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Generalized Laplace Transform and Tempered Ψ-Caputo Fractional Derivative

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Abstract. In this paper, images of the tempered Ψ-Hilfer fractional integral and the tempered Ψ-Caputo fractional derivative under the generalized Laplace transform are derived. The results are applied to find a solution to an initial value problem for a nonhomogeneous linear fractional differential equation with the tempered Ψ-Caputo fractional derivative of an order α for $n-1<\alpha<n\in\mathbb{N}$. An illustrative example is given for $0<\alpha<1$ comparing solutions to the same initial value problem but with different tempering and Ψ.

Keywords: Laplace transform, fractional derivative, fractional differential equation, representation of solution.

AMS Subject Classification: 26A33; 34A08; 44A10.

1 Introduction

Unilateral Laplace transform defined as

$$\mathcal{L}\{f(t)\}(s)=\int_0^\infty \mathrm{e}^{-st}f(t)dt$$

for an exponentially bounded function $f\colon[0,\infty)\to\mathbb{R}$ is a classic tool used in theory of ordinary differential equations to find a solution to an initial value problem. Same transform can be used for differential equations with constant delays (see e.g., [5,12]) or fractional derivative (cf. [8,10,11]). Recently, a modification of the Laplace transform was applied in [7] to differential equations with Riemann-Liouville and Caputo type fractional derivatives derived from Ψ-Hilfer fractional integral. In this paper, we apply this generalized Laplace transform to find solutions of fractional differential equations containing a newly defined [9] tempered Ψ-Caputo fractional derivative which represents a connection between tempered Caputo derivative [13] and Ψ-Caputo fractional derivative [1].

Our results for linear differential equations are in a good agreement with solutions found in [7] if we neglect tempering, as well as with solutions from [9] to homogeneous equations. The results for differential equations can be easily applied to systems (see Remark 3).

We note that generalizations of the classic Caputo fractional derivative such as the above mentioned ones provide more flexible alternatives with practical applications e.g. in viscoelasticity [17] or diffusion processes [3, 18] (see also references therein). Method of Laplace transform was also applied in [15] along with Adomian decomposition to obtaind a semi-analytical solution to a nonlinear fractional differential equation with Caputo–Fabrizio fractional derivative. In [6], ρ-Laplace transform, which is a particular case of the generalized Laplace transform, was used to solve equations containing generalized Caputo derivative.

The paper is organized as follows. In the next section, we collect preliminary results. In Section 3, we derive images of the tempered Ψ-Hilfer fractional integral and the tempered Ψ-Caputo fractional derivative under the generalized Laplace transform. The final section is devoted to Cauchy problems for differential equations with the tempered Ψ-Caputo fractional derivative.

Throughout the paper, we denote by $\mathbb{N}$, $\mathbb{N}_0$ the set of all positive and nonnegative integers, respectively.

2 Preliminary results

In this section, we recall some known and other results that will be helpful for next sections.

First, we recall results from [7] on a generalized Laplace transform.

Definition 1. Let $f \in C[a,\infty)$ and $\Psi \in C^1[a,\infty)$ satisfy the assumption

H $\Psi'(t) > 0$ for all $t \in [a,\infty)$, and $\lim_{t\to\infty} \Psi(t) = \infty$.

The generalized Laplace transform of f is defined by

$$\mathcal{L}_\Psi\{f(t)\}(s) = \int_a^\infty \mathrm{e}^{-s(\Psi(t)-\Psi(a))} f(t)\Psi'(t)dt$$

for all values of s such that the integral is valid.

Theorem 1. *If $f \in C[a,\infty)$ is of $\Psi(t)$-exponential order, i.e., there exist nonnegative constants M, c and $T \geq a$ such that $|f(t)| \leq M\mathrm{e}^{c\Psi(t)}$ for all $t \geq T$, then $\mathcal{L}_\Psi\{f(t)\}$ exists on (c,∞).*

Remark 1. Let $f \in C[a,\infty)$ be of $\Psi(t)$-exponential order for some increasing $\Psi \in C[a,\infty)$, and $M, c \geq 0$, $T \geq a$ be such that $|f(t)| \leq M\mathrm{e}^{c\Psi(t)}$ for all $t \geq T$. Then the continuity of f gives the existence of $M_1 \geq 0$ such that $\max_{t\in[a,T]} |f(t)| \leq M_1$. So taking $\widetilde{M} = \max\{M, M_1\mathrm{e}^{-c\Psi(a)}\}$, the increasing property of Ψ implies $|f(t)| \leq \widetilde{M}\mathrm{e}^{c\Psi(t)}$ for all $t \geq a$. Therefore, whenever Ψ is increasing, we can take $T = a$.

Furthermore, when Ψ is increasing, we can always assume that $c > 0$. Indeed, if $\Psi(t) \geq 0$ for all $t \geq a$, this is obvious. If $\Psi(a) < 0$, we have

$$|f(t)| \leq M\mathrm{e}^{c\Psi(t)} \leq M\mathrm{e}^{c\Psi(t)}\mathrm{e}^{\varepsilon(\Psi(t)-\Psi(a))} = (M\mathrm{e}^{-\varepsilon\Psi(a)})\mathrm{e}^{(c+\varepsilon)\Psi(t)}$$

for all $t \geq a$, where $\varepsilon > 0$ is arbitrary fixed.

The next lemma concludes some of the properties of the generalized Laplace transform (for some other properties see [7]).

Lemma 1. *Let $\Psi \in C^1[a,\infty)$ satisfy assumption* H. *The generalized Laplace transform has the following properties:*

1. $\mathcal{L}_\Psi\{(f *_\Psi g)(t)\} = \mathcal{L}_\Psi\{f(t)\}\mathcal{L}_\Psi\{g(t)\}$ *for appropriate functions f, g of $\Psi(t)$-exponential order, where*

$$(f *_\Psi g)(t) = \int_a^t f(s)g\left(\Psi^{-1}(\Psi(t)+\Psi(a)-\Psi(s))\right)\Psi'(s)ds$$

is a generalized convolution (cf. [7]). We shall call this particular convolution Ψ-convolution.

2. *If $\alpha > 0$, $\lambda \geq 0$, then*

$$\mathcal{L}_\Psi\left\{[\Psi(t)-\Psi(a)]^{\alpha-1}\mathrm{e}^{-\lambda(\Psi(t)-\Psi(a))}\right\}(s) = \frac{\Gamma(\alpha)}{(s+\lambda)^\alpha}$$

for $s > -\lambda$, where $\Gamma(z) = \int_0^\infty t^{z-1}\mathrm{e}^{-t}dt$ is the Euler gamma function (cf. [16]).

Proof. For the proof of statement 1 see [7, Theorem 3.10]. To prove statement 2 we set $q = \Psi(t) - \Psi(a)$ to derive

$$\begin{aligned}
&\mathcal{L}_\Psi\left\{[\Psi(t)-\Psi(a)]^{\alpha-1}\mathrm{e}^{-\lambda(\Psi(t)-\Psi(a))}\right\}(s)\\
&= \int_a^\infty [\Psi(t)-\Psi(a)]^{\alpha-1}\mathrm{e}^{-(s+\lambda)[\Psi(t)-\Psi(a)]}\Psi'(t)dt\\
&= \int_0^\infty q^{\alpha-1}\mathrm{e}^{-(s+\lambda)q}dq = \frac{1}{(s+\lambda)^\alpha}\int_0^\infty \tilde{q}^{\alpha-1}\mathrm{e}^{-\tilde{q}}d\tilde{q} = \frac{\Gamma(\alpha)}{(s+\lambda)^\alpha}.
\end{aligned}$$

Condition $s + \lambda > 0$ was needed when we changed $\tilde{q} = (s+\lambda)q$. □

We shall also need the following result from [7, Corollary 2].

Lemma 2. *Let $\Psi \in C^{n-1}[a,\infty)$ satisfy assumption* H. *Let $f \in C^{n-1}[a,\infty)$ be such that*

$$f_{\Psi}^{[k]}(t) = \left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^k f(t)$$

is of $\Psi(t)$-exponential order for each $k = 1, 2, \dots, n-1$, and $f_{\Psi}^{[n]} \in C[a,\infty)$. Then the generalized Laplace transform of $f_{\Psi}^{[n]}$ exists and

$$\mathcal{L}_{\Psi}\{f_{\Psi}^{[n]}(t)\}(s) = s^n \mathcal{L}_{\Psi}\{f(t)\}(s) - \sum_{k=0}^{n-1} s^{n-k-1} f_{\Psi}^{[k]}(a).$$

In the rest of the paper, we shall denote $x_{\Psi}^{[0]}(t) = x(t)$.

The following definitions of a generalized fractional integral and a corresponding Caputo-like fractional derivative are from [9].

Definition 2. Let $\alpha > 0$, $\lambda \geq 0$, the real function $x(t)$ be continuous on $[a,b]$ and $\Psi \in C^1[a,b]$ be an increasing function. Then the tempered Ψ-Hilfer fractional integral of order α is defined by

$$I_a^{\alpha,\lambda,\Psi} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (\Psi(t) - \Psi(s))^{\alpha-1} \mathrm{e}^{-\lambda(\Psi(t)-\Psi(s))} \Psi'(s) x(s) ds$$

for $t \in [a,b]$.

Definition 3. Let $\Psi \in C^n[a,b]$ be such that $\Psi'(t) > 0$ for all $t \in [a,b]$. For $n-1 < \alpha < n$, $n \in \mathbb{N}$, $\lambda \geq 0$, the tempered Ψ-Caputo fractional derivative of order α is defined by

$${}^C D_a^{\alpha,\lambda,\Psi} x(t) = \frac{\mathrm{e}^{-\lambda\Psi(t)}}{\Gamma(n-\alpha)} \int_a^t (\Psi(t) - \Psi(s))^{n-\alpha-1} \Psi'(s) x_{\lambda,\Psi}^{[n]}(s) ds$$

for $t \in [a,b]$, where

$$x_{\lambda,\Psi}^{[n]}(t) = \left[\frac{1}{\Psi'(t)}\frac{d}{dt}\right]^n \left(\mathrm{e}^{\lambda\Psi(t)} x(t)\right).$$

Note that using the Ψ-convolution we can write

$$I_a^{\alpha,\lambda,\Psi} x(t) = \frac{1}{\Gamma(\alpha)} \left([\Psi(\cdot) - \Psi(a)]^{\alpha-1} \mathrm{e}^{-\lambda(\Psi(\cdot)-\Psi(a))} *_{\Psi} x \right)(t). \tag{2.1}$$

Moreover,

$$x_{\lambda,\Psi}^{[n]}(t) = \left(\mathrm{e}^{\lambda\Psi(\cdot)} x(\cdot)\right)_{\Psi}^{[n]}(t). \tag{2.2}$$

Let us recall that the tempered Ψ-Hilfer fractional integral, $I_a^{\alpha,\lambda,\Psi} x(t)$, and the tempered Ψ-Caputo fractional derivative, ${}^C D_a^{\alpha,\lambda,\Psi} x(t)$, are generalizations of the Ψ-Hilfer fractional integral (known also as a left-sided fractional integral with respect to another function Ψ (see e.g. [8, Chapter 2.5])), $I_a^{\alpha,\Psi} x(t)$, and the Ψ-Caputo fractional derivative, ${}^C D_a^{\alpha,\Psi} x(t)$ (cf. [1]), respectively. More precisely,

$$I_a^{\alpha,\Psi} x(t) = I_a^{\alpha,0,\Psi} x(t), \qquad {}^C D_a^{\alpha,\Psi} x(t) = {}^C D_a^{\alpha,0,\Psi} x(t).$$

The following lemma provides a generalization of [1, Theorem 4].

Lemma 3. *Let $f \in C^n[a,b]$ and $n-1<\alpha<n$ for some $n \in \mathbb{N}$, and $\Psi \in C^n[a,b]$ be such that $\Psi'(t)>0$ for all $t \in [a,b]$. Then*

$$I_a^{\alpha-k,\Psi C}D_a^{\alpha,\Psi}x(t) = x_\Psi^{[k]}(t) - \sum_{j=k}^{n-1}\frac{(\Psi(t)-\Psi(a))^{j-k}}{(j-k)!}x_\Psi^{[j]}(a)$$

for each $k=0,1,\ldots,n-1$ and all $t \in [a,b]$.

Proof. Let us fix any $k \in \{0,1,\ldots,n-1\}$. First, we rewrite the derivative using integral [1], and then we apply a semigroup property of the Ψ-Hilfer fractional integral [8, Lemma 2.26] to get

$$I_a^{\alpha-k,\Psi C}D_a^{\alpha,\Psi}x(t) = I_a^{\alpha-k,\Psi}I_a^{n-\alpha,\Psi}(x_\Psi^{[n]}(t)) = I_a^{n-k,\Psi}(x_\Psi^{[n]}(t)).$$

Consequently, we use the definition of $I_a^{\alpha,\Psi}$ and apply the integration by parts several times to derive

$$\begin{aligned}
I_a^{n-k,\Psi}(x_\Psi^{[n]}(t)) &= \frac{1}{\Gamma(n-k)}\int_a^t(\Psi(t)-\Psi(s))^{n-k-1}\Psi'(s)x_\Psi^{[n]}(s)ds\\
&= \frac{1}{(n-k-1)!}\int_a^t(\Psi(t)-\Psi(s))^{n-k-1}\frac{d}{ds}x_\Psi^{[n-1]}(s)ds\\
&= -\frac{(\Psi(t)-\Psi(a))^{n-k-1}}{(n-k-1)!}x_\Psi^{[n-1]}(a)\\
&\quad+\frac{1}{(n-k-2)!}\int_a^t(\Psi(t)-\Psi(s))^{n-k-2}\Psi'(s)x_\Psi^{[n-1]}(s)ds\\
&= \cdots = -\sum_{j=k+2}^{n-1}\frac{(\Psi(t)-\Psi(a))^{j-k}}{(j-k)!}x_\Psi^{[j]}(a)+\int_a^t(\Psi(t)-\Psi(s))\Psi'(s)x_\Psi^{[k+2]}(s)ds\\
&= -\sum_{j=k+2}^{n-1}\frac{(\Psi(t)-\Psi(a))^{j-k}}{(j-k)!}x_\Psi^{[j]}(a)+\int_a^t(\Psi(t)-\Psi(s))\frac{d}{ds}x_\Psi^{[k+1]}(s)ds\\
&= -\sum_{j=k+1}^{n-1}\frac{(\Psi(t)-\Psi(a))^{j-k}}{(j-k)!}x_\Psi^{[j]}(a)+\int_a^t\Psi'(s)x_\Psi^{[k+1]}(s)ds\\
&= -\sum_{j=k+1}^{n-1}\frac{(\Psi(t)-\Psi(a))^{j-k}}{(j-k)!}x_\Psi^{[j]}(a)+\int_a^t\frac{d}{ds}x_\Psi^{[k]}(s)ds\\
&= x_\Psi^{[k]}(t)-\sum_{j=k}^{n-1}\frac{(\Psi(t)-\Psi(a))^{j-k}}{(j-k)!}x_\Psi^{[j]}(a).
\end{aligned}$$

This completes the proof. □

We end this section with a particular case of [9, Theorem 2]. We remark that originally the lemma was proved for $\alpha \in (0,1)$, but the proof works for any $\alpha > 0$.

Lemma 4. *Let $\lambda > 0$, $\alpha > 0$, $p > 1$, $p(\alpha - 1) + 1 > 0$, $q = \frac{p}{p-1}$, $b \geq 0$, $r \in C[a, \infty)$ be a nonnegative increasing function, $\Psi \in C^1[a, \infty)$ be such that $\Psi'(t) > 0$ for all $t \geq a$, and $u(t)$ be a nonnegative function satisfying*

$$u(t) \leq r(t) + b \int_a^t ((\Psi(t) - \Psi(s))^{\alpha-1} \mathrm{e}^{-\lambda(\Psi(t)-\Psi(s))} \Psi'(s) u(s) ds$$

for all $t \geq a$. Then $u(t) \leq R(t)\mathrm{e}^{B(\Psi(t)-\Psi(a))}$, $t \geq a$, where

$$R(t) = 2^{\frac{1}{p}} r(t), \quad B = \frac{2^{q-1}}{q} \left(\frac{\Gamma(p(\alpha-1)+1)}{(p\lambda)^{p(\alpha-1)+1}} \right)^{\frac{q}{p}} b^q.$$

3 Generalized Laplace transform and tempered Ψ-fractional calculus

In this section, we derive some properties of the generalized Laplace transform usable for tempered Ψ-fractional calculus.

Lemma 5. *Let $\alpha > 0$, $\lambda \geq 0$, $\Psi \in C^1[a, \infty)$ satisfy assumption* H *and $x \in C[a, \infty)$. Then*

$$\mathcal{L}_\Psi\{I_a^{\alpha,\lambda,\Psi} x(t)\}(s) = \frac{\mathcal{L}_\Psi\{x(t)\}(s)}{(s+\lambda)^\alpha}$$

for all $s > -\lambda$ such that the right-hand side exists.

Proof. To simplify the notation, we denote $X_\Psi(s) = \mathcal{L}_\Psi\{x(t)\}(s)$. Using identity (2.1) and Lemma 1, we obtain

$$\mathcal{L}_\Psi\{I_a^{\alpha,\lambda,\Psi} x(t)\}(s) = \frac{1}{\Gamma(\alpha)} \mathcal{L}_\Psi \left\{ \left([\Psi(\cdot) - \Psi(a)]^{\alpha-1} \mathrm{e}^{-\lambda(\Psi(\cdot)-\Psi(a))} *_\Psi x \right)(t) \right\}(s)$$
$$= \frac{1}{\Gamma(\alpha)} \mathcal{L}_\Psi \left\{ [\Psi(t) - \Psi(a)]^{\alpha-1} \mathrm{e}^{-\lambda(\Psi(t)-\Psi(a))} \right\}(s) X_\Psi(s) = \frac{X_\Psi(s)}{(s+\lambda)^\alpha}.$$

□

Lemma 6. *Let $n-1 < \alpha < n$ for some $n \in \mathbb{N}$, $\lambda \geq 0$, and $\Psi \in C^n[a, \infty)$ satisfy assumption* H. *Let $x \in C^n[a, \infty)$ be such that $x_{\lambda,\Psi}^{[k]}$ is of $\Psi(t)$-exponential order for each $k = 1, 2, \ldots, n-1$, and $x_{\lambda,\Psi}^{[n]} \in C[a, \infty)$. Then*

$$\mathcal{L}_\Psi\{{}^C D_a^{\alpha,\lambda,\Psi} x(t)\}(s) = (s+\lambda)^\alpha \mathcal{L}_\Psi\{x(t)\}(s) - \mathrm{e}^{-\lambda\Psi(a)} \sum_{k=0}^{n-1} (s+\lambda)^{\alpha-k-1} x_{\lambda,\Psi}^{[k]}(a).$$

Proof. Using the relations

$${}^C D_a^{\alpha,\lambda,\Psi} x(t) = \mathrm{e}^{-\lambda\Psi(t)} {}^C D_a^{\alpha,\Psi} (\mathrm{e}^{\lambda\Psi(t)} x(t)), \; I_a^{\alpha,\lambda,\Psi} x(t) = \mathrm{e}^{-\lambda\Psi(t)} I_a^{\alpha,\Psi} (\mathrm{e}^{\lambda\Psi(t)} x(t)), \tag{3.1}$$

from [9], formula

$${}^C D_a^{\alpha,\Psi} x(t) = I_a^{n-\alpha,\Psi} \left[\left(\frac{1}{\Psi'(t)} \frac{d}{dt} \right)^n x(t) \right] = I_a^{n-\alpha,\Psi} (x_\Psi^{[n]}(t)),$$

from [2] and relation (2.2), we obtain

$$
\begin{aligned}
{}^{C}D_a^{\alpha,\lambda,\Psi}x(t) =& \mathrm{e}^{-\lambda\Psi(t)}\,{}^{C}D_a^{\alpha,\Psi}(\mathrm{e}^{\lambda\Psi(t)}x(t)) = \mathrm{e}^{-\lambda\Psi(t)}I_a^{n-\alpha,\Psi}([\mathrm{e}^{\lambda\Psi(\cdot)}x(\cdot)]_{\Psi}^{[n]}(t))\\
=& \mathrm{e}^{-\lambda\Psi(t)}I_a^{n-\alpha,\Psi}(x_{\lambda,\Psi}^{[n]}(t)) = I_a^{n-\alpha,\lambda,\Psi}(\mathrm{e}^{-\lambda\Psi(t)}x_{\lambda,\Psi}^{[n]}(t)).
\end{aligned}
$$

Then by Lemma 5, we get

$$
\begin{aligned}
&\mathcal{L}_{\Psi}\{{}^{C}D_a^{\alpha,\lambda,\Psi}x(t)\}(s) = \mathcal{L}_{\Psi}\{I_a^{n-\alpha,\lambda,\Psi}(\mathrm{e}^{-\lambda\Psi(t)}x_{\lambda,\Psi}^{[n]}(t))\}(s)\\
&\quad= \mathcal{L}_{\Psi}\{\mathrm{e}^{-\lambda\Psi(t)}x_{\lambda,\Psi}^{[n]}(t)\}(s)/(s+\lambda)^{n-\alpha}.
\end{aligned}
$$

Next, we apply [7, Theorem 3.2] stating

$$
\mathcal{L}_{\Psi}\{f(t)\}(s) = \mathcal{L}\{f(\Psi^{-1}(t+\Psi(a)))\}(s) \tag{3.2}
$$

for appropriate function f, where $\mathcal{L}$ is the usual unilateral Laplace transform [14], to be able to make use of its translation property [14, Theorem 1.27],

$$
\mathcal{L}\{\mathrm{e}^{\mu t}f(t)\}(s) = \mathcal{L}\{f(t)\}(s-\mu).
$$

So, we have

$$
\begin{aligned}
&\mathcal{L}_{\Psi}\{{}^{C}D_a^{\alpha,\lambda,\Psi}x(t)\}(s) = \mathcal{L}\{\mathrm{e}^{-\lambda(t+\Psi(a))}x_{\lambda,\Psi}^{[n]}(\Psi^{-1}(t+\Psi(a)))\}(s)/(s+\lambda)^{n-\alpha}\\
&\quad= \mathrm{e}^{-\lambda\Psi(a)}\mathcal{L}\{x_{\lambda,\Psi}^{[n]}(\Psi^{-1}(t+\Psi(a)))\}(s+\lambda)/(s+\lambda)^{n-\alpha}\\
&= \frac{\mathrm{e}^{-\lambda\Psi(a)}\mathcal{L}_{\Psi}\{x_{\lambda,\Psi}^{[n]}(t)\}(s+\lambda)}{(s+\lambda)^{n-\alpha}} = \frac{\mathrm{e}^{-\lambda\Psi(a)}\mathcal{L}_{\Psi}\{(\mathrm{e}^{\lambda\Psi(\cdot)}x(\cdot))_{\Psi}^{[n]}(t)\}(s+\lambda)}{(s+\lambda)^{n-\alpha}}.
\end{aligned}
$$

We finish the proof by Lemma 2 to get

$$
\begin{aligned}
&\mathcal{L}_{\Psi}\{{}^{C}D_a^{\alpha,\lambda,\Psi}x(t)\}(s) = \frac{\mathrm{e}^{-\lambda\Psi(a)}}{(s+\lambda)^{n-\alpha}}\Big[(s+\lambda)^n\mathcal{L}_{\Psi}\{\mathrm{e}^{\lambda\Psi(t)}x(t)\}(s+\lambda)\\
&\qquad-\sum_{k=0}^{n-1}(s+\lambda)^{n-k-1}(\mathrm{e}^{\lambda\Psi(\cdot)}x(\cdot))_{\Psi}^{[k]}(a)\Big]\\
&\quad= \mathrm{e}^{-\lambda\Psi(a)}(s+\lambda)^{\alpha}\mathcal{L}\{\mathrm{e}^{\lambda(t+\Psi(a))}x(\Psi^{-1}(t+\Psi(a)))\}(s+\lambda)\\
&\qquad-\mathrm{e}^{-\lambda\Psi(a)}\sum_{k=0}^{n-1}(s+\lambda)^{\alpha-k-1}x_{\lambda,\Psi}^{[k]}(a)\\
&= (s+\lambda)^{\alpha}\mathcal{L}\{x(\Psi^{-1}(t+\Psi(a)))\}(s) - \mathrm{e}^{-\lambda\Psi(a)}\sum_{k=0}^{n-1}(s+\lambda)^{\alpha-k-1}x_{\lambda,\Psi}^{[k]}(a)\\
&= (s+\lambda)^{\alpha}\mathcal{L}_{\Psi}\{x(t)\}(s) - \mathrm{e}^{-\lambda\Psi(a)}\sum_{k=0}^{n-1}(s+\lambda)^{\alpha-k-1}x_{\lambda,\Psi}^{[k]}(a).
\end{aligned}
$$

□

4 Application to differential equations

In this section, we apply the generalized Laplace transform to differential equations with tempered Ψ-Caputo fractional derivative to obtain a formula for the solution. In particular, we consider the following initial value problem

$$ {}^C D_a^{\alpha,\lambda,\Psi} x(t) = F(t, x(t)), \quad t \geq a, \tag{4.1} $$

$$ x_{\lambda,\Psi}^{[k]}(a) = x_a^k, \quad k = 0, 1, \ldots, n-1 \tag{4.2} $$

for some constants x_a^k, $k = 0, 1, \ldots, n-1$, where $n - 1 < \alpha < n \in \mathbb{N}$, $\lambda \geq 0$, $\Psi \in C^n[a, \infty)$ is such that $\Psi'(t) > 0$ for all $t \geq a$, and $F \in C([a, \infty) \times \mathbb{R}, \mathbb{R})$. Let us recall that $x_{\lambda,\Psi}^{[0]}(t) = \mathrm{e}^{\lambda\Psi(t)} x(t)$.

Definition 4. A function $x \in C^n[a, \infty)$ is a solution to the initial value problem (4.1), (4.2), if ${}^C D_a^{\alpha,\lambda,\Psi} x(t)$ exists and is continuous on (a, ∞), and $x(t)$ fulfills Equation (4.1) and initial conditions (4.2).

Theorem 2. *Function x is a solution to initial value problem* (4.1)–(4.2) *if and only if it solves the integral equation*

$$ \begin{aligned} x(t) =& \mathrm{e}^{-\lambda\Psi(t)} \sum_{j=0}^{n-1} \frac{(\Psi(t) - \Psi(a))^j}{j!} x_a^j \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^t (\Psi(t) - \Psi(s))^{\alpha-1} \mathrm{e}^{-\lambda(\Psi(t)-\Psi(s))} \Psi'(s) F(s, x(s)) ds \end{aligned} \tag{4.3} $$

for all $t \geq a$.

Proof. Applying the operator $I_a^{\alpha,\lambda,\Psi}$ to Eq. (4.1) while using [9, Lemma 1] results in

$$ \begin{aligned} I_a^{\alpha,\lambda,\Psi}\, {}^C D_a^{\alpha,\lambda,\Psi} x(t) =& x(t) - \mathrm{e}^{-\lambda\Psi(t)} \sum_{j=0}^{n-1} \frac{(\Psi(t) - \Psi(a))^j}{j!} x_{\lambda,\Psi}^{[j]}(a) \\ =& \frac{1}{\Gamma(\alpha)} \int_a^t (\Psi(t) - \Psi(s))^{\alpha-1} \mathrm{e}^{-\lambda(\Psi(t)-\Psi(s))} \Psi'(s) F(s, x(s)) ds. \end{aligned} $$

So from initial conditions (4.2), Equation (4.3) follows.

On the other side, applying ${}^C D_a^{\alpha,\lambda,\Psi}$ to Equation (4.3) gives

$$ \begin{aligned} {}^C D_a^{\alpha,\lambda,\Psi} x(t) = {}^C D_a^{\alpha,\lambda,\Psi} \left[\mathrm{e}^{-\lambda\Psi(t)} \sum_{j=0}^{n-1} \frac{(\Psi(t) - \Psi(a))^j}{j!} x_a^j \right] \\ + {}^C D_a^{\alpha,\lambda,\Psi} I_a^{\alpha,\lambda,\Psi} F(t, x(t)). \end{aligned} $$

Then, by the first relation of (3.1) and [9, Lemma 1],

$$ {}^C D_a^{\alpha,\lambda,\Psi} x(t) = \mathrm{e}^{-\lambda\Psi(t)} \sum_{j=0}^{n-1} {}^C D_a^{\alpha,\Psi} ((\Psi(t) - \Psi(a))^j) \frac{x_a^j}{j!} + F(t, x(t)). $$

Since ${}^C D_a^{\alpha,\Psi}((\Psi(t)-\Psi(a))^j) = 0$ whenever $j < n$, $j \in \mathbb{N}_0$ (see [1, equation (2)]), Equation (4.1) follows.

It only remains to verify initial conditions (4.2). Note that for any $k \in \mathbb{N}_0$,

$$\left[\mathrm{e}^{-\lambda\Psi(t)}(\Psi(t)-\Psi(a))^j\right]_{\lambda,\Psi}^{[k]}(a) = [(\Psi(t)-\Psi(a))^j]_\Psi^{[k]}(a)$$
$$= \left[\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^k (\Psi(t)-\Psi(a))^j\right]_{t=a} = \begin{cases} \frac{(\Psi(t)-\Psi(a))^{j-k}j!}{(j-k)!}, & k \le j \\ 0, & k > j \end{cases}\Bigg|_{t=a} = \delta_{jk}j!,$$

where δ_{jk} is the Kronecker delta. Furthermore,

$$\left[I_a^{\alpha,\lambda,\Psi}F(t,x(t))\right]_{\lambda,\Psi}^{[k]}(a) = \left[\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^k \left(\mathrm{e}^{\lambda\Psi(t)}I_a^{\alpha,\lambda,\Psi}F(t,x(t))\right)\right]_{t=a}$$
$$= \left[\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^k I_a^{\alpha,\Psi}(\mathrm{e}^{\lambda\Psi(t)}F(t,x(t)))\right]_{t=a} = \left[I_a^{\alpha,\Psi}(\mathrm{e}^{\lambda\Psi(t)}F(t,x(t)))\right]_\Psi^{[k]}(a)$$
$$= \left[\frac{1}{\Gamma(\alpha-k)}\int_a^t (\Psi(t)-\Psi(s))^{\alpha-k-1}\Psi'(s)\mathrm{e}^{\lambda\Psi(s)}F(s,x(s))ds\right]_{t=a} = 0$$

for each $k \le n-1 < \alpha$, $k \in \mathbb{N}_0$ (for the penultimate equality see [2, proof of Theorem 1]). Hence, applying $f \mapsto f_{\lambda,\Psi}^{[k]}(a)$ to (4.3) for some $k = 0, 1, \dots, n-1$, the corresponding initial condition is confirmed. □

Before applying the generalized Laplace transform we need to be sure that all the assumptions of Lemma 6 are satisfied.

Lemma 7. *Let $n-1 < \alpha < n$ for some $n \in \mathbb{N}$, $\lambda \ge 0$, $\Psi \in C^n[a,\infty)$ be such that $\Psi'(t) > 0$ for all $t \ge a$. If the right-hand side $F \in C([a,\infty)\times\mathbb{R},\mathbb{R})$ of* (4.1) *is of Ψ-exponential order, then so are all the functions $x_{\lambda,\Psi}^{[k]}$, $k = 0, 1, \dots, n-1$, where $x(t)$ is a solution to initial value problem* (4.1)–(4.2).

Proof. Let $k \in \{0, 1, \dots, n-1\}$ be arbitrary and fixed. Applying the operator $I_a^{\alpha-k,\lambda,\Psi}$ to Equation (4.1) yields

$$I_a^{\alpha-k,\lambda,\Psi}\,{}^C D_a^{\alpha,\lambda,\Psi}x(t) = \mathrm{e}^{-\lambda\Psi(t)}I_a^{\alpha-k,\Psi}\,{}^C D_a^{\alpha,\Psi}(\mathrm{e}^{\lambda\Psi(t)}x(t))$$
$$= \mathrm{e}^{-\lambda\Psi(t)}\left[x_{\lambda,\Psi}^{[k]}(t) - \sum_{j=k}^{n-1}\frac{(\Psi(t)-\Psi(a))^{j-k}}{(j-k)!}x_{\lambda,\Psi}^{[j]}(a)\right] = I_a^{\alpha-k,\lambda,\Psi}F(t,x(t))$$

due to relations (3.1) and Lemma 3. Therefrom,

$$x_{\lambda,\Psi}^{[k]}(t) = \sum_{j=k}^{n-1}\frac{(\Psi(t)-\Psi(a))^{j-k}}{(j-k)!}x_{\lambda,\Psi}^{[j]}(a) + \mathrm{e}^{\lambda\Psi(t)}I_a^{\alpha-k,\lambda,\Psi}F(t,x(t))$$
$$= \sum_{j=k}^{n-1}\frac{(\Psi(t)-\Psi(a))^{j-k}}{(j-k)!}x_{\lambda,\Psi}^{[j]}(a)$$
$$+ \frac{1}{\Gamma(\alpha-k)}\int_a^t (\Psi(t)-\Psi(s))^{\alpha-k-1}\mathrm{e}^{\lambda\Psi(s)}\Psi'(s)F(s,x(s))ds. \tag{4.4}$$

Now, applying Remark 1, $\Psi(t)$-exponential order of F implies the existence of $M \geq 0$ and $c > 0$ such that $|F(t,y)| \leq M\mathrm{e}^{c\Psi(t)}$ for all $t \geq a$. On the other side,

$$\left|\sum_{j=k}^{n-1} \frac{(\Psi(t)-\Psi(a))^{j-k}}{(j-k)!} x^{[j]}_{\lambda,\Psi}(a)\right| \leq \max_{j=0,1,\dots,n-1} |x_a^j| \sum_{j=0}^{n-k-1} \frac{(\Psi(t)-\Psi(a))^j}{j!}$$
$$\leq \mathrm{e}^{\Psi(t)-\Psi(a)} \max_{j=0,1,\dots,n-1} |x_a^j|$$

with the right-hand side independent of k. Hence, denoting

$$C_{\mathrm{ic}} = \mathrm{e}^{-\Psi(a)} \max_{j=0,1,\dots,n-1} |x_a^j|, \tag{4.5}$$

from (4.4) we obtain

$$|x^{[k]}_{\lambda,\Psi}(t)| \leq C_{\mathrm{ic}}\mathrm{e}^{\Psi(t)} + \frac{M}{\Gamma(\alpha-k)} \int_a^t (\Psi(t)-\Psi(s))^{\alpha-k-1}\mathrm{e}^{\lambda\Psi(s)}\Psi'(s)\mathrm{e}^{c\Psi(s)}ds.$$

Taking the substitution $\Psi(t)-\Psi(s)=q$, we arrive at

$$\begin{aligned}|x^{[k]}_{\lambda,\Psi}(t)| \leq & C_{\mathrm{ic}}\mathrm{e}^{\Psi(t)} + \frac{M\mathrm{e}^{(\lambda+c)\Psi(t)}}{\Gamma(\alpha-k)} \int_0^{\Psi(t)-\Psi(a)} q^{\alpha-k-1}\mathrm{e}^{-(\lambda+c)q}dq \\ \leq & C_{\mathrm{ic}}\mathrm{e}^{\Psi(t)} + \frac{M\mathrm{e}^{(\lambda+c)\Psi(t)}}{\Gamma(\alpha-k)} \int_0^{\infty} q^{\alpha-k-1}\mathrm{e}^{-(\lambda+c)q}dq \\ = & C_{\mathrm{ic}}\mathrm{e}^{\Psi(t)} + \frac{M\mathrm{e}^{(\lambda+c)\Psi(t)}}{(\lambda+c)^{\alpha-k}} \leq \left(C_{\mathrm{ic}} + \frac{M}{(\lambda+c)^{\alpha-k}}\right)\mathrm{e}^{\max\{1,\lambda+c\}\Psi(t)}\end{aligned}$$

for all $t \geq a$. Since k was arbitrary, the proof is complete. □

To show that also a solution is appropriately bounded, we consider the following linear equation

$${}^C D_a^{\alpha,\lambda,\Psi} x(t) = Ax(t) + f(t), \quad t \geq a \tag{4.6}$$

with $A \in \mathbb{R}$ and $f \in C[a,\infty)$, $n-1 < \alpha < n \in \mathbb{N}$.

Lemma 8. *Let $n-1 < \alpha < n$ for some $n \in \mathbb{N}$, $\lambda \geq 0$, $\Psi \in C^n[a,\infty)$ be such that $\Psi'(t) > 0$ for all $t \geq a$. If $f \in C[a,\infty)$ is of Ψ-exponential order, then so is the solution x to initial value problem* (4.6), (4.2).

Proof. From Theorem 2 we have the integral representation of solution x,

$$\begin{aligned}x(t) =& \mathrm{e}^{-\lambda\Psi(t)} \sum_{j=0}^{n-1} \frac{(\Psi(t)-\Psi(a))^j}{j!} x_a^j \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^t (\Psi(t)-\Psi(s))^{\alpha-1}\mathrm{e}^{-\lambda(\Psi(t)-\Psi(s))}\Psi'(s)(Ax(s)+f(s))ds.\end{aligned}$$

Hence, for $M \geq 0$, $c > 0$ such that $|f(t)| \leq M\mathrm{e}^{c\Psi(t)}$ for all $t \geq a$, we get

$$\begin{aligned}|x(t)| \leq & C_{\mathrm{ic}}\mathrm{e}^{(1-\lambda)\Psi(t)} + \frac{|A|}{\Gamma(\alpha)} \int_a^t (\Psi(t)-\Psi(s))^{\alpha-1}\mathrm{e}^{-\lambda(\Psi(t)-\Psi(s))}\Psi'(s)|x(s)|ds \\ &+ \frac{M}{\Gamma(\alpha)} \int_a^t (\Psi(t)-\Psi(s))^{\alpha-1}\mathrm{e}^{-\lambda(\Psi(t)-\Psi(s))}\Psi'(s)\mathrm{e}^{c\Psi(s)}ds\end{aligned}$$

for $C_{\rm ic}$ given by (4.5). Using the substitution $\Psi(t)-\Psi(s)=q$, we can rewrite the last term as

$$\frac{M\mathrm{e}^{c\Psi(t)}}{\Gamma(\alpha)}\int_0^{\Psi(t)-\Psi(a)} q^{\alpha-1}\mathrm{e}^{-(\lambda+c)q}dq \le \frac{M\mathrm{e}^{c\Psi(t)}}{(\lambda+c)^\alpha}.$$

To tackle the case $\lambda=0$ along with $\lambda>0$, we introduce a new positive constant $\nu<\max\{c,1-\lambda\}$. Then $u(t)=\mathrm{e}^{-\nu\Psi(t)}|x(t)|$ satisfies

$$\begin{aligned}u(t)\le&\left(C_{\rm ic}+\frac{M}{(\lambda+c)^\alpha}\right)\mathrm{e}^{(\max\{c,1-\lambda\}-\nu)\Psi(t)}\\&+\frac{|A|}{\Gamma(\alpha)}\int_a^t(\Psi(t)-\Psi(s))^{\alpha-1}\mathrm{e}^{-(\lambda+\nu)(\Psi(t)-\Psi(s))}\Psi'(s)u(s)ds.\end{aligned}$$

Now, we fix $p>1$ such that $p(\alpha-1)+1>0$ and set $q=\frac{p}{p-1}$ to be able to apply Lemma 4. Consequently,

$$u(t)\le 2^{\frac{1}{p}}\left(C_{\rm ic}+\frac{M}{(\lambda+c)^\alpha}\right)\mathrm{e}^{(\max\{c,1-\lambda\}-\nu)\Psi(t)}\mathrm{e}^{B(\Psi(t)-\Psi(a))}$$

with

$$B=\frac{2^{q-1}}{q}\left(\frac{\Gamma(p(\alpha-1)+1)}{(p(\lambda+\nu))^{p(\alpha-1)+1}}\right)^{\frac{q}{p}}\left(\frac{|A|}{\Gamma(\alpha)}\right)^q,$$

which means

$$|x(t)|\le\left[2^{\frac{1}{p}}\left(C_{\rm ic}+\frac{M}{(\lambda+c)^\alpha}\right)\mathrm{e}^{-B\Psi(a)}\right]\mathrm{e}^{(\max\{c,1-\lambda\}+B)\Psi(t)}.$$

This estimation gives that x is of $\Psi(t)$-exponential order. □

Now we can state a result on a solution to (4.6), (4.2).

Theorem 3. *Let $n-1<\alpha<n$ for some $n\in\mathbb{N}$, $\lambda\ge0$, $\Psi\in C^n[a,\infty)$ satisfy assumption* H. *If $f\in C[a,\infty)$ is of $\Psi(t)$-exponential order, then a solution x to initial value problem* (4.6), (4.2) *has the form*

$$\begin{aligned}x(t)=&\,\mathrm{e}^{-\lambda\Psi(t)}\sum_{k=0}^{n-1}(\Psi(t)-\Psi(a))^kE_{\alpha,k+1}((\Psi(t)-\Psi(a))^\alpha A)x_a^k\\&+\int_a^t(\Psi(t)-\Psi(s))^{\alpha-1}\mathrm{e}^{-\lambda(\Psi(t)-\Psi(s))}\Psi'(s)E_{\alpha,\alpha}((\Psi(t)-\Psi(s))^\alpha A)f(s)ds,\end{aligned}\tag{4.7}$$

where $E_{\alpha,\beta}(z)=\sum_{j=0}^\infty\frac{z^j}{\Gamma(\alpha j+\beta)}$ is the Mittag-Leffler function (cf. [4, §18.1]).

Proof. According to Lemmas 7 and 8 we can apply the generalized Laplace transform to Equation (4.6). When we shortly denote $X(s)=\mathcal{L}_\Psi\{x(t)\}(s)$ and $F(s)=\mathcal{L}_\Psi\{f(t)\}(s)$, by Lemma 6 we obtain

$$(s+\lambda)^\alpha X(s)-\mathrm{e}^{-\lambda\Psi(a)}\sum_{k=0}^{n-1}(s+\lambda)^{\alpha-k-1}x_a^k=AX(s)+F(s)$$

for all s sufficiently large. Therefrom,

$$X(s) =((s+\lambda)^\alpha - A)^{-1}\left(\mathrm{e}^{-\lambda\Psi(a)}\sum_{k=0}^{n-1}(s+\lambda)^{\alpha-k-1}x_a^k + F(s)\right)$$
$$=(1-A(s+\lambda)^{-\alpha})^{-1}\left(\mathrm{e}^{-\lambda\Psi(a)}\sum_{k=0}^{n-1}(s+\lambda)^{-k-1}x_a^k + F(s)(s+\lambda)^{-\alpha}\right).$$

Expanding $(1-A(s+\lambda)^{-\alpha})^{-1}$ into series we obtain

$$X(s) = \mathrm{e}^{-\lambda\Psi(a)}\sum_{k=0}^{n-1}\sum_{j=0}^{\infty}(A(s+\lambda)^{-\alpha})^j(s+\lambda)^{-k-1}x_a^k$$
$$+\sum_{j=0}^{\infty}(A(s+\lambda)^{-\alpha})^j F(s)(s+\lambda)^{-\alpha}.$$

Therefore,

$$x(t) = \mathrm{e}^{-\lambda\Psi(a)}\sum_{k=0}^{n-1}A_k x_a^k + A_f, \quad t\geq a,$$

where

$$A_k =\sum_{j=0}^{\infty}A^j\mathcal{L}_\Psi^{-1}\{(s+\lambda)^{-k-1-\alpha j}\}(t), \quad k=0,1,\ldots,n-1,$$
$$A_f =\sum_{j=0}^{\infty}A^j\mathcal{L}_\Psi^{-1}\{(s+\lambda)^{-(j+1)\alpha}F(s)\}(t).$$

Here $\mathcal{L}_\Psi^{-1}$ stands for an inverse of $\mathcal{L}_\Psi$. Due to (3.2), and properties of Ψ, the uniqueness of the inverse in the set of continuous functions follows from the same property of the classic Laplace transform. From statement 2 of Lemma 1 one can see that for each $k=0,1,\ldots,n-1$,

$$A_k =\sum_{j=0}^{\infty}A^j\frac{(\Psi(t)-\Psi(a))^{k+\alpha j}\mathrm{e}^{-\lambda(\Psi(t)-\Psi(a))}}{\Gamma(k+1+\alpha j)}$$
$$=(\Psi(t)-\Psi(a))^k\mathrm{e}^{-\lambda(\Psi(t)-\Psi(a))}\sum_{j=0}^{\infty}\frac{A^j(\Psi(t)-\Psi(a))^{\alpha j}}{\Gamma(k+1+\alpha j)}$$
$$=(\Psi(t)-\Psi(a))^k\mathrm{e}^{-\lambda(\Psi(t)-\Psi(a))}E_{\alpha,k+1}((\Psi(t)-\Psi(a))^\alpha A).$$

Next, we apply both statements of Lemma 1, to derive

$$A_f =\Big(\sum_{j=0}^{\infty}A^j\mathcal{L}_\Psi^{-1}\{(s+\lambda)^{-(j+1)\alpha}\} *_\Psi \mathcal{L}_\Psi^{-1}\{F(s)\}\Big)(t)$$
$$=\Big(f *_\Psi \sum_{j=0}^{\infty}A^j\mathcal{L}_\Psi^{-1}\{(s+\lambda)^{-(j+1)\alpha}\}\Big)(t)$$

$$
\begin{aligned}
&=\Big(f *_\Psi \sum_{j=0}^{\infty} A^j \frac{(\Psi(\cdot)-\Psi(a))^{(j+1)\alpha-1}\mathrm{e}^{-\lambda(\Psi(\cdot)-\Psi(a))}}{\Gamma((j+1)\alpha)}\Big)(t)\\
&=\Big(f *_\Psi \Big((\Psi(\cdot)-\Psi(a))^{\alpha-1}\mathrm{e}^{-\lambda(\Psi(\cdot)-\Psi(a))}\sum_{j=0}^{\infty}\frac{A^j(\Psi(\cdot)-\Psi(a))^{\alpha j}}{\Gamma((j+1)\alpha)}\Big)\Big)(t)\\
&=\Big(f *_\Psi \Big((\Psi(\cdot)-\Psi(a))^{\alpha-1}\mathrm{e}^{-\lambda(\Psi(\cdot)-\Psi(a))}E_{\alpha,\alpha}((\Psi(\cdot)-\Psi(a))^\alpha A)\Big)\Big)(t)\\
&=\int_a^t (\Psi(t)-\Psi(s))^{\alpha-1}\mathrm{e}^{-\lambda(\Psi(t)-\Psi(s))}\Psi'(s)E_{\alpha,\alpha}((\Psi(t)-\Psi(s))^\alpha A)f(s)ds.
\end{aligned}
$$

So we have proved the statement. □

In fact, the statement of Theorem 3 remains valid if we drop the assumption on f:

Theorem 4. *Let $n-1<\alpha<n$ for some $n\in\mathbb{N}$, $\lambda\geq 0$, $\Psi\in C^n[a,\infty)$ satisfy assumption* H, *and $f\in C[a,\infty)$ be a given function. Then a solution x to initial value problem* (4.6), (4.2) *has the form* (4.7).

Proof. We shall show that x of (4.7) satisfies integral Equation (4.3) with $F(t,x(t))=Ax(t)+f(t)$, i.e., $\mathcal{A}+\mathcal{B}=\mathcal{C}+\mathcal{D}+\mathcal{E}+\mathcal{F}$ on $[a,\infty)$, where

$$
\begin{aligned}
\mathcal{A}(t) =&\mathrm{e}^{-\lambda\Psi(t)}\sum_{k=0}^{n-1}(\Psi(t)-\Psi(a))^k E_{\alpha,k+1}((\Psi(t)-\Psi(a))^\alpha A)x_a^k,\\
\mathcal{B}(t) =&\int_a^t (\Psi(t)-\Psi(s))^{\alpha-1}\mathrm{e}^{-\lambda(\Psi(t)-\Psi(s))}\Psi'(s)E_{\alpha,\alpha}((\Psi(t)-\Psi(s))^\alpha A)f(s)ds,\\
\mathcal{C}(t) =&\mathrm{e}^{-\lambda\Psi(t)}\sum_{k=0}^{n-1}\frac{(\Psi(t)-\Psi(a))^k}{k!}x_a^k,\\
\mathcal{D}(t) =&\frac{1}{\Gamma(\alpha)}\int_a^t (\Psi(t)-\Psi(s))^{\alpha-1}\mathrm{e}^{-\lambda(\Psi(t)-\Psi(s))}\Psi'(s)A\mathcal{A}(s)ds,\\
\mathcal{E}(t) =&\frac{1}{\Gamma(\alpha)}\int_a^t (\Psi(t)-\Psi(s))^{\alpha-1}\mathrm{e}^{-\lambda(\Psi(t)-\Psi(s))}\Psi'(s)A\mathcal{B}(s)ds,\\
\mathcal{F}(t) =&\frac{1}{\Gamma(\alpha)}\int_a^t (\Psi(t)-\Psi(s))^{\alpha-1}\mathrm{e}^{-\lambda(\Psi(t)-\Psi(s))}\Psi'(s)f(s)ds.
\end{aligned}
$$

First, using the definition of the Mittag-Leffler function, we have

$$
\begin{aligned}
\mathcal{A}(t) =&\mathrm{e}^{-\lambda\Psi(t)}\sum_{k=0}^{n-1}(\Psi(t)-\Psi(a))^k\sum_{j=0}^{\infty}\frac{(\Psi(t)-\Psi(a))^{\alpha j}A^j x_a^k}{\Gamma(k+1+\alpha j)}\\
=&\mathrm{e}^{-\lambda\Psi(t)}\sum_{k=0}^{n-1}(\Psi(t)-\Psi(a))^k\sum_{j=1}^{\infty}\frac{(\Psi(t)-\Psi(a))^{\alpha j}A^j x_a^k}{\Gamma(k+1+\alpha j)}+\mathcal{C}(t).
\end{aligned}
$$

Next,

$$\mathcal{D}(t) = \frac{\mathrm{e}^{-\lambda\Psi(t)}}{\Gamma(\alpha)} \int_a^t (\Psi(t)-\Psi(s))^{\alpha-1}\Psi'(s)A\sum_{k=0}^{n-1}(\Psi(s)-\Psi(a))^k$$

$$\times \sum_{j=0}^{\infty} \frac{(\Psi(s)-\Psi(a))^{\alpha j}A^j x_a^k}{\Gamma(k+1+\alpha j)}ds = \frac{\mathrm{e}^{-\lambda\Psi(t)}}{\Gamma(\alpha)}\sum_{k=0}^{n-1}\sum_{j=0}^{\infty}$$

$$\times \int_a^t (\Psi(t)-\Psi(s))^{\alpha-1}(\Psi(s)-\Psi(a))^{k+\alpha j}\Psi'(s)ds\, \frac{A^{j+1}x_a^k}{\Gamma(k+1+\alpha j)}$$

and, after the substitution $\Psi(a)+q(\Psi(t)-\Psi(a))=\Psi(s)$, we get

$$\begin{aligned}\mathcal{D}(t) =&\frac{\mathrm{e}^{-\lambda\Psi(t)}}{\Gamma(\alpha)}\sum_{k=0}^{n-1}\sum_{j=0}^{\infty}\int_0^1(1-q)^{\alpha-1}q^{k+\alpha j}dq\,\frac{(\Psi(t)-\Psi(a))^{k+\alpha(j+1)}A^{j+1}x_a^k}{\Gamma(k+1+\alpha j)}\\ =&\frac{\mathrm{e}^{-\lambda\Psi(t)}}{\Gamma(\alpha)}\sum_{k=0}^{n-1}\sum_{j=0}^{\infty}B(\alpha,k+1+\alpha j)\frac{(\Psi(t)-\Psi(a))^{k+\alpha(j+1)}A^{j+1}x_a^k}{\Gamma(k+1+\alpha j)}\\ =&\mathrm{e}^{-\lambda\Psi(t)}\sum_{k=0}^{n-1}\sum_{j=0}^{\infty}\frac{(\Psi(t)-\Psi(a))^{k+\alpha(j+1)}A^{j+1}x_a^k}{\Gamma(k+1+\alpha(j+1))}\\ =&\mathrm{e}^{-\lambda\Psi(t)}\sum_{k=0}^{n-1}(\Psi(t)-\Psi(a))^k\sum_{j=1}^{\infty}\frac{(\Psi(t)-\Psi(a))^{\alpha j}A^j x_a^k}{\Gamma(k+1+\alpha j)},\end{aligned}$$

where $B(t,s)=\int_0^1(1-q)^{t-1}q^{s-1}dq$ is the Euler beta function (cf. [16]). So until now we have proved $\mathcal{A}=\mathcal{C}+\mathcal{D}$. Note that if $f\equiv 0$, the proof is finished.

We continue with the case of general f. In $\mathcal{E}$ we change the order of integration:

$$\begin{aligned}\mathcal{E}(t) =& \frac{1}{\Gamma(\alpha)}\int_a^t\int_a^s(\Psi(t)-\Psi(s))^{\alpha-1}\mathrm{e}^{-\lambda(\Psi(t)-\Psi(q))}\Psi'(s)A\\ &\times(\Psi(s)-\Psi(q))^{\alpha-1}\Psi'(q)E_{\alpha,\alpha}((\Psi(s)-\Psi(q))^\alpha A)f(q)dq\,ds\\ =& \frac{1}{\Gamma(\alpha)}\int_a^t\int_q^t(\Psi(t)-\Psi(s))^{\alpha-1}(\Psi(s)-\Psi(q))^{\alpha-1}\mathrm{e}^{-\lambda(\Psi(t)-\Psi(q))}\Psi'(s)\Psi'(q)\\ &\times AE_{\alpha,\alpha}((\Psi(s)-\Psi(q))^\alpha A)f(q)ds\,dq = \frac{1}{\Gamma(\alpha)}\int_a^t \mathrm{e}^{-\lambda(\Psi(t)-\Psi(q))}\\ &\times\sum_{j=0}^{\infty}\int_q^t(\Psi(t)-\Psi(s))^{\alpha-1}(\Psi(s)-\Psi(q))^{\alpha-1+\alpha j}\Psi'(s)ds\,\frac{A^{j+1}\Psi'(q)f(q)}{\Gamma(\alpha(j+1))}dq.\end{aligned}$$

Then the substitution $\Psi(q)+\sigma(\Psi(t)-\Psi(q))=\Psi(s)$ yields

$$\begin{aligned}\mathcal{E}(t) =&\frac{1}{\Gamma(\alpha)}\int_a^t \mathrm{e}^{-\lambda(\Psi(t)-\Psi(q))}\sum_{j=0}^{\infty}\int_0^1(1-\sigma)^{\alpha-1}\sigma^{\alpha(j+1)-1}d\sigma\\ &\times\frac{(\Psi(t)-\Psi(q))^{\alpha(j+2)-1}A^{j+1}\Psi'(q)f(q)}{\Gamma(\alpha(j+1))}dq,\end{aligned}$$

which after using the beta function gives

$$\begin{aligned}\mathcal{E}(t) &= \int_a^t \mathrm{e}^{-\lambda(\Psi(t)-\Psi(q))}\left(\sum_{j=0}^{\infty}\frac{(\Psi(t)-\Psi(q))^{\alpha(j+2)-1}A^{j+1}}{\Gamma(\alpha(j+2))}\right)\Psi'(q)f(q)dq\\ &= \int_a^t (\Psi(t)-\Psi(q))^{\alpha-1}\mathrm{e}^{-\lambda(\Psi(t)-\Psi(q))}\left(\sum_{j=1}^{\infty}\frac{(\Psi(t)-\Psi(q))^{\alpha j}A^j}{\Gamma(\alpha(j+1))}\right)\Psi'(q)f(q)dq.\end{aligned}$$

On the other side, in $\mathcal{F}$ we apply

$$\frac{1}{\Gamma(\alpha)} = \left[\frac{(\Psi(t)-\Psi(q))^{\alpha j}A^j}{\Gamma(\alpha(j+1))}\right]_{j=0}.$$

Therefore, $\mathcal{B} = \mathcal{E} + \mathcal{F}$ and the proof is complete. □

Remark 2. For $0 < \alpha < 1$ we observe the following:

1. If $\lambda = 0$, solution (4.7) coincides with a solution from [7, Theorem 5.2] of the same problem.
2. Having a zero nonhomogeneity f, solution (4.7) is the same as the solution in [9, Theorem 6].

Example 1. From Theorem 4 we obtain that the Cauchy problem

$$\begin{aligned}&{}^C D_a^{\alpha,\lambda,\Psi} x(t) - \omega x(t) = f(t), \quad t \geq a,\, \omega \in \mathbb{R},\\ &x(a) = x_a\end{aligned} \tag{4.8}$$

for $0 < \alpha < 1$, $\lambda \geq 0$, $\Psi \in C^1[a,\infty)$ satisfying assumption H and $f \in C[a,\infty)$, has the solution

$$\begin{aligned}x(t) &= \mathrm{e}^{-\lambda(\Psi(t)-\Psi(a))}E_{\alpha,1}(\omega(\Psi(t)-\Psi(a))^{\alpha})x_a\\ &+ \int_a^t (\Psi(t)-\Psi(s))^{\alpha-1}\mathrm{e}^{-\lambda(\Psi(t)-\Psi(s))}\Psi'(s)E_{\alpha,\alpha}(\omega(\Psi(t)-\Psi(s))^{\alpha})f(s)ds.\end{aligned} \tag{4.9}$$

Note that this time $x_{\lambda,\Psi}^{[0]}(a) = \mathrm{e}^{\lambda\Psi(a)}x(a) = \mathrm{e}^{\lambda\Psi(a)}x_a$.

In paper [7], the Cauchy problem (4.8) is studied with $\lambda = 0$. The coefficient ω is denoted there by λ. The formula (40) from [7, Theorem 5.2] for the solution to this problem can be obtained by putting $\lambda = 0$ in (4.9). Similarly, solution to (4.8) with $\lambda = 0$, $\Psi(t) = \frac{t^\rho}{\rho}$ from [6, Theorem 4.2] coincides with (4.9). To illustrate the difference, we provide Figure 1 depicting solutions to (4.8) with various values of λ and $\Psi(t)$.

Remark 3. Although in this section we considered scalar differential equations, all the results can be easily rewritten for systems, i.e., with $x_a^k \in \mathbb{R}^N$, $F \in C([a,\infty) \times \mathbb{R}^N, \mathbb{R}^N)$ and matrix $A \in \mathbb{R}^{N\times N}$. In this case, the absolute value should be replaced by a vector norm or the corresponding vector-induced matrix norm. Then the generalized Laplace transform is understood component-wise.

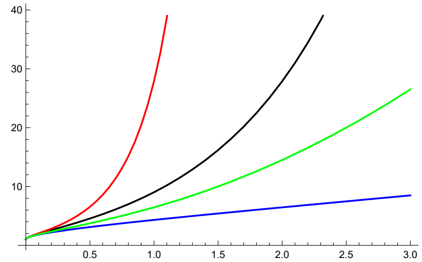

Figure 1. Solutions to (4.8) with $a = 0$, $\omega = 1$, $f \equiv 1$, $\alpha = \frac{1}{2}$, $x_a = 1$ and 1. $\lambda = 0$, $\Psi(t) = t$ (black); 2. $\lambda = 1$, $\Psi(t) = t$ (blue); 3. $\lambda = 0$, $\Psi(t) = t + t^2$ (red); 4. $\lambda = 1$, $\Psi(t) = t + t^2$ (green).

5 Conclusions

Images of the tempered Ψ-Caputo fractional derivative and the tempered Ψ-Hilfer fractional integral were found under the generalized Laplace transform. These results were applied to derive a formula for a solution to an inital value problem for a nonhomogeneous linear differential equation with tempered Ψ-Caputo fractional derivative of a general non-integer order $\alpha > 0$ and any nonhomogeneity (of $\Psi(t)$-exponential order or not). A simple example was given to compare solutions to the same initial value problem but with different tempering and Ψ.

Possible problems to be investigated in the future using the method of this paper include tempered Ψ-Caputo fractional differential equations with one or multiple appropriate delays on the right-hand side, equations with linear differential operator consisting of various tempered Ψ-Caputo derivatives on the left-hand side, stability or asymptotic properties of solutions to such problems, etc.

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