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Existence of Entropy Solution for a Nonlinear Parabolic Problem in Weighted Sobolev Space via Optimization Method

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Abstract. This paper investigates the existence result of entropy solution for some nonlinear degenerate parabolic problem in weighted Sobolov space with Dirichlet type boundary conditions and L^1 data.

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1 Introduction

In the recent years, the study of nonlinear parabolic equations and variational problems with growth conditions has attracted attention of many researchers, that is due to their applications in elastic mechanics, non-Newtonian fluids, gas flows in porous media, nonlinear elasticity, electrorheological fluids, etc. For more details, see, for example, [19, 20, 28]. In this paper, we deal with the following nonlinear parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - div\left(\omega|\nabla u|^{p-2}\nabla u\right) + |u|^{p-2}u = f & \text{in} \quad Q :=]0;T[\times\Omega, \\ u = 0 & \text{on} \quad \Gamma :=]0;T[\times\partial\Omega, \\ u(.,0) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $(N \geq 2)$, $T > 0$, $p > 1$, $f \in L^1(Q)$, $u_0 \in L^1(\Omega)$, ∇u is the gradient of u, and ω is a weight function (i.e., a locally integrable function on $\mathbb{R}^N$, such that $0 < \omega(x) < \infty$ a.e. $x \in \mathbb{R}^N$), satisfied suitable assumptions (see Section 2 for more details). Many papers have dealt with the nonlinear elliptic or parabolic equations involving growth conditions and L^q data, when $1 < q \leq \infty$. For example, in [25], Xu and Zho studied the existence and uniqueness of weak solution for the initial-boundary value problem of a fourth-order nonlinear parabolic equation. In [4], Bhuvaneswari, Lingeshwaran and Balachandran established the existence of weak solution for the degenerate p-Laplacian parabolic by using semi-discretization process. In the case where $p(.)$ is a variable exponent and by variational methods, Ragusa, Razani and Safari proved in [18] the existence of at least one positive radial solution for the generalized $p(.)$-Laplacian problem. Also, in [13], Khaleghi and Razani investigated the existence and multiplicity of weak solution for an elliptic problem involving $p(.)$-Laplacian operator under Steklov boundary condition, the approach was based on variational methods. Moreover, in [2] and by applying Galerkin's method, Antontsev and Shmarev obtained the existence and uniqueness of weak solution with the assumption that the weight ω is bounded. Furthermore, in [21], Singer treats the existence question of weak solutions for some systems of equations of the type (1.1) with two growth conditions. Zhang and Zhou investigated in [27] the existence, uniqueness and long-time behavior of weak solution for fourth-order degenerate parabolic equation with variable exponents.

Recently, in [17] El Ouaarabi, Allalou and Melliani studied the existence of weak solution for a Dirichlet boundary value problems involving the $p(.)$-Laplacian operator depending on three real parameters. For more information, see, for example, the works [7, 12, 16, 22] and references therein.

The usual weak formulations of elliptic or parabolic problems in the case where the initial data are in L^1 do not ensure existence and uniqueness of solution (see, for example, [5] for more details). In [3], Bénilan et al. have been proposed a new solution, called entropy solution. Later on, the notion of entropy solution was then adopted by many authors to study some nonlinear elliptic and parabolic problems. For example, in [6] and via the technique variation method, Cavalheiro proved the existence of entropy solution for the Dirichlet problem

$$\begin{cases} -\,div\left(\omega|\nabla u|^{p-2}\nabla u\right) = f - divG & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ $(N \geq 2)$, $1 < p$, $f \in L^1$, $G/\omega \in [L^{p'}(\Omega,\omega)]^N$ and ω is a weight function, which satisfy some assumptions (see

Section 2 for more details). When $G = 0$ and $p(.)$ is a variable exponent, Zhang treated the same problem in [26]. In addition, in [1], Abbassi, Allalou and Kassidi investigated the existence of an entropy solution to the unilateral problem for a class of nonlinear anisotropic elliptic equations. In [10], El Hachimi, Igbida and Jamea explained the existence of entropy solution of a nonlinear parabolic problem by using a time discretization of continuous problem. Besides that, the existence and uniqueness of degenerate parabolic equations of type (1.1) was proved by Weisheng et al. in [24].

The main purpose of this paper is to extend the result of [6] to the case of parabolic equations. In this paper, we study the existence question of entropy solution for Problem (1.1) with L^1 data, by employing the optimization method combined with a difference scheme and a priori estimates.

This paper is organized as follows. In Section 2, we give some definitions and fundamental properties of weighted Sobolev spaces. Moreover, we recall some known Lemmas to be used in proof of main result. In Section 3, we first employ the difference and variation methods to prove the existence and uniqueness of weak solution for the approximate Problem of (1.1) under appropriate assumptions. In Section 4, we construct an approximate solution sequence and establish some a priori estimates, then, we draw a subsequence to obtain a limit function, and prove this function as an entropy solution.

2 Preliminaries and notations

This section gives some notations and definitions and state some result which we shall use in this work.

Let Ω be a smooth bounded domain in $\mathbb{R}^N$. By weight we mean a locally integrable function ω on $\mathbb{R}^N$ such that $0 < \omega < \infty$ for a.e. $x \in \mathbb{R}^N$. We shall denote by $L^p(\Omega, \omega)$ the set of all measurable functions u on Ω with the finite norm

$$|u|^p_{L^p(\Omega,\omega)} = \left(\int_\Omega \omega(x)\, |u|^p dx\right)^{\frac{1}{p}}, \; 1 \le p < \infty.$$

The weighted Sobolev space $W^{1,p}(\Omega, \omega)$ is defined as the collection of all functions $u \in L^p(\Omega)$ having the derivatives $\nabla u \in L^p(\Omega, \omega)$ with the finite norm

$$|u|_{W^{1,p}(\Omega,\omega)} := |u|_{L^p(\Omega)} + |\nabla u|_{L^p(\Omega,\omega)^N}.$$

The set $C_0^\infty(\Omega)$ denotes the space of all functions with compact support in Ω with continuous derivatives of arbitrary order.
The space $W_0^{1,p}(\Omega, \omega)$ denotes the closure $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega, \omega)$. For a Banach space X and $a < b$, $L^p(a; b; X)$ is the space of measurable functions $u : [a; b] \mapsto$ X such that

$$|u|_{L^p(a,b;X)} := \left(\int_a^b |u(t)|^p_X \mathrm{d}t\right)^{1/p} < \infty.$$

In this work, the function ω satisfies the following hypothesis:

(H) $\omega \in L^1_{loc}(\Omega)$, $\omega^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega)$, $\omega^{-s} \in L^1(\Omega)$, where
$s \in \left(\frac{N}{p}, \infty\right) \cap \left[\frac{1}{p-1}, \infty\right)$.

For more details on weighted Sobolev spaces, see, for example, [11, 14, 15, 23]. For $k > 0$, the cut function T_k (see Proposition 1 for more details) is defined by $T_k : \mathbb{R} \to \mathbb{R}$

$$T_k(s) := \begin{cases} s & \text{if } |s| \leq k, \\ k\frac{s}{|s|} & \text{if } |s| > k. \end{cases}$$

For a function u defined on Ω, the truncated function $T_k u$ is defined by, for every $x \in \Omega$ the value of $T_k u$ at x is just $T_k(u(x))$.

For $k > 0$, the primitive of cut function T_k is a function denoted by S_k and which is defined from $\mathbb{R}$ to $\mathbb{R}^+$ by

$$S_k(x) = \int_0^x T_k(s)ds.$$

And by [9],

$$\int_0^T \langle v_t, T_k(v)\rangle = \int_\Omega S_k(v(T))dx - \int_\Omega S_k(v(0))dx,$$

where $\langle , \rangle$ denotes the duality between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$.

The following proposition gives the definition of the very weak gradient of a measurable function u with $T_k(u) \in W_0^{1,p}(\Omega, \omega)$.

Proposition 1. *[3] For every measurable function u with $T_k(u) \in W_0^{1,p}(\Omega, \omega)$, there exists a unique measurable function $v : \Omega \to \mathbb{R}^{\mathbb{N}}$, which we call the very weak gradient of u and denote $v = \nabla u$, such that*

$$\nabla T_k(u) = v 1_{\{|u|<k\}} \quad \textit{for a.e.} \quad \Omega \quad \textit{and for every} \quad k > 0,$$

where 1_E denotes the characteristic function of a measurable set E. Moreover, if u belongs to $W_0^{1,p}(\Omega, \omega)$, then v coincides with the weak gradient of u.

The notion of the very weak gradient allows us to give the definition of entropy solution for Problem (1.1).

Proposition 2. *[8] Assume that the hypothesis (H) holds, then for $s + 1 \leqslant ps < N(s + 1)$, the following continuous embedding hold true,*

$$W_0^{1,p}(\Omega, \omega) \hookrightarrow W_0^{1,p_1}(\Omega) \hookrightarrow L^q(\Omega), \tag{2.1}$$

where $p_1 = \frac{ps}{1+s}$, $1 \leq q = \frac{Np_1}{N-p_1} = \frac{Nps}{N(s+1)-ps}$, and for $ps \geqslant N(s + 1)$ the embedding (2.1) holds with arbitrary $1 \leq q < \infty$. Moreover, the compact embedding

$$W_0^{1,p}(\Omega, \omega) \hookrightarrow\hookrightarrow L^r(\Omega)$$

holds provided $1 \leq r < q$.

Proposition 3. *[8] (**Hardy-type inequality**) There exists a weight function ω on Ω and a parameter $q, 1 < q < \infty$ such that the inequality*

$$\left(\int_{\Omega} \omega |u(x)|^q dx\right)^{\frac{1}{q}} \leqslant C \left(\int_{\Omega} \omega |\nabla u|^p dx\right)^{\frac{1}{p}} \tag{2.2}$$

holds for every $u \in W_0^{1,p}(\Omega, \omega)$ with a constant $C > 0$ independent of u, moreover the embedding

$$W_0^{1,p}(\Omega, \omega) \hookrightarrow L^q(\Omega, \omega)$$

determined by the inequality (2.2) is compact.

Lemma 1. *For $\xi, \eta \in \mathbb{R}^N$ and $1 < p < \infty$, we have*

$$\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right) \cdot (\xi - \eta) \geq 0.$$

Lemma 2. *For $a \geq 0,\ b \geq 0$ and $1 \leq p < \infty$, we have*

$$(a + b)^p \leq 2^{p-1} \left(a^p + b^p\right).$$

3 Existence and uniqueness of weak solution for the parabolic problem

The goal of this Section is to prove the existence and uniqueness of weak solution for Problem (1.1) with L^∞ data. Firstly, the next definition gives the notion of weak solution for Problem (1.1).

Definition 1. A measurable function u is a weak solution of the parabolic problem (1.1),
if $u \in L^\infty\left((0,T); L^2(\Omega)\right) \cap L^p\left((0,T); W_0^{1,p}(\Omega, \omega)\right) \cap C((0,T); L^2(\Omega))$, $\frac{\partial u}{\partial t} \in L^{p'}\left((0,T); W^{-1,p'}(\Omega)\right)$ and

$$\int_Q \frac{\partial u}{\partial t} \varphi dxdt + \int_Q \omega |\nabla u|^{p-2} \nabla u \nabla \varphi dxdt + \int_Q |u|^{p-2} u \varphi dxdt = \int_Q f \varphi dxdt,$$

for all $\varphi \in L^p((0,T); W_0^{1,p}(\Omega, \omega)) \cap L^\infty\left((0,T); L^2(\Omega)\right) \cap C^1(\bar{Q})$.

Now we state our main result of this section.

Theorem 1. *Let $u_0 \in L^2(\Omega)$, $f \in L^\infty(Q)$ and let hypothesis (H) be satisfied. Then the Problem (1.1) has a unique weak solution.*

The proof of above theorem can be established by investigating the existence and uniqueness of weak solution for the given semi-discrete elliptic problem

$$\begin{cases} \dfrac{u_k - u_{k-1}}{h} - div\left(\omega |\nabla u_k|^{p-2} \nabla u_k\right) + |u_k|^{p-2} u_k = [f]_h\left((k-1)h\right) \text{in } \Omega, \\ \qquad\qquad\qquad\qquad u_k|_{\partial\Omega} = 0 \qquad \text{for } k = 1, \dots, n, \end{cases} \tag{3.1}$$

where $h > 0$, n is a positive integer such that $h = \frac{T}{n}$, and

$$[f]_h (x,t) = \frac{1}{h} \int_t^{t+h} f(x,\tau)\, d\tau.$$

Recall that a function $u \in W_0^{1,p}(\Omega,\omega) \cap L^2(\Omega)$ is a weak solution of (3.1) if and only if for all $\varphi \in W_0^{1,p}(\Omega,\omega) \cap L^2(\Omega)$. We have

$$\int_\Omega \frac{u_k - u_{k-1}}{h} \varphi \mathrm{d}x + \int_\Omega \omega |\nabla u_k|^{p-2} (\nabla u_k \cdot \nabla \varphi)\, \mathrm{d}x + \int_\Omega |u_k|^{p-2} u_k \varphi \mathrm{d}x$$
$$= \int_\Omega [f]_h (0)\, \varphi \mathrm{d}x.$$

Theorem 2. *Let $u_0 \in L^2(\Omega)$, $f \in L^\infty(Q)$ and let hypothesis (H) be satisfied, then the Problem* (3.1) *has a unique weak solution.*

Proof. The first step of proof is to establish the existence of a weak solution for the following elliptic problem:

$$\begin{cases} \dfrac{u-u_0}{h} - div\left(\omega|\nabla u|^{p-2}\nabla u\right) + |u|^{p-2}u = [f]_h(0) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Consider the variational problem $\min\{J(u)\,/u \in V\}$, where $V := W_0^{1,p}(\Omega,\omega) \cap L^2(\Omega)$ and

$$J(u) = \frac{1}{2h}\int_\Omega (u-u_0)^2\, dx + \frac{1}{p}\int_\Omega \omega|\nabla u|^p dx + \frac{1}{p}\int_\Omega |u|^p dx - \int_\Omega |f|_h(0)\, u dx. \tag{3.3}$$

We show that $J(u)$ has a minimizer $u \in V$ and this function is a weak solution of Problem (3.2).
Hölder's and Young's inequalities imply that

$$\left| \int_\Omega [f]_h(0) u \mathrm{d}x \right| \le \varepsilon |u|^p_{L^p(\Omega)} + C(\varepsilon)\, |[f]_h(0)|^{p'}_{L^{p'}(\Omega)}, \qquad \text{for all } \varepsilon > 0. \tag{3.4}$$

This implies that

$$J(u) \ge (\frac{1}{p} - \varepsilon)\|u\|^p_{L^p(\Omega)} - C(\epsilon)\, |[f]_h(0)|^{p'}_{L^{p'}(\Omega)}.$$

Choosing ε very small, then

$$J(u) \ge -C\, |[f]_h(0)|^{p'}_{L^{p'}(\Omega)}.$$

It follows that

$$-C\, |[f]_h(0)|^{p'}_{L^{p'}(\Omega)} \le \inf_{u\in V} J(u) \le \frac{1}{2h} |u_0|^2_{L^2(\Omega)}.$$

Therefore, we can find a minimizing sequence $\{u_m\} \subset V$ such that

$$J(u_m) \leq J(u_0) + 1, \quad \lim_{m\to\infty} J(u_m) = \inf_{u\in V} J(u). \tag{3.5}$$

Then, from (3.3), (3.4) and (3.5), we get

$$\begin{aligned}\frac{1}{2h}\int_\Omega (u_m-u_0)^2 dx + \frac{1}{p}\int_\Omega |u_m|^p dx + \frac{1}{p}\int_\Omega \omega|\nabla u_m|^p dx \\ \leq \varepsilon |u|^p_{L^p(\Omega)} + C(\varepsilon)\,|[f]_h(0)|^{p'}_{L^{p'}(\Omega)} + \frac{1}{2h}\,|u_0|^2_{L^2(\Omega)} + 1.\end{aligned}$$

Choosing ε a small positive number, we obtain

$$\frac{1}{2h}\int_\Omega (u_m-u_0)^2 dx + \frac{1}{p}\int_\Omega \omega|\nabla u_m|^p dx \leq C\,|[f]_h(0)|^{p'}_{L^{p'}(\Omega)} + \frac{1}{2h}\,|u_0|^2_{L^2(\Omega)} + 1.$$

Since $\frac{1}{2}u_m^2 - u_0^2 \leq (u_m-u_0)^2$, then

$$\begin{aligned}\frac{1}{4h}\int_\Omega u_m^2 dx - \frac{1}{2h}\int_\Omega u_0^2 dx + \frac{1}{p}\int_\Omega \omega|\nabla u_m|^p dx \leq C\,|[f]_h(0)|^{p'}_{L^{p'}(\Omega)} \\ + \frac{1}{2h}\,|u_0|^2_{L^2(\Omega)} + 1.\end{aligned}$$

This implies that

$$\frac{1}{4h}|u_m|^2_{L^2(\Omega)} + \frac{1}{p}\,|u_m|^p_{W_0^{1,p}(\Omega,\omega)} \leq C\,|[f]_h\,(0)|^{p'}_{L^{p'}(\Omega)} + \frac{1}{h}\,|u_0|^2_{L^2(\Omega)} + 1.$$

Hence, the above inequality shows that u_m is bounded in V. Since the space V is reflexive, then, there exists a subsequence, still denoted by u_m, and a function $u \in V$ such that $u_m \rightharpoonup u$ in V. Therefore, by using Propositions 2 and 3, we get

$$u_m \rightharpoonup u \text{ weakly in } L^p(\Omega,\omega) \text{ and } L^p(\Omega), \tag{3.6}$$

$$u_m \to u \text{ a.e in } \Omega. \tag{3.7}$$

Now, we show that

$$\liminf_{m\to\infty}\ J(u_m) \geq J(u).$$

By (3.7) and Fatou's Lemma, we have

$$\liminf_{m\to\infty} \frac{1}{2h}\int_\Omega (u_m-u_0)^2\,dx \geq \frac{1}{2h}\int_\Omega (u-u_0)^2\,dx, \tag{3.8}$$

$$\text{and } \liminf_{m\to\infty} \frac{1}{p}\int_\Omega |u_m|^p dx \geq \frac{1}{p}\int_\Omega |u|^p dx. \tag{3.9}$$

Since $u_m \rightharpoonup u$ in $W_0^{1,p}(\Omega,\omega)$, then

$$\liminf_{m\to\infty} \frac{1}{p}\int_\Omega \omega|\nabla u_m|^p dx \geq \frac{1}{p}\int_\Omega \omega|\nabla u|^p dx. \tag{3.10}$$

By (3.6), we get

$$\lim_{m\to\infty}\int_\Omega [f]_h(0)\,u_m \mathrm{d}x=\int_\Omega [f]_h(0)\,u\mathrm{d}x. \tag{3.11}$$

Combining (3.8), (3.9), (3.10) and (3.11), we obtain

$$\liminf_{m\to\infty} J(u_m)\geq J(u)$$

and thus, u is a minimizer of the functional $J(u)$ in V.

Next, we show that u is a weak solution of the elliptic Problem (3.2). Since u is a minimizer of the functional $J(u)$ in V, then for any $v\in V$ we have

$$\begin{aligned}0\leq \frac{J(u+tv)-J(u)}{t}&=\int_\Omega \frac{u-u_0}{h}v\mathrm{d}x+\int_\Omega \frac{|u+tv|^p-|u|^p}{ht}\mathrm{d}x\\&+\int_\Omega \omega\frac{|\nabla u+t\nabla v|^p-|\nabla u|^p}{pt}\mathrm{d}x-\int_\Omega [f]_h(0)\,v\mathrm{d}x.\end{aligned} \tag{3.12}$$

Consider the following function G defined on $[0,1]$ by

$$G(\mu)=\frac{|u+t\mu v|^p-|u|^p}{ht}.$$

Note that G is continuous on [0,1] and differentiable on]0,1[. By mean value theorem, there exists $\gamma\in]0,1[$ such that

$$\frac{|u+tv|^p-|u|^p}{pt}=|u+t\gamma v|^{p-2}(u+t\gamma v)v.$$

Since $\gamma,\ t\in[0,1]$, then by Young's inequality and by Lemma 2, we get

$$|u+t\gamma v|^{p-2}(u+t\gamma v)\,v\leq \frac{1}{p'}|u+t\gamma v|^p+\frac{1}{p}|v|^p\leq \frac{2^{p-1}}{p'}(|u|^p+|v|^p)+\frac{1}{p}|v|^p.$$

On the other hand,

$$\lim_{t\to 0}\frac{|u+tv|^p-|u|^p}{pt}=|u|^{p-2}uv.$$

Hence, by the dominated convergence theorem, we get

$$\lim_{t\to 0}\int_\Omega \frac{|u+tv|^p-|u|^p}{pt}\mathrm{d}x=\int_\Omega |u|^{p-2}uv\mathrm{d}x.$$

Note, if we consider again a function M defined on $[0,1]$ by

$$M(\mu)=\omega\frac{|\nabla u+t\mu\nabla v|^p-|\nabla u|^p}{pt},$$

in the same manner in G, we can show that

$$\lim_{t\to 0}\int_\Omega \omega\frac{|\nabla u+t\nabla v|^p-|\nabla u|^p}{pt}\mathrm{d}x=\int_\Omega \omega|\nabla u|^{p-2}\left(\nabla u\cdot\nabla v\right)\mathrm{d}x.$$

Then, by letting $t \to 0$ in (3.12), we get

$$0 \leq \int_\Omega \frac{1}{h}(u-u_0)\, v\mathrm{d}x + \int_\Omega |u|^{p-2}uv\mathrm{d}x + \int_\Omega \omega |\nabla u|^{p-2}(\nabla u \cdot \nabla v)\,\mathrm{d}x \\ - \int_\Omega [f]_h(0)\, v\mathrm{d}x.$$

This allows us to deduce that

$$\int_\Omega \frac{1}{h}(u-u_0)\, v\mathrm{d}x + \int_\Omega |u|^{p-2}uv\mathrm{d}x + \int_\Omega \omega |\nabla u|^{p-2}(\nabla u \cdot \nabla v)\,\mathrm{d}x \\ = \int_\Omega [f]_h(0)\, v\mathrm{d}x.$$

Now, let's prove that the Problem (3.2) has a unique weak solution. For that, let u_1 and u_2 two weak solutions for Problem (3.2), then

$$\int_\Omega \frac{1}{h}(u_1-u_2)\, v\mathrm{d}x + \int_\Omega \left(|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2\right) v\mathrm{d}x \\ + \int_\Omega \omega \left(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\right) \cdot \nabla v\mathrm{d}x = 0.$$

Let $v = u_1 - u_2$ in (3.2), then the above inequality becomes

$$\int_\Omega \frac{1}{h}(u_1-u_2)^2\,\mathrm{d}x + \int_\Omega \left(|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2\right)(u_1-u_2)\,\mathrm{d}x \\ + \int_\Omega \omega \left(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\right) \cdot (\nabla u_1 - \nabla u_2)\,\mathrm{d}x = 0. \tag{3.13}$$

Lemma 1 allows us to deduce that

$$\left(|\nabla u_1|^{p-2}(\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\right) \cdot (\nabla u_1 - \nabla u_2)\,\mathrm{d}x \geq 0.$$

We recall that $\left(|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2\right)(u_1-u_2)\,\mathrm{d}x \geq 0$.

Then the equality (3.13) implies that

$$\int_\Omega (u_1-u_2)^2\,\mathrm{d}x = 0.$$

Consequently, $u_1 = u_2$ a.e. in Ω, which completes the proof of the existence and uniqueness of the weak solution to Problem (3.2). Let $k=1$, from the Equation (3.2), there exists a weak solution $u_1 \in V$. By induction and in the same above manner, the Problem (3.1) has a unique weak solution $u_k \in V$, where $k = 2, \ldots, n$. $\square$

Proof. [Proof of Theorem 1] Let n be a positive integer and $h = \frac{T}{n}$ and let the function

$$u_h(x,t) = \begin{cases} u_0(x), & t = 0, \\ u_1(x), & 0 < t \leq h, \\ \cdots, & \cdots \\ u_j(x), & (j-1)h < t \leq jh, \\ \cdots, & \cdots \\ u_n(x), & (n-1)h < t \leq nh = T. \end{cases} \tag{3.14}$$

Let u_k be a test function in weak formulation of Problem (3.1), then

$$\int_\Omega \frac{u_k^2}{h}\mathrm{d}x+\int_\Omega \omega|\nabla u_k|^p\mathrm{d}x+\int_\Omega |u_k|^p\mathrm{d}x=\int_\Omega [f]_h\,(k-1)\,u_k\mathrm{d}x+\int_\Omega \frac{u_{k-1}u_k}{h}\mathrm{d}x.$$

Applying Young's inequality, then

$$\begin{aligned}&\frac{1}{h}\int_\Omega u_k^2\mathrm{d}x+\int_\Omega \omega|\nabla u_k|^p\mathrm{d}x+\int_\Omega |u_k|^p\mathrm{d}x\\ &\quad\le \int_\Omega |u_k|^p\mathrm{d}x+\frac{1}{p'}\int_\Omega |[f]_h\,(k-1)\,|^{p'}\mathrm{d}x+\frac{1}{h}\int_\Omega u_{k-1}u_k\mathrm{d}x.\end{aligned}$$

Therefore,

$$\int_\Omega \frac{u_k^2}{h}\mathrm{d}x+\int_\Omega \omega|\nabla u_k|^p\mathrm{d}x\le \frac{1}{p'}\int_\Omega |[f]_h\,(k-1)\,|^{p'}\mathrm{d}x+\frac{1}{h}\int_\Omega u_{k-1}u_k\mathrm{d}x.$$

Since $\quad u_{k-1}u_k\le \frac{u_{k-1}^2+u_k^2}{2}$, then

$$\frac{1}{2}\int_\Omega \frac{u_k^2}{h}\mathrm{d}x+\int_\Omega \omega|\nabla u_k|^p\mathrm{d}x\le \frac{1}{p'}\,|[f]_h\,(k-1)|^{p'}_{L^{p'}(\Omega)}+\frac{1}{2}\int_\Omega \frac{u_{k-1}^2}{h}\mathrm{d}x. \quad (3.15)$$

Note, that for each $t\in\,]0,T]$ there exists $j\in\{0,\dots,n\}$ such that $t\in\,](j-1)h,jh]$. Therefore, by adding the inequality (3.15) from $k=1$ to $k=j$, we get

$$\frac{1}{2}\int_\Omega u_j^2\mathrm{d}x+h\sum_{k=1}^{j}\int_\Omega \omega|\nabla u_k|^p\mathrm{d}x\le \frac{h}{p'}\sum_{i=1}^{j}|[f]_h\,(k-1)|^{p'}_{L^{p'}(\Omega)}+\frac{1}{2}\int_\Omega u_0^2\mathrm{d}x.$$

Then, (3.14) implies that

$$\frac{1}{2}\,|u_h(t)|^2_{L^2(\Omega)}+\int_0^t\int_\Omega \omega|\nabla u_h(t)|^p\mathrm{d}x\mathrm{d}t\le \frac{1}{2}\,|u_0|^2_{L^2(\Omega)}+\frac{1}{p'}\int_0^t |f(t)|^{p'}_{L^{p'}(\Omega)}\mathrm{d}t.$$

This implies that

$$\begin{array}{lll} u_h\rightharpoonup u, & \text{weakly} * \quad \text{in} & L^\infty\left(0,T\,;L^2(\Omega)\right),\\ u_h\rightharpoonup u, & \text{weakly} \quad \text{in} & L^p\left(0,T\,;L^p\left(\Omega,\omega\right)\right),\\ |\nabla u_h|^{p-2}\,\nabla u_h\rightharpoonup \xi, & \text{weakly} \quad \text{in} & L^{p'}\left(0,T\,;L^{p'}\left(\Omega,\omega^{1-p'}\right)\right).\end{array}$$

Next, we prove that u is a weak solution for the Problem (1.1). Let $\varphi\in C^1(\bar{Q})$ with $\varphi(.,T)=0$ and $\varphi(x,t)_\Gamma=0$. By taking $\varphi(x,kh)$ as test function for every $k\in\{1,\dots,n\}$, we get

$$\begin{aligned}&\int_\Omega \frac{u_k-u_{k-1}}{h}\varphi\mathrm{d}x+\int_\Omega \omega|\nabla u_k|^{p-2}\left(\nabla u\cdot\nabla\varphi\right)\mathrm{d}x+\int_\Omega |u_k|^{p-2}u_k\varphi\mathrm{d}x\\ &=\int_\Omega [f]_h\left((k-1)\,h\right)\varphi\left(x,kh\right)\mathrm{d}x.\end{aligned}$$

Then, by summing the above equalities, we have

$$\sum_{k=0}^{n-1}\int_{\Omega} u_k\left(\varphi\left(x,kh\right)-\varphi\left(x,(k+1)h\right)\right)\mathrm{d}x-\int_{\Omega}u_0\varphi(x,0)\mathrm{d}x$$
$$+h\sum_{k=1}^{n}\int_{\Omega}|u_k|^{p-2}u_k\varphi\left(x,kh\right)\mathrm{d}x+h\sum_{k=1}^{n}\int_{\Omega}\omega\left|\nabla u_k\right|^{p-2}\left(\nabla u_k\cdot\nabla\varphi\left(x,kh\right)\right)\mathrm{d}x$$
$$=h\sum_{k=1}^{n}\int_{\Omega}[f]_h((k-1)\,h)\varphi\left(x,kh\right)\mathrm{d}x. \tag{3.16}$$

On the other hand,

$$\sum_{k=0}^{n-1}\int_{\Omega}u_k(x)[\varphi(x,kh)-\varphi\left(x,(k+1)\,h\right)]dx$$
$$=-\sum_{k=0}^{n-1}\int_{kh}^{(k+1)h}\int_{\Omega}u_h(x,t)\frac{\partial\varphi(x,t)}{\partial t}\mathrm{d}x\mathrm{d}t$$
$$=-\int_{Q}u_h(x,t)\frac{\partial\varphi(x,t)}{\partial t}\mathrm{d}x\mathrm{d}t\to-\int_{Q}u(x,t)\frac{\partial\varphi(x,t)}{\partial t}\mathrm{d}x\mathrm{d}t\quad\text{as } h\to 0,$$
$$h\sum_{k=1}^{n}\int_{\Omega}\omega\left|\nabla u_h\right|^{p-2}\left(\nabla u_h(x,kh)\cdot\nabla\varphi(x,kh)\right)\mathrm{d}x=\int_{Q}\omega|\nabla u_h|^{p-2}\big(\nabla u_h\left(x,t\right)$$
$$\cdot\nabla\varphi\left(x,t\right)\big)\mathrm{d}x\ \mathrm{d}t+\sum_{k=1}^{n}\int_{(k-1)h}^{kh}\int_{\Omega}\omega\left|\nabla u_h\right|^{p-2}\nabla\, u_h\left(x,t\right)\cdot\big(\nabla\varphi\left(x,kh\right)$$
$$-\nabla\varphi\left(x,t\right)\big)\mathrm{d}x\mathrm{d}t\to\int_{Q}\omega\xi\cdot\nabla\varphi(x,\tau)\mathrm{d}x\mathrm{d}\tau,\text{as}\quad h\to 0.$$

And also

$$h\sum_{k=1}^{n}\int_{\Omega}[f]_h\left(x,(k-1)\,h\right)\varphi\left(x,kh\right)\mathrm{d}x$$
$$=\sum_{k=1}^{n}\int_{(k-1)h}^{kh}\int_{\Omega}f\left(x,\right)\varphi\left(x,kh\right)\mathrm{d}x\mathrm{d}t\to\int_{Q}f\varphi\mathrm{d}x\mathrm{d}t\qquad as\ h\to 0,$$
$$h\sum_{k=1}^{n}\int_{\Omega}|u_k|^{p-2}u_k\varphi\left(x,kh\right)\mathrm{d}x=-\sum_{k=1}^{n}\int_{(k-1)h}^{kh}\int_{\Omega}|u_h|^{p-2}u_h\big(\varphi\left(x,t\right)$$
$$-\varphi\left(x,kh\right)\big)\mathrm{d}x\mathrm{d}t+\int_{Q}|u_h|^{p-2}u_h\varphi\left(x,t\right)\mathrm{d}x\mathrm{d}t\to\int_{Q}|u|^{p-2}u\varphi\ \mathrm{d}x\mathrm{d}t\quad as\ h\to 0.$$

Then, for $h\to 0$ in (3.16),

$$-\int_{Q}u\frac{\partial\varphi}{\partial t}\mathrm{d}x\mathrm{d}t-\int_{\Omega}u_0\left(x\right)\varphi\left(x,0\right)\mathrm{d}x+\int_{Q}|u|^{p-2}u\varphi\mathrm{d}x\mathrm{d}t+\int_{Q}\omega\xi\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t$$
$$=\int_{Q}f\varphi\mathrm{d}x\mathrm{d}t. \tag{3.17}$$

For $\varphi \in C_c^\infty(Q)$, the above inequality becomes

$$-\int_Q u\frac{\partial\varphi}{\partial t}\mathrm{d}x\mathrm{d}t+\int_Q u|^{p-2}u\varphi\mathrm{d}x\mathrm{d}t+\int_Q \omega\xi\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t=\int_Q f\varphi\mathrm{d}x\mathrm{d}t. \qquad (3.18)$$

This implies that $\frac{\partial u}{\partial t}\in L^{p'}\left((0,T);W^{-1,p'}(\Omega)\right)$.

Now, we prove that $\xi=|\nabla u|^{p-2}\nabla u$. Let $Au:=|\nabla u|^{p-2}\nabla u$ and $v\in L^p\left(0;T\,;W_0^{1,p}(\Omega;\omega)\right)\cap L^\infty\left(0,T\,;L^2(\Omega)\right)$, by summing the above inequalities (3.15) for $k=1,\dots,n$, we get

$$\frac{1}{2}\int_\Omega u_h^2(T)dx+\int_Q\omega Au_h\cdot\nabla u_h dxdt+\int_Q|u_h|^p dxdt\le\int_Q fu_h dxdt+\frac{1}{2}\int_\Omega u_0^2dx.$$

The application of Lemma 1 implies that

$$\int_Q\omega\left(Au_h-Av\right)\cdot\left(\nabla u_h-\nabla v\right)dxdt\ge 0.$$

Then, it follows from (3.17) that

$$\frac{1}{2}\int_\Omega u_h^2(T)dx+\int_Q\omega\left(Au_h\right)\cdot\nabla vdxdt+\int_Q\omega\left(Av\right)\left(\nabla u_h-\nabla v\right)dxdt+\int_Q|u_h|^p dxdt$$
$$\le\frac{1}{2}\int_\Omega u_0^2dx+\int_Q fu_h dxdt.$$

This implies for $h\to 0$ that

$$\frac{1}{2}\int_\Omega u^2(T)dx+\int_Q\omega\left(Au\right)\cdot\nabla vdxdt+\int_Q\omega(Av)(\nabla u-\nabla v)dxdt+\int_Q|u|^p dxdt$$
$$\le\frac{1}{2}\int_\Omega u_0^2dx+\int_Q fudxdt. \qquad (3.19)$$

Let $\varphi=u$ in inequality (3.18), then

$$-\frac{1}{2}\int_\Omega u^2(T)\mathrm{d}x+\int_0^T\int_\Omega|u|^p\mathrm{d}x\mathrm{d}t+\int_Q\omega\xi\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t=\int_Q fu\mathrm{d}x\mathrm{d}t. \qquad (3.20)$$

Combining (3.19) with (3.20) to get

$$\int_Q\omega(\xi-Av)\cdot(\nabla v-\nabla u)dxdt\le 0.$$

For $v=u-\lambda\Psi$ for any $\lambda>0,\Psi\in L^p\left(0;T\,;W_0^{1,p}(\Omega;\omega)\right)\cap L^\infty\left(0,T\,;L^2(\Omega)\right)$ in above inequality, it follows that

$$\int_Q\omega\left(\xi-A\left(u-\lambda\Psi\right)\right)\cdot\nabla\Psi\mathrm{d}x\mathrm{d}\tau\ge 0.$$

Passing to limits as $\lambda \to 0^+$ and using Lebesgue's dominated convergence theorem to get

$$\int_Q \omega(\xi - Au) \cdot \psi \mathrm{d}x \,\mathrm{d}\tau \geq 0, \quad \text{for all } \psi \in \left(L^p\left(0;T\,;W_0^{1,p}(\Omega;\omega)\right)\right)^N.$$

Hence, $\xi = Au$, a.e. in Q. Therefore, for all $\varphi \in L^p\left((0,T);W_0^{1,p}(\Omega;\omega)\right) \cap L^\infty\left((0,T);L^2(\Omega)\right) \cap C^1(\bar{Q})$

$$-\int_Q u\frac{\partial\varphi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \int_Q |u|^{p-2}u\varphi \mathrm{d}x\mathrm{d}\,t + \int_Q \omega\,|\nabla u|^{p-2}\,\nabla u \cdot \nabla\varphi \mathrm{d}x\,\mathrm{d}t = \int_Q f\varphi \mathrm{d}x\,\mathrm{d}t.$$

On the other hand, the fact that $u \in L^p\left(0,T;W_0^{1,p}(\Omega)\right) \cap L^\infty\left((0,T);L^2(\Omega)\right)$ and $\frac{\partial u}{\partial t} \in L^{p'}\left((0,T);W^{-1,p'}(\Omega)\right)$ implies that u belongs to $C\left((0,T);L^2(\Omega)\right)$, hence the existence of weak solution of the Problem (1.1). To show that this weak solution is unique, let u and v two weak solution for Problem (1.1), then for all $\varphi \in L^p\left((0,T)\,;W_0^{1,p}(\Omega,\omega)\right) \cap L^\infty\left((0,T);L^2(\Omega)\right)$,

$$\begin{aligned} -\int_Q (u-v)\frac{\partial\varphi}{\partial t}\mathrm{d}x\mathrm{d}t + \int_Q \left(|u|^{p-2}u - |v|^{p-2}v\right)\varphi \mathrm{d}x\mathrm{d}t \\ + \int_Q \omega\left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t = 0. \end{aligned}$$

Let $u - v$ as a test function in the weak formulation of Problem (1.1), then

$$\begin{aligned} \frac{1}{2}\int_\Omega (u(t)-v(t))^2\,\mathrm{d}x + \int_Q \left(|u|^{p-2}u - |v|^{p-2}v\right)(u-v)\,\mathrm{d}x\mathrm{d}t \\ + \int_Q \omega\left(|\nabla u|^{p-2}\,\nabla u - |\nabla v|^{p-2}\,\nabla v\right)\cdot(\nabla u - \nabla v)\,\mathrm{d}x\,\mathrm{d}t = 0. \end{aligned}$$

This implies that

$$\frac{1}{2}\int_\Omega (u(t)-v(t))^2\,\mathrm{d}x = 0.$$

Therefore, $u = v$ a.e. in Q, this completes the proof of uniqueness. □

4 Entropy solution of continuous problem

The aim of this section is the proof of the main result of this article, it is the existence of an entropy solution of the Problem (1.1).

Definition 2. Let $f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$. A measurable function u defined on Q is an entropy solution of Problem (1.1) if and only if

$u \in C\left((0,T);L^1(\Omega)\right), T_k(u) \in L^p((0,T);W_0^{1,p}(\Omega,\omega)$ and for all $k>0$,

$$\int_\Omega S_k(u-\phi)(T)dx - \int_\Omega S_k(u-\phi)(0)dx + \int_0^T \left\langle \frac{\partial \phi}{\partial s}, T_k(u-\phi)\right\rangle ds$$
$$+ \int_Q |u|^{p-2}uT_k(u-\phi)dxds + \int_Q |\nabla u|^{p-2}\nabla u \nabla T_k(u-\phi)dxds$$
$$\leq \int_Q fT_k(u-\phi)dxds,$$

for all $\phi \in L^p((0,T);W_0^{1,p}(\Omega)\cap L^\infty(Q)\cap C\left((0,T);L^1(\Omega)\right)$ and
$\frac{\partial \phi}{\partial t} \in L^{p'}((0,T);W^{-1,p'}(\Omega)$.

Next, we give the main result of this paper.

Theorem 3. *Let $f \in L^1(Q)$, $u_0 \in L^1(\Omega)$ and let the hypothesis (H) holds, the Problem (1.1) has an entropy solution.*

Proof. Let the approximation problem

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - div\left(\omega|\nabla u_n|^{p-2}\nabla u_n\right) + |u_n|^{p-2}u_n = f & \text{in} \quad Q :=]0;T[\times\Omega, \\ u_n = 0 & \text{on} \quad \Gamma :=]0;T[\times\partial\Omega, \\ u_n(.,0) = u_{0n} & \text{in} \quad \Omega, \end{cases} \tag{4.1}$$

where $f_n \in L^\infty(Q)$ such that $\|f_n\|_{L^1(Q)} \leq \|f\|_{L^1(Q)}$, $f_n \to f$ strongly in $L^1(Q)$ and $u_{0n} \in L^2(\Omega)$ such that $\|u_{0n}\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)}$, $u_{0n} \to u_0$ strongly in $L^1(\Omega)$. By Theorem 1, the Problem (4.1) has a weak solution u_n. To prove that (1.1) has an entropy solution, it suffices to show the following lemmas. □

Lemma 3. *Let u_n be a solution of approximate Problem (4.1) and let $k>0$, we have*

$$|T_k(u_n)|_{L^p\left(0,T,W_0^{1,p}(\Omega,\omega)\right)} \leq Ck^{1/p} \quad \textit{for all } n \in \mathbb{N},$$

where C is a constant independent of n.

Proof. Taking $T_k(u_n)$ as a test function in (4.1) for get

$$\int_\Omega S_k(u_n)(T)dx + \int_Q |u_n|^{p-2}u_nT_k(u_n)dxds + \int_Q \omega|\nabla u_n|^{p-2}\nabla u_n\nabla T_k(u_n)dxds$$
$$= \int_\Omega S_k(u_0)\,dx + \int_Q fT_k(u_n)dxds.$$

This implies that

$$\int_\Omega S_k(u_n)(T)dx + \int_Q |u_n|^{p-2}u_nT_k(u_n)dxds + \int_Q |\nabla u_n|^{p-2}\nabla u_n\nabla T_k(u_n)dxds$$
$$\leq \int_\Omega S_k(u_0)\,dx + k\,||\,f\,||_{L^1(Q)}\,.$$

Note that, $|u_n|^{p-2}u_nT_k(u_n) \geq 0$, $S_k \geq 0$ and $S_k(r) \leq k|r|$, therefore,

$$\int_Q \omega|\nabla T(u_n)|^p \leq k\left((\|u_{0n}\|_{L^1(Q)} + \|f\|_{L^1(Q)}\right) \text{ for all } k \geq 1.$$

Thus,

$$|T_k\left(u_n\right)|_{L^p\left(0,T,W_0^{1,p}(\Omega,\omega)\right)} \leq Ck^{1/p} \text{ for all } n \in \mathbb{N}.$$

□

Lemma 4. *Let u_n be a solution of approximate Problem* (4.1)*, then there exists subsequence, still denoted u_n, such that*
(i) $u_n \to u$ a.e. in Q; (ii) $\nabla u_n \to u$ in Q;
(iii) $u_n \to u$ in $C\left((0,T);L^1(\Omega)\right)$.

Proof. (i) Let $k > 0$ be large enough. We have by Markov's inequality, Proposition 3 and Lemma 3,

$$\text{meas}\left\{|u_n| > k\right\} \leq \frac{\|T_k\left(u_n\right)\|^p_{L^p(Q,\omega)}}{k^p} \leq \frac{C_1|T_k\left(u_n\right)|^p_{L^p\left(0,T);W_0^{1,p}(\Omega,\omega)\right)}}{k^p} \leq \frac{C_2}{k^{p-1}}.$$

It yields

$$\text{meas}\left\{|u_n| > k\right\} \to 0, \qquad \text{as } k \to +\infty. \tag{4.2}$$

Let $\delta > 0$, $k > 0$ and let the following sets

$$E_1 := \left\{|u_n| > k\right\}, E_2 := \left\{|u_m| > k\right\}, E_3 := \left\{|T_k\left(u_n\right) - T_k\left(u_m\right)| > \delta\right\}.$$

Then,

$$\text{meas}\left\{|u_n - u_m| > \delta\right\} \leq \text{meas}\left(E_1\right) + \text{meas}\left(E_2\right) + \text{meas}\left(E_3\right). \tag{4.3}$$

Let $\varepsilon > 0$, by (4.2), we can choose $k = k(\varepsilon)$ such that

$$\text{meas}\left(E_1\right) \leq \varepsilon/3 \quad \text{and} \quad \text{meas}\left(E_2\right) \leq \varepsilon/3. \tag{4.4}$$

Since $T_k\left(u_n\right)$ is bounded in $L^p\left((0,T);W_0^{1,p}(\Omega,\omega)\right)$, then there exists some η_k in $L^p\left((0,T);W_0^{1,p}(\Omega,\omega)\right)$ such that $T_k\left(u_n\right) \rightharpoonup \eta_k$ in $L^p\left((0,T);W_0^{1,p}(\Omega,\omega)\right)$ as $n \to \infty$ and by the embedding compact, it follows that

$$T_k\left(u_n\right) \to \eta_k \quad \text{in } L^p(Q,\omega) \text{ and a.e. in } \Omega. \tag{4.5}$$

Consequently, $\left(T_k\left(u_n\right)\right)_n$ is a Cauchy sequence in measure in Ω. Thus, for all $n,m \geq n_0(\delta,\varepsilon)$,

$$\text{meas}\left(E_3\right) \leq \varepsilon/3. \tag{4.6}$$

Finally, from (4.3), (4.4) and (4.6), we obtain, for all $n,m \geq n_0(\delta,\varepsilon)$,

$$\text{meas }\left\{|u_n - u_m| > \delta\right\} \leq \varepsilon.$$

This implies that (u_n) is a Cauchy sequence in measure, then $u_n \to u$ in measure, up to a subsequence and we can assume that $u_n \to u$ a.e. in Q.

(ii) Let $\delta > 0$ and let the following sets

$$\begin{aligned}&E_{11} := \{|u_n| > h\} \cup \{|u_m| > h\}, E_{22} := \{|u_n - u_m| > 1\},\\ &E_{33} := \{|\nabla T_k(u_n)| > h\} \cup \{|\nabla T_k(u_m)| > h\},\\ &E_{44} := \{|\nabla T_k(u_n)| \le h,\ |\nabla T_k(u_m)| \le h,\ |u_n - u_m| \le 1,\ |\nabla u_n - \nabla u_m| > \delta\}.\end{aligned}$$

It is obvious that

$$\{|\nabla u_n - \nabla u_m| > \delta\} \subset E_{11} \cup E_{22} \cup E_{33} \cup E_{44}. \tag{4.7}$$

Let $\varepsilon > 0$, we have by (i) and for h sufficiently large that

$$\text{meas}\,(E_{11}) \le \varepsilon/4, \text{ for all } n, m \ge 0. \tag{4.8}$$

On the other hand, by (i), (u_n) is a Cauchy sequence in measure, then there exists $N_1(\varepsilon) \in \mathbb{N}$ such that

$$\text{meas}\,(E_{22}) \le \varepsilon/4, \text{ for all } n, m \ge N_1(\varepsilon). \tag{4.9}$$

Since $u_n \to u$ a.e. in Q and by (4.5),

$$\begin{aligned}T_k(u_n) &\to T_k(u) \quad \text{in } L^p((0,T); W_0^{1,p}(\Omega,\omega)),\\ T_k(u_n) &\to T_k(u) \quad \text{in } L^p(Q,\omega) \text{ and a.e. in } Q.\end{aligned} \tag{4.10}$$

Therefore, by using (4.10) and for h sufficiently large, we obtain

$$\text{meas}\, E_{33} \le \varepsilon/4 \text{ for all } n, m \ge 0.$$

Now, let the following function $\mathcal{D}$ and the following set $\mathbf{K}$

$$\begin{aligned}&\mathcal{D} : (\xi,\eta) \mapsto \omega(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta).(\xi - \eta),\\ &\mathbf{K} := \left\{(\xi,\eta) \in \mathbb{R}^N \times \mathbb{R}^N, |\xi| \le k, |\eta| \le k, |\xi - \eta| > s\right\}.\end{aligned}$$

Note, that $\mathcal{D}$ is continuous and $\mathbf{K}$ is compact, so by using the following inequality

$$\omega(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta).(\xi - \eta) > 0. \tag{4.11}$$

The function $\mathcal{D}$ attains its minimum on set $\mathbf{K}$, denoted it by β. It is easily to see that $\beta > 0$ and

$$\begin{aligned}\int_{E_{44}} \beta dx &\le \int_{E_{44}} \omega\left[|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m\right] \cdot \nabla T_l(u_n - u_m) dxds\\ &\le \int_Q \omega\left[|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m\right] \cdot \nabla T_l(u_n - u_m) dxds.\end{aligned}$$

Let $T_l(u_n - u_m)$ as a test function in (4.1), with $t \le T$, therefore,

$$\begin{aligned}&\int_Q \left[|u_n|^{p-2}u_n - |u_m|^{p-2}u_m\right] T_l(u_n - u_m) dxds + \int_\Omega S_1(u_n - u_m)(t) dx\\ &\quad + \int_Q \omega\left[|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m\right] \cdot \nabla T_l(u_n - u_m) dxds\\ &= \int_Q (f_n - f_m) T_1(u_n - u_m) dxds + \int_\Omega S_1(u_n - u_m)(0) dx.\end{aligned}$$

Using the fact that $\left[|u_n|^{p-2}u_n - |u_m|^{p-2}u_m\right] T_l(u_n - u_m) \geq 0$, $S_l(x) \geq 0$ and $S_l(x) \leq l|x|$ for all $x \in \Omega$, to get

$$\begin{aligned}
&\int_Q \omega(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m) \cdot \nabla T_l(u_n - u_m) dxds \\
&\leq \int_Q (f_n - f_m) T_1 (u_n - u_m) dxds + \int_\Omega S_1 (u_n - u_m)(0) dx \\
&\leq 2l \left(\|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}\right). \qquad (4.12)
\end{aligned}$$

The minimum $\beta > 0$ of the function $\mathcal{D}$ on $\mathbf{K}$ is strictly positive, then, the above inequality (4.12) implies that

$$\beta \operatorname{meas}(E_{44}) \leq 2l(\|f\|_{L^1(\Omega)} + \|u_0\|_{L^1(\Omega)}).$$

Hence,

$$\operatorname{meas}(E_{44}) \leq \varepsilon/4, \qquad (4.13)$$

for every $m, n \in \mathbb{N}$, provided that l is sufficiently small. Thus, the inequalities (4.7), (4.8), (4.9) and (4.13) tell us that (∇u_n) is actually a Cauchy sequence in measure. As a consequence, there exists a subsequent, still denoted by (∇u_n), such that $\nabla u_n \to \nabla u$ a.e. in Q.

(iii) The sequence (u_n) is a Cauchy sequence in $C\left((0,T); L^1(\Omega)\right)$, then there exists subsequence still denoted (u_n) such that u_n converges to u and $u \in C\left((0,T); L^1(\Omega)\right)$.

Let $T_l(u_n - u_m)$ as a test function in (4.1), with $t \leq T$, then,

$$\begin{aligned}
&\int_\Omega S_1 (u_n - u_m)(t) dx + \int_Q \left[|u_n|^{p-2}u_n - |u_m|^{p-2}u_m\right] T_l(u_n - u_m) dxds \\
&\quad + \int_0^t \int_\Omega \omega \left[|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m\right] \cdot \nabla T_l(u_n - u_m) dxds \\
&\leq T \int_\Omega |f_n - f_m| dx + \int_\Omega S_1 (u_{0n} - u_{0m}) dx := b_{n,m}.
\end{aligned}$$

Moreover, by using Lemma 1 and (4.11), we obtain

$$\int_\Omega S_1 (u_n - u_m)(t) dx \leq b_{n,m}. \qquad (4.14)$$

Since

$$\int_{|u_n - u_m|<1} |u_n - u_m|^2 (t) + \int_{|u_n - u_m|>1} \frac{|u_n - u_m|(t)}{2} \leq \int_\Omega S_1 (u_n - u_m)(t),$$

then, (4.14) implies that

$$\int_{|u_n - u_m|<1} |u_n - u_m|^2 (t) + \int_{|u_n - u_m|>1} \frac{|u_n - u_m|(t)}{2} \leq b_{n,m},$$

which yields

$$\int_{\Omega} |u_n - u_m|(t) = \int_{|u_n-u_m|<1} |u_n - u_m|(t) + \int_{|u_n-u_m|>1} |u_n - u_m|(t)$$

$$\leq \left(\int_{|u_n-u_m|<1} |u_n-u_m|^2(t)\right)^{\frac{1}{2}} \text{meas}(\Omega)^{\frac{1}{2}}+2b_{n,m} \leq (2 \text{ meas } (\Omega))^{\frac{1}{2}} b_{n,m}^{\frac{1}{2}}+2b_{n,m}.$$

Since (f_n) and (u_n) converge in $L^1(Q)$, then $b_{n,m} \to 0$ for m and $n \to \infty$. Thus, (u_n) is a Cauchy sequence in $C\left((0,T);L^1(\Omega)\right)$. Moreover, there exists a subsequence, still denoted (u_n), such that $u_n \to u$ in $C\left((0,T);L^1(\Omega)\right)$ and $u \in C\left((0,T);L^1(\Omega)\right)$. Now, we can show that u is an entropy solution.

Let $\varphi \in L^p((0,T);W_0^{1,p}(\Omega)\cap L^\infty(Q)\cap C\left((0,T);L^1(\Omega)\right)$, choosing $T_k(u_n-\varphi)$ as a test function in (4.1), then

$$\int_{\Omega} S_k(u_n-\varphi)(T)dx + \int_{\Omega} S_k\left(u_{0n}-\varphi(0)\right)dx + \int_0^T \left\langle \frac{\partial\varphi}{\partial s}, T_k(u_n-\varphi)\right\rangle ds$$
$$+ \int_Q |u_n|^{p-2}u_n T_k(u_n-\varphi)dxds + \int_Q |\nabla u_n|^{p-2}\nabla u_n \nabla T_k(u_n-\varphi)dxds$$
$$= \int_Q f_n T_k(u_n-\varphi)dxds. \tag{4.15}$$

The results of Lemma 4 allow us to conclude that the function S_k is $k-$ *Lipschitz*, thus,

$$\int_{\Omega} S_k(u_n-\varphi)(T)\mathrm{d}x + \int_{\Omega} S_k(u_{0n}-\varphi)(0)\mathrm{d}x$$
$$\to \int_{\Omega} S_k(u-\varphi)(T)\mathrm{d}x + \int_{\Omega} S_k\left(u_0-\varphi(0)\right)\mathrm{d}x. \tag{4.16}$$

Therefore, the fact that $\frac{\partial\varphi}{\partial s} \in L^{p'}\left((0,T);W^{-1,p'}(\Omega)\right)$, implies for $n \to \infty$ that

$$\int_0^T < \frac{\partial\varphi}{\partial t}, T_k(u_n-\varphi) > ds \to \int_0^T < \frac{\partial\varphi}{\partial s}, T_k(u-\varphi) > ds. \tag{4.17}$$

Let $M = \|\varphi\|_\infty$, $G_{n,k} = \{|T_{k+M}(u_n) - \varphi| \leq k\}$, $G_k = \{|T_{k+M}(u) - \varphi| \leq k\}$ then,

$$\int_Q \omega|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla T_k(u_n-\varphi)dxds = \int_Q \omega|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla T_k(T_{k+M}(u_n)$$
$$-\varphi)dxds = \int_Q \omega|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla T_{k+M}(u_n)1_{G_{n,k}}dxds$$
$$- \int_Q \omega|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\varphi 1_{G_{n,k}}dxds.$$

The sequel $(T_{k+M}(u_n))$ is bounded in $L^p\left(0,T;W_0^{1,p}(\Omega,\omega)\right)$ and $\nabla u_n \to \nabla u$ a.e. in Q, then $\nabla T_{k+M}(u_n) \to \nabla T_{k+M}(u)$ a.e. in Q and Lebesgue's theorem implies that

$$\int_Q \omega\left|\nabla u_n\right|^{p-2} \nabla u_n \cdot \nabla T_{k+M}\left(u_n\right) 1_{G_{n,k}} dxds$$
$$\to \int_Q \omega|\nabla u|^{p-2}\nabla u \cdot \nabla T_{k+M} 1_{G_k} dxds,$$
$$\int_Q \omega\left|\nabla u_n\right|^{p-2} \nabla u_n \cdot \nabla \varphi 1_{G_{n,k}} dxds \to \int_Q \omega|\nabla u|^{p-2}\nabla u \cdot \nabla \varphi 1_{G_k} dxds.$$

Thus, implies that

$$\int_Q \omega|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla T_k(u_n-\varphi)dxds \to \int_Q \omega|\nabla u|^{p-2}\nabla u \cdot \nabla T_k(u-\varphi)dxds. \tag{4.18}$$

On the other hand, we have

$$\int_Q |u_n|^{p-2}u_n T_k\left(u_n-\varphi\right)dxds = \int_Q \left(\left|u_n\right|^{p-2} u_n - |\varphi|^{p-2}\varphi\right)$$
$$\times T_k\left(u_n-\varphi\right)dxds + \int_Q |\varphi|^{p-2}\varphi T_k\left(u_n-\varphi\right)dxds.$$

Note that $(|u_n|^{p-2}u_n - |\varphi|^{p-2}\varphi)T_k\left(u_n-\varphi\right) \geq 0$ and converges to $(|u|^{p-2}u - |\varphi|^{p-2}\varphi)T_k\left(u-\varphi\right)$ a.e. in Q, then, so the use of Fatou's lemma implies that

$$\liminf_{n\to\infty} \int_Q \left(\left|u_n\right|^{p-2} u_n - |\varphi|^{p-2}\varphi\right) T_k\left(u_n-\varphi\right)dxds$$
$$\geq \int_Q (|u|^{p-2}u - |\varphi|^{p-2}\varphi)T_k\left(u-\varphi\right)dxds.$$

Since $T_k(u_n-\varphi)$ converges weakly $*$ to $T_k(u-\varphi)$ in $L^\infty(Q)$ and $|\varphi|^{p-2}\varphi \in L^1(Q)$, then

$$\int_Q |\varphi|^{p-2}\varphi T_k(u_n-\varphi)dxds \to \int_Q |\varphi|^{p-2}\varphi T_k(u-\varphi)dxds.$$

Hence,

$$\liminf_{n\to\infty} \int_Q |u_n|^{p-2}u_n T_k\left(u_n-\varphi\right)dxds \geq \int_Q |u|^{p-2}u T_k\left(u-\varphi\right)dxds. \tag{4.19}$$

For the last term, as we know that $T_k(u_n-\varphi)$ converges weakly $*$ to $T_k(u-\varphi)$ in $L^\infty(Q)$ and $f_n \to f$ in $L^1(Q)$, then

$$\int_Q f_n T_k(u_n-\varphi)dxds \to \int_Q f T_k(u-\varphi)dxds. \tag{4.20}$$

Finally, by passing to limit, as $n\to\infty$, in (4.15) and by using the results (4.16), (4.17), (4.18), (4.19) and (4.20), we deduce that u is an entropy solution of the Problem (1.1). □

5 Conclusion and perspectives

In this work, we study the question of existence of entropy solution for the parabolic Problem (1.1) in weighted Sobolev space with Dirichlet type boundary condition, by using optimization method combined with a difference scheme and a priori estimates. Other questions are still being processed, it is the question of uniqueness entropy solution of this problem and the question of existence and uniqueness solution of this problem in the case where the data are in L^1 and the exponent is variable.

References

[1] A. Abbassi, C. Allalou and A. Kassidi. Existence of entropy solutions for anisotropic elliptic nonlinear problem in weighted Sobolev space. In *The International Congress of the Moroccan Society of Applied Mathematics*, pp. 102–122. Springer, 2019. https://doi.org/10.1007/978-3-030-62299-2_8.

[2] S. Antontsev and S. Shmarev. Anisotropic parabolic equations with variable nonlinearity. *Publicacions Matematiques*, pp. 355–399, 2009. https://doi.org/10.5565/PUBLMAT_53209_04.

[3] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre and J. Vázquez. An l^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, **22**(2):241–273, 1995.

[4] V. Bhuvaneswari, S. Lingeshwaran and K. Balachandran. Weak solutions for p-Laplacian equation. *Advances in Nonlinear Analysis*, **1**(4):319–334, 2012. https://doi.org/10.1515/anona-2012-0009.

[5] L. Boccardo and Th. Gallouët. Non-linear elliptic and parabolic equations involving measure data. *Journal of Functional Analysis*, **87**(1):149–169, 1989. https://doi.org/10.1016/0022-1236(89)90005-0.

[6] A. Cavalheiro. Weighted Sobolev spaces and degenerate elliptic equations. *Boletim da Sociedade Paranaense de Matemática*, **26**(1-2):117–132, 2008. https://doi.org/10.5269/bspm.v26i1-2.7415.

[7] A. Cianchi and V. Maz'ya. Second-order regularity for parabolic p-Laplace problems. *The Journal of Geometric Analysis*, **30**(2):1565–1583, 2020. https://doi.org/10.1007/s12220-019-00213-3.

[8] P. Drabek, A. Kufner and V. Mustonen. Pseudo-monotonicity and degenerated or singular elliptic operators. *Bulletin of the Australian Mathematical Society*, **58**(2):213–221, 1998. https://doi.org/10.1017/S0004972700032184.

[9] G. Gagneux and M. Madaune-Tort. *Analyse mathématique de modéles non linéaires de l'ingénierie pétroliére*, volume 22. Springer Science & Business Media, 1995.

[10] A. El Hachimi, J. Igbida and A. Jamea. Existence result for nonlinear parabolic problems with l^1-data. *Appl. Math.(Warsaw)*, **37**(4):483–508, 2010. https://doi.org/10.4064/am37-4-6.

[11] J. Heinonen, T. Kilpelanen and O. Martio. *Nonlinear Potential Theory of Degenerate Elliptic Equations.* Dover Publications, Inc., Mineola, New York, 2006.

[12] L. Hmidouch, A. Jamea and M. Laghdir. Optimization method for a nonlinear degenerate parabolic problem in weighted Sobolev spaces. *Discussiones Mathematicae: Differential Inclusions, Control & Optimization*, **41**(1), 2021. https://doi.org/10.7151/dmdico.1225.

[13] A. Khaleghi and A. Razani. Existence and multiplicity of solutions for $p(x)$-Laplacian problem with Steklov boundary condition. *Boundary Value Problems*, **2022**(1):1–11, 2022. https://doi.org/10.1186/s13661-022-01624-y.

[14] M. El Ouaarabi, A. Abbassi and C. Allalou. Existence result for a Dirichlet problem governed by nonlinear degenerate elliptic equation in weighted Sobolev spaces. *Journal of Elliptic and Parabolic Equations*, **7**(1):221–242, 2021. https://doi.org/10.1007/s41808-021-00102-3.

[15] M. El Ouaarabi, A. Abbassi and C. Allalou. Existence and uniqueness of weak solution in weighted Sobolev spaces for a class of nonlinear degenerate elliptic problems with measure data. *International Journal of Nonlinear Analysis and Applications*, **13**(1):2635–2653, 2022.

[16] M. El Ouaarabi, C. Allalou and A. Abbassi. On the Dirichlet problem for some nonlinear degenerated elliptic equations with weight. In *2021 7th International Conference on Optimization and Applications (ICOA)*, pp. 1–6. IEEE, 2021. https://doi.org/10.1109/ICOA51614.2021.9442620.

[17] M. El Ouaarabi, C. Allalou and S. Melliani. Existence of weak solution for a class of $p(x)$-Laplacian problems depending on three real parameters with Dirichlet condition. *Boletín de la Sociedad Matemática Mexicana*, **28**(2):1–16, 2022. https://doi.org/10.1007/s40590-022-00427-6.

[18] M. Ragusa, A. Razani and F. Safari. Existence of radial solutions for $p(x)$-Laplacian Dirichlet problem. *Advances in Difference Equations*, **2021**(1):1–14, 2021. https://doi.org/10.1186/s13662-021-03369-x.

[19] K. Rajagopal and M. Ružička. Mathematical modeling of electrorheological materials. *Continuum mechanics and thermodynamics*, **13**(1):59–78, 2001. https://doi.org/10.1007/s001610100034.

[20] M. Růžička. Electrorheological fluids: modeling and mathematical theory. *Lecture Notes in Math*, **1748**:16–38, 2000. https://doi.org/10.1007/BFb0104029.

[21] T. Singer. Existence of weak solutions of parabolic systems with p, q-growth. *manuscripta mathematica*, **151**(1):87–112, 2016. https://doi.org/10.1007/s00229-016-0827-1.

[22] M. Tsutsumi. Existence and nonexistence of global solutions for nonlinear parabolic equations. *Publications of the Research Institute for Mathematical Sciences*, **8**(2):211–229, 1972. https://doi.org/10.2977/prims/1195193108.

[23] B.O. Turesson. *Nonlinear potential theory and weighted Sobolev spaces*, volume 1736. Springer Science & Business Media, 2000. https://doi.org/10.1007/BFb0103908.

[24] N. Weisheng, M. Qing and C. Xiaojuan. Asymptotic behavior for nonlinear degenerate parabolic equations with irregular data. *Applicable Analysis*, **100**(16):3391–3405, 2021. https://doi.org/10.1080/00036811.2020.1721470.

[25] M. Xu and S. Zhou. Existence and uniqueness of weak solutions for a generalized thin film equation. *Nonlinear Analysis: Theory, Methods & Applications*, **60**(4):755–774, 2005. https://doi.org/10.1016/j.na.2004.01.013.

[26] C. Zhang. Entropy solutions for nonlinear elliptic equations with variable exponents. *Electronic Journal of Differential Equations*, **92**(2014):1–14, 2014.

[27] C. Zhang and S. Zhou. A fourth-order degenerate parabolic equation with variable exponent. *J. Partial Differ. Equ*, **22**(4):376–392, 2009. https://doi.org/10.4208/jpde.v22.n4.6.

[28] V. Zhikov. On density of smooth functions in Sobolev–Orlich spaces. *Zapiski Nauchnykh Seminarov POMI*, **310**:67–81, 2004.