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A Collocation Method for Fredholm Integral Equations of the First Kind via Iterative Regularization Scheme

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Abstract. To solve the ill-posed integral equations, we use the regularized collocation method. This numerical method is a combination of the Legendre polynomials with non-stationary iterated Tikhonov regularization with fixed parameter. A theoretical justification of the proposed method under the required assumptions is detailed. Finally, numerical experiments demonstrate the efficiency of this method.

Keywords: ill-posed problems, iterative regularization scheme, Legendre collocation method, integral equations of the first kind.

AMS Subject Classification: 47A52; 65R30.

1 Introduction

Integral equations are used in various disciplines of science and engineering. Several physical models, such as spectroscopy, image processing, cosmic radiation, machine learning, and radiography can be modeled as a Fredholm integral equation of the first kind [11, 18, 23].

Now, we consider the integral equation of the first kind

$$Tx(s) = \int_a^b k(s,t)x(t)dt = y(s), \quad s \in [a,b], \tag{1.1}$$

where $k(.,.) \in C([a,b]^2)$, $y \in \mathcal{H} := L^2([a,b],\mathbb{R})$ are known functions and x is the unknown function to be determined in $\mathcal{H}$.

We assume that the kernel $k(.,.)$ of T is non-degenerate. Hence, T is a compact operator of infinite rank from $\mathcal{H}$ to itself. In this case, T does not closed range $\mathcal{R}(T)$, that is, $\mathcal{R}(T) \neq \overline{\mathcal{R}(T)}$, and therefore the problem of finding the best approximate solution x^+ of problem (1.1) is ill-posed, in the sense that, even minor perturbations of y can change the solution drastically. For more details, we refer the reader to [4, 10, 13]. As a result, it is important to high-precision methods for numerically solving them.

Many regularization procedures have been employed to estimate the solution of ill-posed equation (1.1) in recent years, including multi-scale methods [2, 17, 24], projection methods [6, 16, 20, 21], multilevel methods [5, 12], collocation methods [14, 15], and so on.

Collocation methods, as we all know, are a strong tool for solving integral and differential equations in a variety of fields. These numerical methods are considered one of the most famous approximate methods for solving well-posed problems because of their ease of application.

In this paper, we consider the issue of numerically solving these integral equations by applying regularized collocation method. The proposed method is based on the combination of iterative regularization scheme and Legendre collocation method and it leads to fast convergence of the solutions of the discrete equations, and it is different from the collocation methods proposed in [14, 15]. The methods in [14, 15] are mainly based on the quadrature formula to approximate the integral equation (1.1) at collocation points $s_1, s_2, ..., s_n \in [a, b]$ and employing the Tikhonov regularization to treat the ill-posedness of the discrete equation to obtain a stable approximate solution.

In the present work, we will rely mainly on discrete Legendre expansion at collocation points $s_1, s_2, ..., s_n$ to get the ill-posed discrete equation $T_n x = y_n$ and apply the nonstationary iterated Tikhonov regularization with fixed parameter for obtaining a stable approximate solution.

To extract valuable and relevant information from the model presented by ill-posed equations (1.1), the numerical solution of these equations necessitates the application of discretization techniques. This can be accomplished in one of two ways: regularization-discretization (RD) or discretization-regularization (DR). The first method is well studied in the literature for general linear ill-posed problems, we recommend the reader to [4] for further information. In this work, we will adopt the second strategy. In general, the collocation methods are treating linear ill-posed problems by converting these problems into finite-dimensional systems. This discretization gives rise to very ill-conditioned linear systems of algebraic equations. In most instances, the obtained linear systems must be regularized to compute a meaningful approximation solution possible. One of the most often used regularization approaches is iterative regularization scheme [3, 7, 8, 9, 22].

This paper is organized as follows. We introduce the concept of Legendre polynomials and describe the iterative regularization scheme for linear ill-posed equations in Section 2. In Section 3, we present a collocation method for solving the corresponding integral equation of the first kind. We study the convergence of approximate solutions and develop a priori choice of the regularization parameter strategy in Section 4. Finally, in Section 5, numerical examples are given, which illustrate the efficiency of our method and confirm the theoretical analysis of this paper.

2 Preliminaries

This section provides a brief description of the properties of Legendre polynomials and presents some results of nonstationary iterated Tikhonov regularization that we will apply in our study.

2.1 Legendre polynomials

In this subsection, we discuss some properties of Legendre polynomials. We suggest the reader to [1, 19] for more information.

The Legendre polynomials L_k, are defined by the recursion relation

$$L_{k+1}(t) = \frac{2k+1}{k+1} t L_k(t) - \frac{k}{k+1} L_{k-1}(t), k = 1, 2, ...,$$

where $L_0(t) = 1$ and $L_1(t) = t$. The set $\{L_k\}_{k\in\mathbb{N}}$ is a complete orthogonal system in the Hilbert space $L^2(-1, 1)$.

The normalized shifted Legendre polynomials $\widehat{L}_k$ are given on the interval $[a, b]$ by

$$\widehat{L}_k(t) = \sqrt{\frac{2k+1}{b-a}} L_k(\frac{2}{b-a} t - \frac{a+b}{b-a}), \ \ t \in [a, b].$$

Moreover, the set of normalized shifted Legendre polynomials is complete orthonormal system in the Hilbert space $\mathcal{H}$.

For any function $f \in \mathcal{H}$ its formal series in terms of the system $\left\{\widehat{L}_k\right\}_{k\in\mathbb{N}}$ is

$$f(t) = \sum_{k=0}^{\infty} f_k \widehat{L}_k(t), \ \ f_k = \int_a^b f(s)\widehat{L}_k(s)ds.$$

Thus, the orthogonal projection of order n for the function f is defined by

$$P_n f(t) = \sum_{k=0}^{n} f_k \widehat{L}_k(t). \tag{2.1}$$

By completeness of the system $\left\{\widehat{L}_k\right\}_{k\in\mathbb{N}}$, we have the property

$$\|f - P_n f\|_{\mathcal{H}} \to 0 \ \text{ as } n \to \infty. \tag{2.2}$$

Further, if $f \in \mathcal{H}^s$ for some $s > 0$, then

$$\|f - P_n f\|_{\mathcal{H}} \leq c n^{-s} \|f\|_{\mathcal{H}^s}, \tag{2.3}$$

where $c > 0$ and $\mathcal{H}^s = \{h \in \mathcal{H},\, h^{(l)} \in \mathcal{H} \text{ for } l = 1, 2, ..., s\}$ is the Sobolev space whose norm is

$$\|h\|_{\mathcal{H}^s} = \sqrt{\sum_{l=0}^{s} \left\|h^{(l)}\right\|_{\mathcal{H}}^2}.$$

We review the Legendre Gauss (LG) formula that can be used to calculate the numerical value of an integral in the interval $[-1,1]$. In this case, the nodes $\left\{\widehat{t}_k\right\}_{k=0}^{n}$ are the roots of the Legendre polynomial L_{n+1} and the weights $\{\widehat{w}_k\}_{k=0}^{n}$ are given by

$$\widehat{w}_k = \frac{2}{(1-\widehat{t}_k^2)\left[L'_{n+1}(\widehat{t}_k)\right]^2}, \quad k=0,1,...,n,$$

where n is a positive integer. Thus, the nodes and the weights over an interval $[a,b]$ are given by

$$t_k = \frac{2}{b-a}\widehat{t}_k - \frac{a+b}{b-a}, \quad w_k = \frac{b-a}{2}\widehat{w}_k \quad \text{for} \quad k=0,1,...,n.$$

Now, we turn to the discrete Legendre approximation. For this, we can define discrete semi-inner product and its corresponding semi-norm in $C\left([a,b]\right)$ as follows:

$$\langle f,g\rangle_{n,w} = \sum_{k=0}^{n} w_k f(t_k) g(t_k), \ \|f\|_{n,w} = \sqrt{\sum_{k=0}^{n} w_k f^2(t_k)}.$$

We have

$$\left\langle \widehat{L}_i, \widehat{L}_j \right\rangle_{n,w} = \begin{cases} 0, & i\neq j, \\ 1, & i=j. \end{cases} \tag{2.4}$$

2.2 Iterative regularization scheme

In this work, we assume $y\in\mathcal{R}(T)$ (The range space of T). If T is not injective, then the integral equation (1.1) will have several solutions, and in that case, one looks for the minimal norm solution (best-approximate solution) $x^+\in\mathcal{N}(T)^\perp$ which satisfies $Tx^+=y$, that is, the unique $x^+\in\mathcal{H}$ such that

$$\left\|x^+\right\|_{\mathcal{H}} = \min_{x\in\mathcal{S}_y} \|x\|_{\mathcal{H}},$$

where $\mathcal{N}(T)$ denotes the null space of T and $\mathcal{S}_y = \{x\in\mathcal{H} : Tx=y\}$.

Let T^+ be the generalized inverse of T which associates each $y\in\mathcal{D}(T^+) = \mathcal{R}(T)\oplus\ \mathcal{R}(T)^\perp$ to the best-approximate solution $x^+\in\mathcal{R}(T^+) = \mathcal{N}(T)^\perp$. For obtaining a stable approximation solution to the best-approximate solution $x^+ := T^+y$, we replace the generalized inverse T^+ by bounded operators of the form R_mT^*, such that

$$\lim_{m\to\infty} R_m T^* y = T^+ y, \quad y\in\mathcal{D}(T^+),$$

where T^* denotes the adjoint of T defined by

$$T^*y(t) = \int_a^b k(s,t)y(s)ds, \quad t\in[a,b].$$

The family $\{R_mT^*\}_{m\in\mathbb{N}}$ of bounded operators is called a regularization family for T, and $x_m := R_mT^*y$ is called a regularized solution of the ill-posed problem (1.1). Now we will consider the iterated regularization method (see, [7,9,22])

$$x_0 = 0,\ \ (T^*T + \alpha I)x_k = \alpha x_{k-1} + T^*y \quad \text{for} \quad k = 1, 2, ..., m, \tag{2.5}$$

where the number of iteration steps m is used as a regularization parameter and $\alpha > 0$ a fixed parameter.

Remark 1. If T is a self-adjoint operator then the iterated regularization method (2.5) may be replaced by

$$x_0 = 0,\ \ (T + \alpha I)x_k = \alpha x_{k-1} + y \text{ for } k = 1, 2, ..., m. \tag{2.6}$$

Since $T^*y = T^*Tx^+$ we'll start by pointing out that,

$$x^+ - x_m = r_{\alpha,m}(T^*T)x^+ \text{ where } r_{\alpha,m}(\lambda) = (\alpha/(\lambda+\alpha))^m .$$

An easy calculation shows that

$$\sup_{\lambda\in[0,\infty)} r_{\alpha,m}(\lambda) \le 1 \text{ and } \sup_{\lambda\in[0,\infty)} r_{\alpha,m}(\lambda)\lambda^u \le \alpha^u u^u m^{-u},\ 0 < u < m. \tag{2.7}$$

If the solution x^+ of (1.1) fulfills a source condition

$$x^+ = (T^*T)^v z, \quad \|z\|_{\mathcal{H}} \le \rho < \infty,\ 0 < v < m, \tag{2.8}$$

then

$$\left\|x^+ - x_m\right\|_{\mathcal{H}} \le \rho \sup_{\lambda\in[0,\infty)} |r_{\alpha,m}(\lambda)\lambda^v| \le \rho\alpha^v v^v m^{-v}. \tag{2.9}$$

We denote by (σ_i, u_i, v_i) the singular system of T, i.e., (σ_i) is a sequence of positive real numbers such that $\sigma_i \to 0$ and $\{u_i\}$, $\{v_i\}$ are orthonormal basis of orthogonal complements to the null space $\mathcal{N}(T)$ and $\overline{\mathcal{R}(T)}$ respectively.

Since $Tu_i = \sigma_i v_i$, $T^*v_i = \sigma_i u_i$, we have

$$x_m = R_mT^*y = \sum_{j=1}^{\infty} \frac{1 - r_{\alpha,m}(\sigma_j^2)}{\sigma_j} \langle y, v_j\rangle u_j.$$

Suppose y^δ is the noisy data satisfying $\left\|y - y^\delta\right\|_{\mathcal{H}} \le \delta$ for a known error bound $\delta > 0$. For this case the m-th step iterative approximate solution x_m^δ is given by the following relation:

$$x_m^\delta = R_mT^*y^\delta = \sum_{j=1}^{\infty} \frac{1 - r_{\alpha,m}(\sigma_j^2)}{\sigma_j} \left\langle y^\delta, v_j\right\rangle u_j.$$

Moreover, straightforward calculations show that

$$x_m^\delta - x_m = \sum_{j=1}^{\infty} \frac{1 - r_{\alpha,m}(\sigma_j^2)}{\sigma_j} \left\langle y^\delta - y, v_j\right\rangle u_j.$$

Because $r'_{\alpha,m}$ is monotonically non-decreasing on $[0,\infty)$, $r_{\alpha,m}$ is a convex function. Therefore $\frac{1-r_{\alpha,m}(\lambda)}{\lambda} \leq -r'_{\alpha,m}(0) = \frac{m}{\alpha}$. Thus,

$$\begin{aligned}\left\|x_m^\delta - x_m\right\|_{\mathcal{H}}^2 &= \sum_{j=1}^{\infty}\left(\frac{1-r_{\alpha,m}(\sigma_j^2)}{\sigma_j}\right)^2 \left|\left\langle y^\delta - y, v_j\right\rangle\right|^2 \\ &\leq \sup_{\lambda\in[0,\infty)}\left(\frac{1-r_{\alpha,m}(\lambda)}{\lambda}\right)\sup_{\lambda\in[0,\infty)}\left(1-r_{\alpha,m}(\lambda)\right)\delta^2 \\ &\leq \frac{m}{\alpha}\delta^2,\end{aligned}$$

and therefore,

$$\left\|x_m^\delta - x_m\right\|_{\mathcal{H}} \leq \sqrt{\frac{m}{\alpha}}\delta. \tag{2.10}$$

3 Collocation method

In this section, we discuss the collocation regularized method for solving ill-posed integral equations of the first kind (1.1). To this end, for any positive integer n, let $\{t_k\}_{k=0}^n$ and $\{w_k\}_{k=0}^n$ the LG nodes and LG weights respectively.

Now, we define the operators $Q_n : \mathcal{H} \to \mathbb{R}^{n+1}$ by

$$Q_n h = \left(\sqrt{w_0}P_n h(t_0), \sqrt{w_1}P_n h(t_1), ..., \sqrt{w_n}P_n h(t_n)\right)^T,$$

where $h \in \mathcal{H}$, and P_n is the orthogonal projection defined in (2.1).

Lemma 1. *Let n is a positive integer and let $\{t_k\}_{k=0}^n$, $\{w_k\}_{k=0}^n$ the LG nodes and LG weights respectively. Then, the operator Q_n is a bounded operator of norm almost 1 and its adjoint operator is given by*

$$Q_n^* : \mathbb{R}^{n+1} \to \mathcal{H},\ \ Q_n^* u(.) = \sum_{i=0}^{n} u_i\sqrt{w_i}\sum_{j=0}^{n}\widehat{L}_j(t_i)\widehat{L}_j(.),$$

where $u \in \mathbb{R}^{n+1}$ and $\widehat{L}_j, j = 0, 1, ...n$ are normalized shifted Legendre polynomials.

Proof. Let $\|.\|_2$ be the Euclidean norm in $\mathbb{R}^{n+1}$. For every $h \in \mathcal{H}$,

$$\|Q_n h\|_2^2 = \sum_{i=0}^{n}\left(\sqrt{w_i}P_n h(t_i)\right)^2 = \sum_{i=0}^{n} w_i\left(P_n h(t_i)\right)^2.$$

By (2.1), we have

$$\begin{aligned}\|Q_n h\|_2^2 = \sum_{i=0}^{n} w_i\left(\sum_{k=0}^{n} h_k\widehat{L}_k(t_i)\right)^2 &= \sum_{i=0}^{n} w_i\sum_{k=0}^{n} h_k\widehat{L}_k(t_i)\sum_{j=0}^{n} h_j\widehat{L}_j(t_i) \\ &= \sum_{k=0}^{n} h_k\sum_{j=0}^{n} h_j\sum_{i=0}^{n} w_i\widehat{L}_k(t_i)\widehat{L}_j(t_i).\end{aligned}$$

Using the formula (2.4),

$$\|Q_n h\|_2^2 = \sum_{k=0}^{n} h_k \sum_{j=0}^{n} h_j \left\langle \widehat{L}_k, \widehat{L}_j \right\rangle_{n,w} = \sum_{k=0}^{n} h_k^2.$$

Since $\|h\|_{\mathcal{H}}^2 = \sum\limits_{k=0}^{\infty} h_k^2$, we have $\|Q_n h\|_2 \le \|h\|_{\mathcal{H}}$, so that Q_n is a bounded operator and $\|Q_n\|_2 \le 1$. Moreover,

$$\langle Q_n h, u\rangle_2 = \sum_{i=0}^{n} u_i \sqrt{w_i} P_n h(t_i) = \sum_{i=0}^{n} u_i \sqrt{w_i} \sum_{j=0}^{n} h_j \widehat{L}_j(t_i),$$

where $h_j = \int_a^b h(s)\widehat{L}_j(s)ds$. Therefore,

$$\langle Q_n h, u\rangle_2 = \sum_{i=0}^{n} u_i \sqrt{w_i} \sum_{j=0}^{n} \int_a^b h(s)\widehat{L}_j(s)ds\widehat{L}_j(t_i) = \int_a^b h(s) \sum_{i=0}^{n} u_i \sqrt{w_i}$$
$$\times \sum_{j=0}^{n} \widehat{L}_j(t_i)\widehat{L}_j(s)ds = \left\langle h, \sum_{i=0}^{n} u_i \sqrt{w_i} \sum_{j=0}^{n} \widehat{L}_j(t_i)\widehat{L}_j \right\rangle_{\mathcal{H}} = \langle h, Q_n^* u\rangle_{\mathcal{H}}.$$

This completes the proof. □

Remark 2. The product of operators Q_n^* and Q_n is P_n. Indeed for $h \in C([a,b])$, we have

$$Q_n^* Q_n h(.) = \sum_{i=0}^{n} \left(\sqrt{w_i} \sum_{k=0}^{n} h_k \widehat{L}_k(t_i) \right) \sqrt{w_i} \sum_{j=0}^{n} \widehat{L}_j(t_i)\widehat{L}_j(.)$$
$$= \sum_{k=0}^{n} \sum_{j=0}^{n} h_k \widehat{L}_j(.) \left\langle \widehat{L}_k, \widehat{L}_j \right\rangle_{n,w} = P_n h(.).$$

Assume that y is a continuous function. Then, analogous to the method in [14, 15], we replace the original ill-posed equation (1.1) with the discretized problem

$$T_n x = Q_n T x = Q_n y = y_n. \tag{3.1}$$

Lemma 2. *The operator $T_n : \mathcal{H} \to \mathbb{R}^{n+1}$ is a bounded operator and its adjoint operator is given by*

$$T_n^* : \mathbb{R}^{n+1} \to \mathcal{H},\ T_n^* u(.) = \sum_{i=0}^{n} u_i \sqrt{w_i} \sum_{j=0}^{n} \widehat{L}_j(t_i) T^* \widehat{L}_j(.).$$

Proof. For every $h \in \mathcal{H}$, by Lemma 1 we have

$$\|T_n h\|_2 = \|T Q_n h\|_2 \le \|T\| \, \|h\|_{\mathcal{H}}.$$

Moreover,

$$T_n^* u(.) = T^* Q_n^* u(.),$$

where $u \in \mathbb{R}^{n+1}$. This completes the proof. $\square$

Since the best-approximate solution x^+ satisfies the Equation (1.1), we have $T_n x^+ = y_n$, equivalently,

$$\begin{aligned}
&\sqrt{w_l}\sum_{i=0}^{n}\left\langle x^+, T^*\widehat{L}_i\right\rangle_{\mathcal{H}} \widehat{L}_i(t_l) = \sqrt{w_l}\sum_{i=0}^{n}\left\langle y, \widehat{L}_i\right\rangle_{\mathcal{H}} \widehat{L}_i(t_l), \quad l = 0,1,...,n,\\
&\sum_{i=0}^{n}\left\langle x^+, T^*\widehat{L}_i\right\rangle_{\mathcal{H}} \left\langle \widehat{L}_i, \widehat{L}_j\right\rangle_{n,w} = \sum_{i=0}^{n}\left\langle y, \widehat{L}_i\right\rangle_{\mathcal{H}} \left\langle \widehat{L}_i, \widehat{L}_j\right\rangle_{n,w},\\
&\left\langle x^+, T^*\widehat{L}_i\right\rangle_{\mathcal{H}} = \left\langle y, \widehat{L}_i\right\rangle_{\mathcal{H}}, \quad i = 0,1,...,n. \qquad (3.2)
\end{aligned}$$

Let x_n^+ the best-approximate solution of Eq. (3.1), thus $x_n^+ \in \mathcal{N}(T_n)^\perp = \mathcal{R}(T_n^*)$. Then, there exists $\mathbf{d} = (d_0, d_1, ..., d_n)^T \in \mathbb{R}^{n+1}$, such that

$$x_n^+(.) = \sum_{i=0}^{n} d_i T^*\widehat{L}_i(.).$$

Also, by (3.2), we obtain

$$\begin{aligned}
\left\langle \sum_{i=0}^{n} d_i T^*\widehat{L}_i, T^*\widehat{L}_j\right\rangle_{\mathcal{H}} &= \left\langle y, \widehat{L}_j\right\rangle_{\mathcal{H}}, \quad j = 0,1,...,n,\\
\sum_{i=0}^{n} d_i \left\langle T^*\widehat{L}_i, T^*\widehat{L}_j\right\rangle_{\mathcal{H}} &= \left\langle y, \widehat{L}_j\right\rangle_{\mathcal{H}}, \quad j = 0,1,...,n,
\end{aligned}$$

and therefore, the coefficients d_i, $i = 0,1,...,n$ can be found from the linear algebraic system

$$\mathbf{A}\mathbf{d} = \mathbf{g}, \qquad (3.3)$$

where

$$\mathbf{A} = [\mathbf{A}_{i,j}]_{i,j=0}^{n} = \left[\left\langle T^*\widehat{L}_i, T^*\widehat{L}_j\right\rangle_{\mathcal{H}}\right]_{i,j=0}^{n},$$

is a symmetric matrix in $\mathbb{R}^{(n+1)\times(n+1)}$ and $\mathbf{g} \in \mathbb{R}^{(n+1)}$, $\mathbf{g}_j = \left\langle y, \widehat{L}_j\right\rangle_{\mathcal{H}}$.

Since the original problem (1.1) is ill-posed, the problem of finding the best approximate solution x_n^+ of the finite-dimensional system (3.3) is unstable. For this, we regularize this system by iterated regularization as in (2.6), resulting in

$$\mathbf{d}^{(0)} = 0, \ \ (\mathbf{A} + \alpha\mathbf{I})\ \mathbf{d}^{(k)} = \alpha\mathbf{d}^{(k-1)} + \mathbf{g} \text{ for } k = 1,2,...,m. \qquad (3.4)$$

For $i = 0,1,...,n$, the iterative formula (3.4) can be written as

$$d_i^{(0)} = 0, \ \ (\sum_{j=0}^{n}\mathbf{A}_{i,j} + \alpha)d_i^{(k)} = \alpha d_i^{(k-1)} + \mathbf{g}_i \text{ for } k = 1,2,...,m.$$

Thus, for $k = 1, 2, ...m$, we have

$$\sum_{i=0}^{n}(\sum_{j=0}^{n}\mathbf{A}_{i,j} + \alpha)d_i^{(k)}T^*\widehat{L}_i(.) = \alpha\sum_{i=0}^{n}d_i^{(k-1)}T^*\widehat{L}_i(.) + \sum_{i=0}^{n}\mathbf{g}_iT^*\widehat{L}_i(.),$$

$$\sum_{i=0}^{n}d_i^{(0)}T^*\widehat{L}_i(.) = 0.$$

Define $x_n^{(k)}$ by

$$x_n^{(k)} = \sum_{i=0}^{n}d_i^{(k)}T^*\widehat{L}_i(.),\ k = 0, 1, ..., m.$$

Then, it follows that

$$x_n^{(0)} = 0,\ (T_n^*T_n + \alpha I)x_n^{(k)} = \alpha x_n^{(k-1)} + T_n^*y_n.$$

Now, let us consider iterated regularization by replacing exact data y by inexact data y^δ which is satisfying $\left\|y - y^\delta\right\|_{\mathcal{H}} \leq \delta$ for a known error bound $\delta > 0$. In this situation, we have

$$\mathbf{d}^{(0),\delta} = 0,\ \ (\mathbf{A} + \alpha\mathbf{I})\mathbf{d}^{(k),\delta} = \alpha\mathbf{d}^{(k-1),\delta} + \mathbf{g}^\delta \text{ for } k = 1, 2, ..., m,$$

so that

$$x_n^{(0),\delta} = 0,\ (T_n^*T_n + \alpha I)x_n^{(k),\delta} = \alpha x_n^{(k-1),\delta} + T_n^*y_n^\delta \text{ for } k = 1, 2, ..., m,$$

where $\mathbf{g}_i^\delta = \left\langle y^\delta, \widehat{L}_i\right\rangle_{\mathcal{H}}$, $i = 0, 1, ..., n$ and

$$x_n^{(k),\delta}(.) = \sum_{i=0}^{n}d_i^{(k),\delta}T^*\widehat{L}_i(.) \text{ for } k = 0, 1, ..., m.$$

4 Convergence and error estimates

The goal of this section is to discuss the convergence rate for our numerical method. Now, let $y \in \mathcal{R}(T)$ and assume the exact solution x^+ fulfils the smoothness source condition (2.8).

Since $y_n = T_nx^+$ and $y = Tx^+$, we will start by pointing out that,

$$x^+ - x_n^{(m)} = r_{\alpha,m}(F_n)x^+ \text{ and } x^+ - x_m = r_{\alpha,m}(F)x^+,$$

where $F_n = T_n^*T_n$ and $F = T^*T$. Our first goal is to estimate the quantity $\|x_m - x_n^{(m)}\|_{\mathcal{H}}^2$. We use the following expression to accomplish this

$$x_n^{(m)} - x_m = \alpha^m(F_n + \alpha I)^{-m}((F_n + \alpha I)^m - (F + \alpha I)^m)(F + \alpha I)^{-m}x^+,$$

we get

$$x_n^{(m)} - x_m = \alpha^m(F_n + \alpha I)^{-m}\Big[\sum_{i=1}^{m}(F_n + \alpha I)^{m-i}\widetilde{F}_n(F + \alpha I)^{i-1}\Big](F + \alpha I)^{-m}x^+, \tag{4.1}$$

where $\widetilde{F}_n = F_n - F$.

Lemma 3. *Assume that the exact solution x^+ fulfils the smoothness source condition (2.8). If $0 < v < 1$, then*

$$\left\|x_n^{(m)} - x_m\right\|_{\mathcal{H}} \leq \rho\alpha^{v-1}\frac{v^v}{1-v}\left\|\widetilde{F}_n\right\| m^{1-v}. \tag{4.2}$$

Moreover, if $1 \leq v < m$, then we have

$$\left\|x_n^{(m)} - x_m\right\|_{\mathcal{H}} \leq \rho\left\|\widetilde{F}_n\right\|\|F\|^{v-\frac{v}{m}}v^{\frac{v}{m}}\left(\frac{m}{\alpha}\right)^{1-\frac{v}{m}}. \tag{4.3}$$

Proof. Let $0 < v < 1$, by using (4.1), we have

$$\begin{aligned} x_n^{(m)}-x_m &= \alpha^m\left(\sum_{i=1}^{m}(F_n+\alpha I)^{-i}\widetilde{F}_n(F+\alpha I)^{i-m-1}\right)F^v z \\ &= \alpha^m\left(\sum_{i=1}^{m}(F_n+\alpha I)^{-i}\widetilde{F}_n\left[r_{\alpha,m+1-i}(F)F^v\right]\alpha^{-m-1+i}\right)z. \end{aligned}$$

By using (2.7), we also have

$$\begin{aligned} \left\|x_n^{(m)}-x_m\right\|_{\mathcal{H}} &\leq \rho\alpha^m\sum_{i=1}^{m}\left\|(F_n+\alpha I)^{-1}\right\|^i \\ &\times\left\|\widetilde{F}_n\right\|\left(\alpha^v v^v(m+1-i)^{-v}\right)\alpha^{-m-1+i} \leq \rho\alpha^{v-1}v^v\left\|\widetilde{F}_n\right\|\sum_{i=1}^{m}\frac{1}{i^v}. \end{aligned}$$

Because $1/i^v \leq \int_{i-1}^{i} t^{-v}dt$, then

$$\left\|x_n^{(m)} - x_m\right\|_{\mathcal{H}} \leq \rho\alpha^{v-1}v^v\left\|\widetilde{F}_n\right\|\int_0^m\frac{dt}{t^v} = \rho\alpha^{v-1}\frac{v^v}{1-v}m^{1-v}\left\|\widetilde{F}_n\right\|.$$

Now, let $1 \leq v < m$, we have

$$\begin{aligned} x_n^{(m)} - x_m &= \alpha^m\left(\sum_{i=1}^{m}(F_n+\alpha I)^{-i}\widetilde{F}_n(F+\alpha I)^{i-m-1}\right)F^v z \\ &= \alpha^m\left(\sum_{i=1}^{m}(F_n+\alpha I)^{-i}\widetilde{F}_n\left[r_{\alpha,m+1-i}(F)F^{\frac{m+1-i}{m}v}\right]\alpha^{-m-1+i}F^{\frac{i-1}{m}v}\right)z. \end{aligned}$$

By using (2.7), we also have

$$\begin{aligned} \left\|r_{\alpha,m+1-i}(F)F^{\frac{m+1-i}{m}}\right\| &\leq \sup_{\lambda\in[0,\infty)} r_{\alpha,m+1-i}(\lambda)\lambda^{\frac{m+1-i}{m}v} \\ &\leq \alpha^{u_i}u_i^{u_i}(m+1-i)^{-u_i},\ i = 1,2,...,m\ , \end{aligned}$$

where $u_i = \frac{m+1-i}{m} v$ and therefore,

$$\begin{aligned}\left\|x_n^{(m)} - x_m\right\|_{\mathcal{H}} &\leq \alpha^m \left(\sum_{i=1}^{m} \alpha^{-i} \left\|\widetilde{F}_n\right\| \left(\alpha^{u_i} v^{u_i} (\frac{mu_i}{v})^{-u_i}\right) \alpha^{-m-1+i} \left\|F\right\|^{\frac{i-1}{m} v} \right) \rho \\ &\leq \alpha^{-1} \rho \left\|\widetilde{F}_n\right\| \sum_{i=1}^{m} \alpha^{u_i} u_i^{u_i} (\frac{mu_i}{v})^{-u_i} \left\|F\right\|^{\frac{i-1}{m} v} \\ &\leq \alpha^{-1} \rho \left\|\widetilde{F}_n\right\| \left\|F\right\|^{\frac{-v}{m}} \left(\frac{\alpha v}{m}\right)^{\frac{(m+1)v}{m}} \sum_{i=1}^{m} \left(\frac{m\left\|F\right\|}{\alpha v}\right)^{\frac{iv}{m}} \\ &\leq \alpha^{-1} \rho \left\|\widetilde{F}_n\right\| \left\|F\right\|^{\frac{-v}{m}} \left(\frac{\alpha v}{m}\right)^{\frac{(m+1)v}{m}} m \left(\frac{\alpha v}{m\left\|F\right\|}\right)^{-v}.\end{aligned}$$

Finally, we have (4.3). This completes the proof of Lemma 3. □

The estimations in Lemma 3 indicate that selecting the number of discretizations n to get $\left\|\widetilde{F}_n\right\| \to 0$ as $n \to 0$ is acceptable. To this end, we consider the following theorem.

Theorem 1. *Then*

$$\left\|\widetilde{F}_n\right\| \to 0 \text{ as } n \to \infty.$$

Moreover, if $k(.,.) \in C^s([a,b]^2)$, then we have the following estimates

$$\left\|\widetilde{F}_n\right\| \leq c_1 \varepsilon(n), \tag{4.4}$$

where $\varepsilon(n) = n^{-s}$ and c_1 is a positive constant independent of n.

Proof. Let $h \in \mathcal{H}$, by Remark 2 we have

$$\left\|\widetilde{F}_n h\right\|_{\mathcal{H}} \leq \left\|T^*\right\| \left\|(I - P_n) T h\right\|_{\mathcal{H}}. \tag{4.5}$$

Using (2.2) and the compactness of T, we obtain $\left\|(I - P_n) T\right\| \to 0$ as $n \to \infty$, and hence, from (4.5), we have $\left\|\widetilde{F}_n\right\| \to 0$ as $n \to \infty$.

Now, let $k(.,.) \in C^s([a,b]^2)$, then for all (i,j) such that $i+j \leq s$, $\frac{\partial^{i+j} k(s,t)}{\partial^i s \partial^j t}$ is a continuous function on $[a,b]^2$. Under this regularity of the kernel $k(.,.)$ we have $\mathcal{R}(T) \subset \mathcal{H}^s([a,b]^2)$, where $\mathcal{H}^s([a,b]^2)$ is the Sobolev space

$$\mathcal{H}^s([a,b]^2) = \{f \in L^2([a,b]^2), \frac{\partial^{i+j} f(s,t)}{\partial^i s \partial^j t} \in L^2([a,b]^2) \text{ for } i+j \leq s\},$$

equipped with the norm

$$\|f\|_{\mathcal{H}^s([a,b]^2)} = \sqrt{\sum_{i+j=0}^{s} \left\|\frac{\partial^{i+j}}{\partial^i s \partial^j t} f\right\|_{L^2}^2}.$$

Denoting

$$\left\| \frac{\partial^{i+j}}{\partial^i s \partial^j t} k \right\|_{L^2} = \gamma_{i,j} \text{ and } \|k\|_{\mathcal{H}^s([a,b]^2)} = \sqrt{\sum_{i+j=0}^{s} \gamma_{i,j}^2} = \gamma_s < \infty.$$

For $j = 0$ and $h \in \mathcal{H}$, we have

$$\frac{d^i}{ds^i} Th(s) = \int_a^b \frac{\partial^i}{\partial s^i} k(s,t) h(t) dt,\ s \in [a,b],$$

and by Cauchy–Schwarz inequality, we have

$$\left| \frac{d^i}{ds^i} Th(s) \right| \leq \int_a^b \left| \frac{\partial^i}{\partial s^i} k(s,t) \right| |h(t)|\, dt \leq \sqrt{\int_a^b \left| \frac{\partial^i}{\partial s^i} k(s,t) \right|^2 dt}\, \|h\|_{\mathcal{H}}.$$

Therefore,

$$\left\| \frac{d^i}{ds^i} Th \right\|_{\mathcal{H}} \leq \sqrt{\int_a^b \int_a^b \left| \frac{\partial^i}{\partial s^i} k(s,t) \right|^2 dtds}\, \|h\|_{\mathcal{H}} \leq \gamma_{i,0} \|h\|_{\mathcal{H}}.$$

Thus, we obtain

$$\|Tx\|_{\mathcal{H}^s} \leq \gamma_s \|h\|_{\mathcal{H}}.$$

Using the estimations (2.3) and (4.5), we have

$$\left\| \widetilde{F}_n h \right\|_{\mathcal{H}} \leq \|T^*\|\, cn^{-s} \|Th\|_{\mathcal{H}^s} \leq \|T^*\|\, cn^{-s} \gamma_s \|h\|_{\mathcal{H}}.$$

Therefore, we have the estimation (4.4) with $c_1 = \|T^*\|\, c\gamma_s$ and the proof of Theorem 1 is complete. □

In the following theorem we obtain estimates for the error $\left\| x^+ - x_n^{(m)} \right\|_{\mathcal{H}}$ under certain assumptions.

Theorem 2. *Let $y \in \mathcal{R}(T)$, $k(.,.) \in C^s([a,b]^2)$ and assume that the exact solution x^+ fulfils the smoothness source condition (2.8). If $0 < v < 1$, then*

$$\left\| x^+ - x_n^{(m),\delta} \right\|_{\mathcal{H}} \leq c_2 \left(m^{-v} + \sqrt{m}\delta + m^{1-v} \varepsilon(n) \right). \tag{4.6}$$

Moreover, if $1 \leq v < m$, then we have the following estimates

$$\left\| x^+ - x_n^{(m),\delta} \right\|_{\mathcal{H}} \leq c_3 \left(m^{-v} + \sqrt{m}\delta + m\varepsilon(n) \right), \tag{4.7}$$

where c_2 and c_3 be positive constants.

Proof. Since

$$\left\| x^+ - x_n^{(m),\delta} \right\|_{\mathcal{H}} \leq \left\| x^+ - x_m \right\|_{\mathcal{H}} + \left\| x_n^{(m)} - x_m \right\|_{\mathcal{H}} + \left\| x_n^{(m)} - x_n^{(m),\delta} \right\|_{\mathcal{H}},$$

from the relations (2.9) and (2.10), we have

$$\left\|x^{+}-x_{n}^{(m),\delta}\right\|_{\mathcal{H}}\leq\rho\alpha^{v}v^{v}m^{-v}+\sqrt{\frac{m}{\alpha}}\delta+\left\|x_{n}^{(m)}-x_{m}\right\|_{\mathcal{H}}.$$

Let $0<v<1$, from (4.2) and by using the estimate in Theorem 1, we obtain

$$\begin{aligned}\left\|x^{+}-x_{n}^{(m),\delta}\right\|_{\mathcal{H}}&\leq\rho\alpha^{v}v^{v}m^{-v}+\sqrt{\frac{m}{\alpha}}\delta+\rho c_{1}\varepsilon\left(n\right)\frac{\left(\alpha v\right)^{v}}{\alpha\left(1-v\right)}m^{1-v}\\&\leq\max(\rho\alpha^{v}v^{v},\frac{1}{\sqrt{\alpha}},\frac{\rho c_{1}\left(\alpha v\right)^{v}}{\alpha\left(1-v\right)})\left(m^{-v}+\sqrt{m}\delta+m^{1-v}\varepsilon\left(n\right)\right).\end{aligned}$$

Let $1\leq v<m$, from (4.3) and by using the estimate in Theorem 1, we obtain

$$\begin{aligned}\left\|x^{+}-x_{n}^{(m),\delta}\right\|_{\mathcal{H}}&\leq\rho\alpha^{v}v^{v}m^{-v}+\sqrt{\frac{m}{\alpha}}\delta+\rho c_{1}\varepsilon\left(n\right)\left\|F\right\|^{v}v\left(\frac{m}{\alpha}\right)^{1-\frac{v}{m}}.\\&\leq\max(\rho\alpha^{v}v^{v},\frac{1}{\sqrt{\alpha}},\frac{\rho c_{1}\left\|F\right\|^{v}v}{\alpha})\left(m^{-v}+\sqrt{m}\delta+m\varepsilon\left(n\right)\right),\end{aligned}$$

which completes the proof. □

4.1 A priori choice of the regularization parameter

We can have an a priori parameter choice that leads to the best convergence rate based on the estimates in Theorem 2. This theorem allows us to estimate the best possible order of convergence that our numerical approach can achieve. In particular, if $m=m(\delta)$ is chosen such that $\sqrt{m\left(\delta\right)}\delta\rightarrow0$ as $\delta\rightarrow0$ and $m\left(\delta\right)\varepsilon\left(n\right)\rightarrow0$ as $\delta\rightarrow0$ (if $0<v<1$) or $\left(m\left(\delta\right)\right)^{1-v}\varepsilon\left(n\right)\rightarrow0$ as $\delta\rightarrow0$ (if $1\leq v<m\left(\delta\right)$).

Theorem 3. *Suppose that conditions of Theorem 2 hold and we assume that $m\left(\delta\right)=\lceil\delta^{-\frac{2}{2v+1}}\rceil$. If $0<v<1$ and let n be the least positive integer such that $\varepsilon\left(n\right)\leq\delta^{\frac{2}{2v+1}}$, then*

$$\left\|x^{+}-x_{n}^{(m(\delta)),\delta}\right\|_{\mathcal{H}}=O(\delta^{\frac{2v}{2v+1}}).$$

Moreover, if $1\leq v<m$ and let n be the least positive integer such that $\varepsilon\left(n\right)\leq\delta^{\frac{2v+2}{2v+1}}$, then

$$\left\|x^{+}-x_{n}^{(m(\delta)),\delta}\right\|_{\mathcal{H}}=O(\delta^{\frac{2v}{2v+1}}),$$

where $\lceil a\rceil$ is the integer part of a.

Proof. Let $0<v<1$ and n be the least positive integer such that $\varepsilon\left(n\right)\leq\delta^{\frac{2}{2v+1}}$. By using the estimate (4.6) in Theorem 2, we have

$$\left\|x^{+}-x_{n}^{(m),\delta}\right\|_{\mathcal{H}}\leq c_{2}\left(\delta^{\frac{2v}{2v+1}}+\delta^{\frac{-1}{2v+1}}\delta+\delta^{-\frac{2-2v}{2v+1}}\delta^{\frac{2}{2v+1}}\right)\leq3c_{2}\delta^{\frac{2v}{2v+1}}.$$

Now, $1 \leq v < m$ and n be the least positive integer such that $\varepsilon(n) \leq \delta^{\frac{2v+2}{2v+1}}$. By using the estimate (4.7) in Theorem 2, we have

$$\left\| x^{+} - x_{n}^{(m),\delta} \right\|_{\mathcal{H}} \leq c_3 \left(\delta^{\frac{2v}{2v+1}} + \delta^{\frac{-1}{2v+1}} \delta + \delta^{\frac{-2}{2v+1}} \delta^{\frac{2v+2}{2v+1}} \right) \leq 3c_3 \delta^{\frac{2v}{2v+1}}.$$

The proof of the theorem finished. $\square$

From the Theorem 3 it follows that this is the case when m is large enough, we have the order of convergence $O(\delta^{l})$ where $l \approx 1$.

5 Numerical examples

In this section, several numerical examples are given to approximate the solution of Fredholm integral equations of the first kind using the numerical method described in this paper. The numerical experiments are implemented in Matlab R2013a software. In all experiments, we choose the parameter of regularization $m = m(\delta)$ by a priori parameter choice strategy described in Theorem 3. We introduce the relative error by the notation $E_n(\alpha)$ as follows:

$$E_n(\alpha) = \frac{\left\| x^{+} - x_{n}^{(m(\delta)),\delta} \right\|_{\mathcal{H}}}{\|x^{+}\|_{\mathcal{H}}}.$$

Example 1. Consider the following integral equation of the first kind:

$$Tx^{+}(s) = \int_{0}^{\frac{\pi}{2}} \sin(s) t^{2} x^{+}(t) dt = \frac{\pi^{5}}{160} \sin(s), \quad 0 \leq s \leq \frac{\pi}{2},$$

with the exact solution $x^{+}(t) = t^{2}$. Moreover, $x^{+} = T^{*}z$, $z = 1$, which means $v = \frac{1}{2}$.

The numerical results of Example 1 are presented in Figure 1 and Tables 1–3

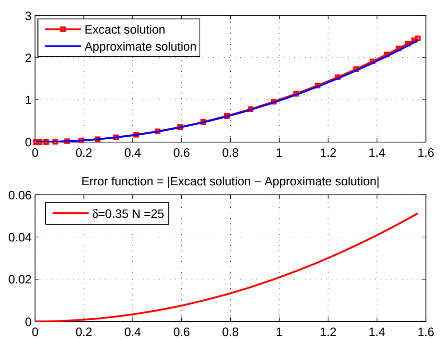

Figure 1. Example 1: exact and computed approximate solutions, absolute errors with $n = 25$ and noise level 35%.

Table 1. Numerical results for Examples 1–3 with noise level 10%.

	Example 1			Example 2			Example 3		
n	m	$E_n(0.5)$	$E_n(0.1)$	m	$E_n(0.5)$	$E_n(0.1)$	m	$E_n(0.5)$	$E_n(0.1)$
3	11	$4.6E-2$	$2.0E-2$	3	$9.2E-3$	$9.1E-3$	5	$4.4E-1$	$1.5E-1$
6	11	$1.3E-2$	$1.1E-2$	3	$2.3E-6$	$1.4E-7$	5	$4.2E-1$	$1.2E-1$
9	11	$9.2E-3$	$3.4E-3$	3	$1.1E-7$	$2.4E-9$	5	$4.1E-1$	$1.0E-1$

Table 2. Numerical results for Examples 1–3 with noise level 1%.

	Example 1			Example 2			Example 3		
n	m	$E_n(0.1)$	$E_n(0.05)$	m	$E_n(0.1)$	$E_n(0.05)$	m	$E_n(0.1)$	$E_n(0.05)$
3	101	$1.1E-3$	$7.3E-4$	7	$9.1E-3$	$9.1E-3$	22	$6.3E-2$	$5.9E-2$
6	101	$6.1E-4$	$4.0E-4$	7	$1.4E-7$	$1.4E-7$	22	$5.7E-2$	$5.7E-2$
9	101	$3.2E-4$	$3.2E-4$	7	$3E-10$	$5E-11$	22	$4.6E-2$	$4.5E-2$

Example 2. As the second example, the following integral equation is considered

$$Tx^+(s) = \int_0^1 e^{2s+3t}x^+(t)dt = \frac{1}{1728}e^{2s}(e^3-1)(e^4-1)^2(e^6-1)^2, \quad 0 \le s \le 1,$$

with the exact solution $x^+(t) = \frac{1}{288}e^{3t}(e^3-1)(e^4-1)^2(e^6-1)$. Moreover, $x^+ = (T^*T)^2 z$, $z = 1$, which means $v = 2$.

The numerical results of Example 2 are presented in Tables 1–3 and Figure 2.

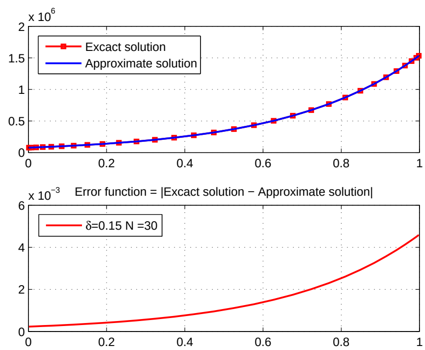

Figure 2. Example 2: exact and computed approximate solutions, absolute errors with $n = 30$ and noise level 15%.

Example 3. Consider the following Fredholm integral equation

$$Tx^+(s) = \int_0^1 (2s^4+st^3+4t-1)x^+(t)dt = \frac{1931}{360}s^4 + \frac{60\,563}{50\,400}s + \frac{3593}{720}, \quad 0 \le s \le 1,$$

Table 3. Numerical results for Examples 1–3 with noise level 0.1%.

	Example 1			Example 2			Example 3		
n	m	$E_n(0.8)$	$E_n(0.1)$	m	$E_n(0.8)$	$E_n(0.1)$	m	$E_n(0.8)$	$E_n(0.1)$
3	1001	$2.5E-4$	$1.1E-4$	16	$9.1E-3$	$9.1E-3$	100	$6.2E-2$	$5.8E-2$
6	1001	$9.9E-5$	$6.4E-5$	16	$1.4E-7$	$1.4E-7$	100	$6.1E-2$	$5.1E-2$
9	1001	$5.8E-5$	$3.2E-5$	16	$3E-11$	$1E-11$	100	$5.8E-2$	$2.6E-2$

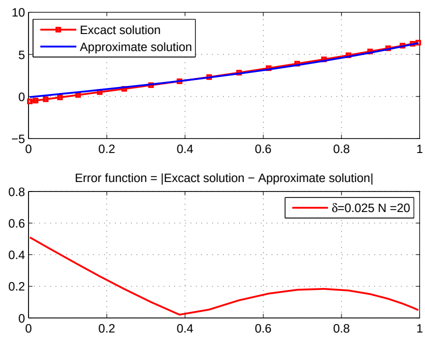

Figure 3. Example 3: exact and computed approximate solutions, absolute errors with $n = 20$ and noise level 2.5%.

with the exact solution $x^+(t) = \frac{11}{12}t^3 + \frac{61}{10}t - \frac{43}{72}$. Moreover, $x^+ = T^*Tz$, $z = 1$, which means $v = 1$.

The numerical results of Example 3 are presented in Tables 1–3 and Figure 3.

The comparisons between the approximate solutions of Examples 1–3 (see Tables 1–3) both with their exact solutions where α in $(0, 1)$ confirmed the validity and accuracy of the new regularized-collocation method. These results show that when the parameter α is small, the relative error is lower. However, the difference between the values of the parameters does not significantly affect the relative errors.

6 Conclusions

In this work, we have employed an efficient method to solve integral equations of the first kind. This method is a combination of Legendre collocation method and the iterative regularization method. The numerical experiments have demonstrated the validity and the applicability of the suggested method. However, our method shows good results with large noise levels.

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References

[1] C. Canuto, M.Y. Hussaini, A. Quarteroni and T.A. Zang. *Spectral methods: fundamentals in single domains.* Springer Science & Business Media, 2007. https://doi.org/10.1007/978-3-540-30726-6.

[2] Z. Chen, S. Cheng, G. Nelakanti and H. Yang. A fast multiscale Galerkin method for the first kind ill-posed integral equations via Tikhonov regularization. *International Journal of Computer Mathematics*, **87**(3):565–582, 2010. https://doi.org/10.1080/00207160802155302.

[3] M. Donatelli. On nondecreasing sequences of regularization parameters for nonstationary iterated Tikhonov. *Numerical Algorithms*, **60**(4):651–668, 2012.

[4] H.W. Engl, M. Hanke and A. Neubauer. *Regularization of inverse problems*, volume 375. Springer Science & Business Media, 1996.

[5] L. Fanchun, Y. Suhua, L. Xingjun and P. Yubing. Multilevel iterative algorithm for solving Fredholm integral equation of the first kind. *Mathematica Numerica Sinica*, **35**(3):225–238, 2013. https://doi.org/10.12286/jssx.2013.3.225.

[6] U. Hamarik, E. Avi and A. Ganina. On the solution of ill-posed problems by projection methods with a posteriori choice of the discretization level. *Mathematical Modelling and Analysis*, **7**(2):241–252, 2002. https://doi.org/10.3846/13926292.2002.9637196.

[7] M. Hanke and C.W. Groetsch. Nonstationary iterated Tikhonov regularization. *Journal of Optimization Theory and Applications*, **98**(1):37–53, 1998. https://doi.org/10.1023/A:1022680629327.

[8] G. Huang, L. Reichel and F. Yin. Projected nonstationary iterated Tikhonov regularization. *BIT Numerical Mathematics*, **56**(2):467–487, 2016. https://doi.org/10.1007/s10543-015-0568-7.

[9] A.V. Kryanev. An iterative method for solving incorrectly posed problems. *USSR Computational Mathematics and Mathematical Physics*, **14**(1):24–35, 1974. https://doi.org/10.1016/0041-5553(74)90133-5.

[10] S. Lu and S.V. Pereverzev. *Regularization theory for ill-posed problems.* de Gruyter, Berlin, Boston, 2013. https://doi.org/10.1515/9783110286496.

[11] Y. Lu, L. Shen and Y. Xu. Multi-parameter regularization methods for high-resolution image reconstruction with displacement errors. *IEEE Transactions on Circuits and Systems I: Regular Papers*, **54**(8):1788–1799, 2007. https://doi.org/10.1109/TCSI.2007.902535.

[12] X. Luo, L. Fan, Y. Wu and F. Li. Fast multi-level iteration methods with compression technique for solving ill-posed integral equations. *Journal of Computational and Applied Mathematics*, **256**:131–151, 2014. https://doi.org/10.1016/j.cam.2013.07.043.

[13] M.T. Nair. *Linear operator equations: approximation and regularization.* World Scientific, 2009.

[14] M.T. Nair. Quadrature based collocation methods for integral equations of the first kind. *Advances in Computational Mathematics*, **36**(2):315–329, 2012. https://doi.org/10.1007/s10444-011-9196-1.

[15] M.T. Nair and S.V. Pereverzev. Regularized collocation method for Fredholm integral equations of the first kind. *Journal of Complexity*, **23**(4):454–467, 2007. https://doi.org/10.1016/j.jco.2006.09.002.

[16] B. Neggal, N. Boussetila and F. Rebbani. Projected Tikhonov regularization method for Fredholm integral equations of the first kind. *Journal of Inequalities and Applications*, **2016**(195):1–21, 2016. https://doi.org/10.1186/s13660-016-1137-6.

[17] B. Nemati Saray. Sparse multiscale representation of Galerkin method for solving linear-mixed Volterra-Fredholm integral equations. *Mathematical Methods in the Applied Sciences*, **43**(5):2601–2614, 2020. https://doi.org/doi.org/10.1002/mma.6068.

[18] T. Poggio and C.R. Shelton. On the mathematical foundations of learning. *Bulletin of the American Mathematical Society*, **39**(1):1–49, 2002. https://doi.org/10.1090/S0273-0979-01-00923-5.

[19] A. Quarteroni and A. Valli. *Numerical approximation of partial differential equations*, volume 23. Springer Science & Business Media, 2008.

[20] M.P. Rajan. A modified convergence analysis for solving Fredholm integral equations of the first kind. *Integral Equations and Operator Theory*, **49**(4):511–516, 2004. https://doi.org/10.1007/s00020-002-1213-9.

[21] B. Tahar, B. Nadjib and R. Faouzia. A variant of projection-regularization method for ill-posed linear operator equations. *International Journal of Computational Methods*, **18**(04):2150008, 2021. https://doi.org/10.1142/S0219876221500080.

[22] J. Thomas King and D. Chillingworth. Approximation of generalized inverses by iterated regularization. *Numerical Functional Analysis and Optimization*, **1**(5):499–513, 1979. https://doi.org/10.1080/01630567908816031.

[23] C.R. Vogel and M.E. Oman. Fast, robust total variation-based reconstruction of noisy, blurred images. *IEEE transactions on image processing*, **7**(6):813–824, 1998. https://doi.org/10.1109/83.679423.

[24] S. Yang, X. Luo, F. Li and G. Long. A fast multiscale Galerkin method for the first kind ill-posed integral equations via iterated regularization. *Applied Mathematics and Computation*, **219**(21):10527–10537, 2013. https://doi.org/10.1016/j.amc.2013.04.029.