_2\int_a^c u\overline{v}\omega dx+\int_c^b u\overline{v}\omega dx+\frac{\gamma_1\gamma_2}{\rho_1}u_1\overline{v}_1+\frac{1}{\rho_2}u_2\overline{v}_2+\frac{\gamma_1}{\delta}u_3\overline{v}_3)\\
&= \gamma_1\gamma_2[(pu')\overline{v} - (p\overline{v}')u]|_a^c + [(pu')\overline{v} - (p\overline{v}')u]|_c^b\\
&\quad - (\gamma_1\gamma_2\int_a^c \frac{1}{p}[(pu')(p\overline{v}') - (pu')(\tilde{p}\overline{v}')]dx + \int_c^b \frac{1}{p}[(pu')(p\overline{v}') - (pu')(\tilde{p}\overline{v}')]dx).
\end{aligned}$$

Via (1.2)–(1.5), the above equality can be expressed as

$$\begin{aligned}
&\big[\lambda(\frac{1}{p} + h) - \lambda(\frac{1}{p})\big]\big(\gamma_1\gamma_2\int_a^c u\overline{v}\omega dx + \int_c^b u\overline{v}\omega dx + \frac{\gamma_1\gamma_2}{\rho_1}u_1\overline{v}_1 + \frac{1}{\rho_2}u_2\overline{v}_2\\
&\quad + \frac{\gamma_1}{\delta}u_3\overline{v}_3)\big) = -(\gamma_1\gamma_2\int_a^c \frac{1}{p}[(pu')(p\overline{v}') - (pu')(\tilde{p}\overline{v}')]dx + \int_c^b \frac{1}{p}[(pu')(p\overline{v}')\\
&\quad - (pu')(\tilde{p}\overline{v}')]dx) = -\gamma_1\gamma_2\int_a^c [(p-\tilde{p})u'\overline{v}']dx - \int_c^b [(p-\tilde{p})u'\overline{v}']dx\\
&\quad = -\gamma_1\gamma_2\int_a^c p\tilde{p}hu'\overline{v}'dx - \int_c^b ph\tilde{p}u'\overline{v}'dx.
\end{aligned}$$

Thus,

$$d\lambda_{\frac{1}{p}}(h) = -\gamma_1\gamma_2\int_a^c |pu'|^2 hdx - \int_c^b |pu'|^2 hdx,\ h \in L(I).$$

Using the similar method, (4) and (5) hold. □

Theorem 7. *(1) Fix all the data of Z except a and let $\lambda = \lambda(a), u = u(\cdot, a)$. we know that λ is differentiable and*

$$\lambda'(a) = \gamma_1\gamma_2[\frac{1}{p(a)}|(pu')(a,a)|^2 + (\lambda(a)\omega(a) - q(a))|u(a,a)|^2].$$

(2) Fix all the data of Z except b and let $\lambda = \lambda(b), u = u(\cdot, b)$. we know that λ is differentiable and

$$\lambda'(b) = -[\frac{1}{p(b)}|(pu')(b,b)|^2 + (\lambda(b)\omega(b) - q(b))|u(b,b)|^2].$$

Proof. Direct computation yields that

$$\begin{aligned}
&[\lambda(a+\varepsilon)-\lambda(a)](\gamma_1\gamma_2\int_a^c u\overline{v}\omega dx+\int_c^b u\overline{v}\omega dx+\frac{\gamma_1\gamma_2}{\rho_1}u_1\overline{v}_1 + \frac{1}{\rho_2}u_2\overline{v}_2 + \frac{\gamma_1}{\delta}u_3\overline{v}_3)\\
&= -\gamma_1\gamma_2[u,v](c-) + \gamma_1\gamma_2[u,v](a) - [u,v](b) + [u,v](c+)\\
&\quad + [\lambda(a+\varepsilon) - \lambda(a)]\frac{\gamma_1\gamma_2}{\rho_1}u_1\overline{v}_1 + [\lambda(a+\varepsilon) - \lambda(a)]\frac{1}{\rho_2}u_2\overline{v}_2\\
&\quad + [\lambda(a+\varepsilon) - \lambda(a)]\frac{\gamma_1}{\delta}u_3\overline{v}_3, \qquad (4.7)
\end{aligned}$$

$$[\lambda(a+\varepsilon)-\lambda(a)]\frac{\gamma_1\gamma_2}{\rho_1}u_1\overline{v}_1=\gamma_1\gamma_2[(p\overline{u}')(a,a+\varepsilon)u(a,a)-(pu')(a,a)\overline{u}(a,a+\varepsilon)], \tag{4.8}$$

$$[\lambda(a+\varepsilon)-\lambda(a)]\frac{1}{\rho_2}u_2\overline{v}_2=[u,v](b), \tag{4.9}$$

$$[\lambda(a+\varepsilon)-\lambda(a)]\frac{\gamma_1}{\delta}u_3\overline{v}_3=\gamma_1\gamma_2[u,v](c-)-[u,v](c+). \tag{4.10}$$

Combining (4.7)–(4.10), we have

$$\begin{aligned}&[\lambda(a+\varepsilon)-\lambda(a)]\big[\gamma_1\gamma_2\int_a^c u\overline{v}\omega dx+\int_c^b u\overline{v}\omega dx+\frac{\gamma_1\gamma_2}{\rho_1}u_1\overline{v}_1+\frac{1}{\rho_2}u_2\overline{v}_2+\frac{\gamma_1}{\delta}u_3\overline{v}_3\big]\\&\quad=\gamma_1\gamma_2[(pu')(a,a)(\overline{u}(a+\varepsilon,a+\varepsilon)-\overline{u}(a,a+\varepsilon))\\&\qquad-u(a,a)((p\overline{u}')(a+\varepsilon,a+\varepsilon)-(p\overline{u}')(a,a+\varepsilon))].\end{aligned}$$

By Theorems 3.2 and 3.3 of [11], we get

$$\begin{aligned}&\lim_{\varepsilon\to0}(v(a+\varepsilon)-v(a))/\varepsilon=(pu')(a)/p(a),\\&\lim_{\varepsilon\to0}\big((pv')(a+\varepsilon)-(pv')(a)\big)/\varepsilon=[q(a)-\lambda(a)\omega(a)]u(a).\end{aligned}$$

Combining above two equation, we get

$$\begin{aligned}&[\lambda(a+\varepsilon)-\lambda(a)](\gamma_1\gamma_2\int_a^c u\overline{v}\omega dx+\int_c^b u\overline{v}\omega dx+\frac{\gamma_1\gamma_2}{\rho_1}u_1\overline{v}_1+\frac{1}{\rho_2}u_2\overline{v}_2+\frac{\gamma_1}{\delta}u_3\overline{v}_3)\\&\quad=\gamma_1\gamma_2[\frac{1}{p(a)}|(pu')(a,a)|^2+(\lambda(a)\omega(a)-q(a))|u(a,a)|^2].\end{aligned}$$

Let $h\to0$, we can get the desired result. Using the similarly method, we can get that (2) holds. □

Next we consider the derivative formula of λ with respect to the inner discontinuity point c. Let $c_1=c-$, $c_2=c+$. Then the following conclusions can be obtained.

Theorem 8. *(1) Fix all the data of Z except c_1 and let $\lambda=\lambda(c_1), u=u(\cdot,c_1)$. and $v=u(\cdot,c_1+\varepsilon)$. Then, λ is differentiable about c_1 and*

$$\lambda'(c_1)=\gamma_1\gamma_2[\frac{1}{p(c_1)}|(pu')(c_1,c_1)|^2+(\lambda(c_1)\omega(c_1)-q(c_1))|u(c_1,c_1)|^2].$$

(2) Fix all the data of Z except c_2 and let $\lambda=\lambda(c_2), u=u(\cdot,c_2)$. and $v=u(\cdot,c_2+\varepsilon)$. Then, λ is differentiable about c_2 and

$$\lambda'(c_2)=-[\frac{1}{p(c_2)}|(pu')(c_2,c_2)|^2+(\lambda(c_2)\omega(c_2)-q(c_2))|u(c_2,c_2)|^2].$$

Proof. We prove the first conclusion of the theorem. Fix all the data except

c_1 and let $u = u(\cdot, c_1)$ and $v = u(\cdot, c_1 + \varepsilon)$, since

$$\begin{aligned}
&[\lambda(c_1+\varepsilon)-\lambda(c_1)](\gamma_1\gamma_2\int_a^c u\overline{v}\omega dx+\int_c^b u\overline{v}\omega dx+\frac{\gamma_1\gamma_2}{\rho_1}u_1\overline{v}_1+\frac{1}{\rho_2}u_2\overline{v}_2+\frac{\gamma_1}{\delta}u_3\overline{v}_3)\\
&\quad = -\gamma_1\gamma_2[u, v](c-) + \gamma_1\gamma_2[(p\overline{u}')(c_1 + \varepsilon, c_1 + \varepsilon)u(c_1, c_1)]\\
&\quad = -\gamma_1\gamma_2[(pu')(c_1, c_1)(\overline{u}(c_1 + \varepsilon, c_1 + \varepsilon) - \overline{u}(c_1, c_1 + \varepsilon))\\
&\qquad - u(c_1, c_1)((p\overline{u}')(c_1 + \varepsilon, c_1 + \varepsilon) - (p\overline{u}')(c_1, c_1 + \varepsilon))].
\end{aligned}$$

Let $\varepsilon \to 0$, and the desired result can be obtained. The second conclusion can be obtained by using the similar method. $\square$

Theorem 9. *Let $Z=(\frac{1}{p}, q, \omega, \alpha_1, \alpha_2, \alpha_1', \alpha_2', \beta_1, \beta_2, \beta_1', \beta_2', \gamma_1, \gamma_2, \delta, a, b, c-, c+) \in \Omega_1$ with $\lambda = \lambda(Z)$ be an eigenvalue of operator $\mathcal{T}$, and let (u, u_1, u_2, u_3) be a normalized eigenvector corresponding to $\lambda(Z)$. Then λ is differential with respect to all the parameters in Z, the differential expression of λ for each parameter are given below:*

(1) Fix all the parameters of Z except α_1'. Then,

$$\lambda'(\alpha_1') = \lambda\gamma_1\gamma_2|u(a)|^2/\big(\lambda\alpha_2' - \alpha_2\big),$$

where $\lambda\alpha_2' - \alpha_2 \neq 0$.

(2) Fix all the parameters of Z except α_1. Then,

$$\lambda'(\alpha_1) = -\gamma_1\gamma_2|u(a)|^2/(\lambda\alpha_2' - \alpha_2),$$

where $\lambda\alpha_2' - \alpha_2 \neq 0$.

(3) Fix all the parameters of Z except α_2'. Then,

$$\lambda'(\alpha_2') = -\lambda\gamma_1\gamma_2|(pu')(a)|^2/(\lambda\alpha_1' - \alpha_1),$$

where $\lambda\alpha_1' - \alpha_1 \neq 0$.

(4) Fix all the parameters of Z except α_2. Then,

$$\lambda'(\alpha_2) = \gamma_1\gamma_2|(pu')(a)|^2/(\lambda\alpha_1' - \alpha_1),$$

where $\lambda\alpha_1' - \alpha_1 \neq 0$.

(5) Fix all the parameters of Z except β_1'. Then,

$$\lambda'(\beta_1') = -\lambda|u(a)|^2/(\lambda\beta_2' + \beta_2),$$

where $\lambda\beta_2' + \beta_2 \neq 0$.

(6) Fix all the parameters of Z except β_1. Then,

$$\lambda'(\beta_1) = -|u(a)|^2/(\lambda\beta_2' + \beta_2),$$

where $\lambda\beta_2' + \beta_2 \neq 0$.

(7) Fix all the parameters of Z except β_2'. Then,

$$\lambda'(\beta_2') = \lambda|(pu')(a)|^2/(\lambda\beta_1' + \beta_1),$$

where $\lambda\beta_1' + \beta_1 \neq 0$.

(8) Fix all the parameters of Z except β_2. Then,

$$\lambda'(\beta_2') = |(pu')(a)|^2/(\lambda\beta_1' + \beta_1),$$

where $\lambda\beta_1' + \beta_1 \neq 0$.

(9) Fix all the parameters of Z except γ_1. Then,

$$d\lambda_{\gamma_1}(h) = (\gamma_1 + h)(pu')(c+)u(c+)/(\gamma_1 + h), \quad h \in \mathbb{R},$$

where $\gamma_1 + h \neq 0$.

(10) Fix all the parameters of Z except γ_2. Then,

$$\lambda'(\gamma_2) = \gamma_1(pu')(c-)u(c-),$$

(11) Fix all the parameters of Z except δ. Then,

$$\lambda'(\delta) = -\lambda\delta^2|u(a)|^2/\gamma_1,$$

where $\gamma_1 \neq 0$.

Proof. (1) Let $h \in \mathbb{R}$ and fix all data except α_1' , then direct caculation yields that

$$\begin{aligned}
&[\lambda(\alpha_1' + h) - \lambda(\alpha_1')][\gamma_1\gamma_2 \int_a^c u\overline{v}\omega dx + \int_c^b u\overline{v}\omega dx + \frac{\gamma_1\gamma_2}{\rho_1}u_1\overline{v}_1 + \frac{1}{\rho_2}u_2\overline{v}_2 \\
&\quad + \frac{\gamma_1}{\delta}u_3\overline{v}_3] = -\gamma_1\gamma_2[u,v](c-) + \gamma_1\gamma_2[u,v](a) - [u,v](b) + [u,v](c+) \\
&\quad + [\lambda(\alpha_1' + h) - \lambda(\alpha_1')]\frac{\gamma_1\gamma_2}{\rho_1}u_1\overline{v}_1 + [\lambda(\alpha_1' + h) - \lambda(\alpha_1')]\frac{1}{\rho_2}u_2\overline{v}_2 \\
&\quad + [\lambda(\alpha_1' + h) - \lambda(\alpha_1')]\frac{\gamma_1}{\delta}u_3\overline{v}_3
\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}
&[\lambda(\alpha_1' + h) - \lambda(\alpha_1')]\frac{\gamma_1\gamma_2}{\rho_1}u_1\overline{v}_1 \\
&= \frac{\gamma_1\gamma_2}{\rho_1}[(\alpha_1\overline{v}(a) - \alpha_2(p\overline{v}')(a))(\alpha_1'u(a) - \alpha_2'(pu')(a)) \\
&\quad - (\alpha_1u(a) - \alpha_2(pu')(a))((\alpha_1' + h)\overline{v}(a) - \alpha_2'(p\overline{v}')(a)] \\
&= \frac{\gamma_1\gamma_2}{\rho_1}[-h\alpha_1u(a)\overline{v}(a) + h\alpha_2(pu')(a)\overline{v}(a) \\
&\quad + (\alpha_1'\alpha_2 - \alpha_1\alpha_2')(pu')(a)\overline{v}(a) - (\alpha_1'\alpha_2 - \alpha_1\alpha_2')u(a)(p\overline{v}')(a))] \\
&= \frac{\gamma_1\gamma_2}{\rho_1}[-h\alpha_1u(a)\overline{v}(a) + h\alpha_2\frac{\lambda\alpha_1' - \alpha_1}{\lambda\alpha_2' - \alpha_2} - \rho_1[u,v](a)] \\
&= \frac{\gamma_1\gamma_2h\lambda}{\lambda\alpha_2' - \alpha_2}u(a)\overline{v}(a) - \gamma_1\gamma_2[u,v](a),
\end{aligned}$$

$$\begin{aligned}
&[\lambda(\alpha_1'+h)-\lambda(\alpha_1')]\frac{1}{\rho_2}u_2\overline{v}_2=\frac{1}{\rho_2}[(\beta_1'u(b)-\beta_2'(pu')(b))(\beta_2(p\overline{v}')(b)\\
&\quad-\beta_1\overline{v}(b))-(\beta_2(pu')(b)-\beta_1u(b))(\beta_1'\overline{v}(b)-\beta_2'(p\overline{v}')(b))]\\
&=\frac{1}{\rho_2}[(\beta_1'\beta_2-\beta_1\beta_2')u(b)(p\overline{v}')(b))-(\beta_1'\beta_2-\beta_1\beta_2')(pu')(b))\overline{v}(b)]\\
&=[u,v](b),\\
&[\lambda(\alpha_1'+h)-\lambda(\alpha_1')]\frac{\gamma_1}{\delta}u_3\overline{v}_3]=\frac{\gamma_1}{\delta}[(\gamma_2(p\overline{v}')(c-)-(p\overline{v}')(c+))\delta u(c-)\\
&\quad-\gamma_2(pu')(c-)-(pu')(c+)\delta\overline{v}(c-)]=\gamma_1\gamma_2[(p\overline{v}')(c-)u(c-)\\
&\quad-(pu')(c-)\overline{v}(c-)]-\gamma_1[(pu')(c+)\overline{v}(c-)-(p\overline{v}')(c+)u(c-)]\\
&=\gamma_1\gamma_2[u,v](c-)-[u,v](c+). \qquad (4.12)
\end{aligned}$$

Combining (4.11)–(4.12), we have

$$\begin{aligned}
&[\lambda(\alpha_1'+h)-\lambda(\alpha_1')][\gamma_1\gamma_2\int_a^c u\overline{v}\omega dx+\int_c^b u\overline{v}\omega dx\\
&\quad+\frac{\gamma_1\gamma_2}{\rho_1}u_1\overline{v}_1+\frac{1}{\rho_2}u_2\overline{v}_2+\frac{\gamma_1}{\delta}u_3\overline{v}_3]=\frac{\gamma_1\gamma_2 h\lambda}{\lambda\alpha_2'-\alpha_2}u(a)\overline{v}(a). \qquad (4.13)
\end{aligned}$$

Dividing both sides of Equation (4.13) by h and let $h\to 0$, we get

$$\lambda'(\alpha_1')=\frac{\lambda\gamma_1\gamma_2}{\lambda\alpha_2'-\alpha_2}|u(a)|^2,$$

where $\lambda\alpha_2'-\alpha_2\neq 0$. □

The proof for part (2) to part (11) can be given similarly.

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References

[1] Z. Akdoğan, M. Demirci and O.Sh. Mukhtarov. Sturm-Liouville problems with eigendependent boundary and transmission conditions. *Acta Mathematica Scientia.*, **25**(4):731–740, 2005. https://doi.org/10.1016/S0252-9602(17)30213-8.

[2] Z. Akdoğan, M. Demirci and O.Sh. Mukhtarov. Green function of discontinuous boundary-value problem with transmission conditions. *Mathematical Methods in the Applied Sciences*, **30**(14):1719–1738, 2007. https://doi.org/10.1002/mma.867.

[3] B.P. Allahverdiev. Spectral analysis of singular Hamiltonian systems with an eigenparameter in the boundary condition. *Electronic Journal of Differential Equations*, **2019**(2):1–14, 2019. Available from Internet: `https://digital.library.txstate.edu/handle/10877/14645`.

[4] K. Aydemir and O.Sh. Mukhtarov. Class of Sturm-Liouville problems with eigenparameter dependent transmission conditions. *Numerical Functional Analysis and Optimization*, **38**(10):1260–1275, 2017. https://doi.org/10.1080/01630563.2017.1316995.

[5] Y. Bai, W. Wang, K. Li and Z. Zheng. Eigenvalues of a class of eigenparameter dependent third-order differential operators. *Journal of Nonlinear Mathematical Physics*, **29**(3):477–492, 2022. https://doi.org/10.1007/s44198-022-00032-1.

[6] P. Bailey, W. Everitt and A. Zettl. The SLEIGN2 Sturm-Liouville code. *ACM Transactions on Mathematical Software*, **27**(2):143–192, 2001. https://doi.org/10.1145/383738.383739.

[7] P.A. Binding, P.J. Browne and B.A. Watson. Equivalence of inverse Sturm-Liouville problems with boundary conditions rationally dependent on the eigenparameter. *Journal of Mathematical Analysis and Applications*, **291**(1):246–261, 2004. https://doi.org/10.1016/j.jmaa.2003.11.025.

[8] C.T. Fulton. Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, **77**(3-4):293–308, 1977. https://doi.org/10.1017/S030821050002521X.

[9] N.J. Guliyev. Schrödinger operators with distributional potentials and boundary conditions dependent on the eigenvalue parameter. *Journal of Mathematical Physics*, **60**(6):063501, 2019. https://doi.org/10.1063/1.5048692.

[10] Q. Kong, H. Wu and A. Zettl. Dependence of eigenvalues on the problem. *Mathematische Nachrichten*, **188**(1):173–201, 1997. https://doi.org/10.1002/mana.19971880111.

[11] Q. Kong and A. Zettl. Dependence of eigenvalues of Sturm-Liouville problems on the boundary. *Journal of Differential Equations*, **126**(2):389–407, 1996. https://doi.org/10.1006/jdeq.1996.0056.

[12] Q. Kong and A. Zettl. Eigenvalues of regular Sturm-Liouville problems. *Journal of Differential Equations*, **131**(1):1–19, 1996. https://doi.org/10.1006/jdeq.1996.0154.

[13] K. Li, J. Sun and X. Hao. Eigenvalues of regular fourth-order Sturm-Liouville problems with transmission conditions. *Mathematical Methods in the Applied Sciences*, **40**(10):3538–3551, 2017. https://doi.org/10.1002/mma.4243.

[14] A.V. Likov and Y. Mikhailov. *The Theory of Heat and Mass Transfer*. Qosenergaizdat, 1963. (in Russian)

[15] O.Sh. Mukhtarov and K. Aydemir. Minimization principle and generalized Fourier series for discontinuous Sturm-Liouville systems in direct sum spaces. *Journal of Applied Analysis & Computation*, **8**(5):1511–1523, 2018. https://doi.org/10.11948/2018.1511.

[16] O.Sh. Mukhtarov and K. Aydemir. Oscillation properties for non-classical Sturm-Liouville problems with additional transmission conditions. *Mathematical Modelling and Analysis.*, **26**(3):432–443, 2021. https://doi.org/10.3846/mma.2021.13216.

[17] O.Sh. Mukhtarov and K. Aydemir. Two-linked periodic Sturm-Liouville problems with transmission conditions. *Mathematical Methods in the Applied Sciences*, **44**(18):14664–14676, 2021. https://doi.org/10.1002/mma.7734.

[18] H. Olğar and O.Sh. Mukhtarov. Weak eigenfunctions of two-interval Sturm-Liouville problems together with interaction conditions. *Journal of Mathematical Physics*, **58**(4):042201, 2017. https://doi.org/10.1063/1.4979615.

[19] E. Şen. A class of second-order differential operators with eigenparameter-dependent boundary and transmission conditions. *Mathematical Methods in the Applied Sciences*, **37**(18):2952–2961, 2014. https://doi.org/10.1002/mma.3033.

[20] E. Tunç and O.Sh. Mukhtarov. Fundamental solutions and eigenvalues of one boundary-value problem with transmission conditions. *Applied Mathematics and Computation*, **157**(2):347–355, 2004. https://doi.org/10.1016/j.amc.2003.08.039.

[21] E. Uğurlu. Regular third-order boundary value problems. *Applied Mathematics and Computation*, **343**:247–257, 2019. https://doi.org/10.1016/j.amc.2018.09.046.

[22] E. Uğurlu. Third-order boundary value transmission problems. *Turkish Journal of Mathematics*, **43**(3):1518–1532, 2019. https://doi.org/10.3906/mat-1812-36.

[23] E. Uğurlu and E. Bairamov. Spectral analysis of eigenparameter dependent boundary value transmission problems. *Journal of Mathematical Analysis and Applications*, **413**(1):482–494, 2014. https://doi.org/10.1016/j.jmaa.2013.11.022.

[24] N.N. Voitovich, B.Z. Katsenelbaum and A.N. Sivov. Generalized method of eigen-vibration in the theory of diffraction. *Nakua, Mockow.*, **373**(22):1901–2000, 1997. https://doi.org/10.1016/j.physleta.2009.03.060.

[25] X.-C. Xu and C.-F Yang. Inverse spectral problems for the Sturm-Liouville operator with discontinuity. *Journal of Differential Equations*, **262**(3):3093–3106, 2017. https://doi.org/10.1016/j.jde.2016.11.024.

[26] A. Zettl. *Sturm-Liouville Theory.* American Mathematical Society, 2005.

[27] H-Y. Zhang, J-J. Ao and D. Mu. Eigenvalues of discontinuous third-order boundary value problems with eigenparameter-dependent boundary conditions. *Journal of Mathematical Analysis and Applications.*, **506**(2):125680, 2022. https://doi.org/10.1016/j.jmaa.2021.125680.

[28] M. Zhang and K. Li. Dependence of eigenvalues of Sturm–Liouville problems with eigenparameter dependent boundary conditions. *Applied Mathematics and Computation*, **378**:125214, 2020. https://doi.org/10.1016/j.amc.2020.125214.

[29] M. Zhang, J. Sun and A. Zettl. Eigenvalues of limit-point Sturm–Liouville problems. *Journal of Mathematical Analysis and Applications*, **419**(1):627–642, 2014. https://doi.org/10.1016/j.jmaa.2014.05.021.

[30] M. Zhang and Y. Wang. Dependence of eigenvalues of Sturm-Liouville problems with interface conditions. *Applied Mathematics and Computation*, **265**:31–39, 2015. https://doi.org/10.1016/j.amc.2015.05.002.