Mathematical Modelling and Analysis
Volume 28, Issue 2, 218–236, 2023
https://doi.org/10.3846/mma.2023.16359

Collocation Based Approximations for a Class of Fractional Boundary Value Problems

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Received January 28, 2022; revised January 4, 2023; accepted January 4, 2023

Abstract. A boundary value problem for fractional integro-differential equations with weakly singular kernels is considered. The problem is reformulated as an integral equation of the second kind with respect to $z = D^{\alpha}_{\mathrm{Cap}}y$, the Caputo fractional derivative of y of order α, with $1 < \alpha < 2$, where y is the solution of the original problem. Using this reformulation, the regularity properties of both y and its Caputo derivative z are studied. Based on this information a piecewise polynomial collocation method is developed for finding an approximate solution z_N of the reformulated problem. Using z_N, an approximation y_N for y is constructed and a detailed convergence analysis of the proposed method is given. In particular, the attainable order of convergence of the proposed method for appropriate values of grid and collocation parameters is established. To illustrate the performance of our approach, results of some numerical experiments are presented.

Keywords: fractional weakly singular integro-differential equation, Caputo derivative, boundary value problem, collocation method, graded grid.

AMS Subject Classification: 34A08; 65L10; 65R20.

1 Introduction

Differential equations containing derivatives of fractional (i.e., of any real positive) order are useful in various fields of science and engineering, especially when modelling real-life processes with memory properties [3, 23, 26]. For the fundamental theory of fractional derivatives and equations containing them we refer the reader to [8, 11, 21, 22], see also [25]. Since it is rarely possible to find the solution of a given fractional differential equation in a closed form [16, 21], the analysis and development of numerical methods for fractional differential equations has become a very active area of research. In particular, a number of studies have used collocation based methods, see, for example, [9,14,15,18,27]. This approach also forms the basis of our research in the present paper. A comprehensive survey of different numerical methods for various classes of fractional differential equations including a brief summary about the convergence behaviour of the methods is given in the monograph [3], see also [8,21]. For various other types of studies, we direct the reader to [1,4,7,10,12].

However, considerably less research has been done on fractional integro-differential equations, especially those with weakly singular kernels. For instance, in [28] initial value problems and in [17,19,20] boundary value problems for weakly singular integro-differential equations with Caputo fractional differential operators are investigated. In [17,19], the highest order of the fractional differential operator belongs to $(0,1)$. In the present paper we will consider the case, where the highest order of the underlying fractional differential operator belongs to $(1,2)$. More precisely, by using some ideas of [17] (see also [20]), we construct a high order method for the numerical solution of fractional weakly singular integro-differential equations in the form

$$(D^{\alpha}_{\mathrm{Cap}}y)(t) + h(t)y(t) + \int_0^t L_\kappa(t,s)y(s)ds = f(t),\ 0 \le t \le b,\ 0 < b < \infty, \quad (1.1)$$

subject to the conditions

$$a_{11}y(0) + a_{12}y(b_1) = \gamma_1, \quad a_{21}y'(0) + a_{22}y(b_1) = \gamma_2. \quad (1.2)$$

Here y is the unknown function, $D^{\alpha}_{\mathrm{Cap}}$ is the Caputo fractional differential operator of order α with $1 < \alpha < 2$, $b_1 \in (0,b]$ and $a_{11}, a_{12}, a_{21}, a_{22}, \gamma_1, \gamma_2 \in \mathbb{R} = (-\infty,\infty)$. For our approach below we assume that $b_1a_{11}a_{22} + a_{11}a_{21} + a_{12}a_{21} \neq 0$. The functions h, f belong to $C[0,b]$ and the function L_κ is defined by the formula

$$L_\kappa(t,s) = \begin{cases} [1 + \log(t-s)]K(t,s) & \text{for } \kappa = 0, \\ (t-s)^{-\kappa}K(t,s) & \text{for } 0 < \kappa < 1, \end{cases} \quad (1.3)$$

where $K \in C(\Delta)$ and $\Delta = \{(t,s) : 0 \le s \le t \le b\}$.

By $C^m[0,b]$ and $C^m(\Delta)$, with $m \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $\mathbb{N} = \{1,2,\dots\}$, we denote the sets of m times continuously differentiable functions on $[0,b]$ and Δ, respectively; for $m = 0$ we set $C^0[0,b] = C[0,b]$ and $C^0(\Delta) = C(\Delta)$. In particular, $C[0,b]$ will denote the Banach space of continuous functions $u : [0,b] \to \mathbb{R}$ with the norm $\|u\|_\infty = \sup\{|u(t)| : 0 \le t \le b\}$. Note that for $a_{12} = a_{22} = 0$ the problem (1.1)–(1.2) takes the form of an initial value problem for equation (1.1) and for $b_1 = b$ a two-point boundary value problem for equation (1.1). To simplify the presentation we have restricted ourselves to conditions (1.2). However, the proposed approach below can also be applied in the case where the conditions associated with equation (1.1) are given in a more general form.

We are interested in solutions $y \in C^1[0,b]$ of problem (1.1)–(1.2) such that $D^{\alpha}_{\mathrm{Cap}}y \in C[0,b]$, $\alpha \in (1,2)$. Note that in [25] necessary and sufficient conditions for the existence of $D^{\alpha}_{\mathrm{Cap}} y \in C[0,b]$ $(1<\alpha<2)$ for a function $y \in C^1[0,b]$ have been derived.

The Caputo fractional differential operator $D^{\delta}_{\mathrm{Cap}}$ of order $\delta \in (1,2)$ can be defined by formula (see, e.g., [8, 25])

$$(D^{\delta}_{\mathrm{Cap}} y)(t) = (D^2 J^{2-\delta}(y - T_1 y))(t), \quad t \in [0,b], \quad y \in C^1[0,b],$$

where

$$(T_1 y)(t) = y(0) + y'(0)t$$

is the Taylor polynomial of degree 1 at the point 0. The classical differential operator $\left(\frac{d}{dt}\right)^m$ of order $m \in \mathbb{N}_0$ is denoted by D^m (with $D^0 = I$) and J^{δ} is the Riemann-Liouville integral operator of order δ, defined by

$$(J^{\delta} y)(x) = \frac{1}{\Gamma(\delta)} \int_0^x (x-t)^{\delta-1} y(t)dt, \ x \in [0,b], \ y \in L^{\infty}(0,b), \ \delta > 0; \ J^0 = I, \quad (1.4)$$

where I is the identity mapping and Γ the Euler gamma function:

$$\Gamma(\eta) = \int_0^{\infty} s^{\eta-1} e^{-s} ds, \quad \eta > 0.$$

By $L^{\infty}(0,b)$ we denote the space of all essentially bounded measurable functions $y : (0,b) \to \mathbb{R}$ such that

$$\|y\|_{L^{\infty}(0,b)} = \inf_{\mathbf{meas}(\Omega)=0} \sup_{t \in (0,b)\setminus\Omega} |y(t)| < \infty,$$

where $\mathbf{meas}(\Omega) = 0$ means that the Lebesgue measure of the set $\Omega \subset (0,b)$ is equal to 0.

Note that, for any $y \in L^{\infty}(0,b)$ we have (see [8, 11, 22])

$$J^{\delta} J^{\eta} y = J^{\delta+\eta} y, \delta > 0, \quad \eta > 0, \quad (1.5)$$

$$D^1 J^1 y = y, D^{\eta}_{\mathrm{Cap}} J^{\eta} y = y, \quad 1 < \eta < 2, \quad (1.6)$$

$$D^k (J^{\eta} y) \in C[0,b], (D^k J^{\eta} y)(0) = 0, \quad \eta > 0, \ k = 0, \ldots, \lceil \eta \rceil - 1, \quad (1.7)$$

where $\lceil \eta \rceil$ denotes the smallest integer greater than or equal to real number η.

Due to [8] we cannot expect that a solution of a fractional differential equation with Caputo differential operators will be smooth on the closed interval of integration and this is a challenge for constructing high order methods for the numerical solution of such equations. Therefore, using an integral equation reformulation of problem (1.1)–(1.2), we first study the possible singular behaviour of the exact solution y to (1.1)–(1.2). We observe that usual derivatives of y may be unbounded near the left endpoint of the interval of integration $[0,b]$, even if $L_{\kappa} = 0$ and the functions h and f are infinitely differentiable on $[0,b]$ (see Theorem 1 below). It is our aim, in the present paper, to construct and justify a high order method for solving (1.1)–(1.2) which takes into account the possible singular behaviour of the exact solution y to (1.1)–(1.2).

The rest of the paper is organised in the following matter. In Section 2, we reformulate the problem (1.1)–(1.2) and study the existence, uniqueness and regularity of the exact solution to (1.1)–(1.2). In Section 3, we introduce a collocation based method for finding approximate solutions of problem (1.1)–(1.2). In Section 4, we study the convergence and convergence order of the proposed method. In Section 5, we test our theoretical results with three numerical experiments. The main results of the article are formulated by Theorems 1–3.

2 Existence, uniqueness and smoothness of the solution

We start by reformulating (1.1)–(1.2) as an integral equation. Let $y \in C^1[0,b]$ be an arbitrary function such that $D^\alpha_{\text{Cap}} y \in C[0,b]$, where $\alpha \in (1,2)$.

Denote $z = D^\alpha_{\text{Cap}} y$. Then (cf. [8])

$$y(t) = k_1 + k_2 t + (J^\alpha z)(t), \quad t \in [0,b], \quad k_1, k_2 \in \mathbb{R}, \tag{2.1}$$

where J^α is the Riemann-Liouville integral operator of order α (see (1.4)). With the help of (1.5)–(1.7) it is easy to check that a function of the form (2.1) satisfies the conditions (1.2) if and only if

$$y(t) = (J^\alpha z)(b_1)k_{00} + k_{01} + [(J^\alpha z)(b_1)k_{10} + k_{11}]t + (J^\alpha z)(t), \quad t \in [0,b], \tag{2.2}$$

where

$$\begin{aligned} k_{00} &= \frac{-a_{21}a_{12}}{b_1 a_{11} a_{22} + a_{11}a_{21} + a_{12}a_{21}}, \quad k_{10} = \frac{-a_{22}a_{11}}{b_1 a_{11} a_{22} + a_{11}a_{21} + a_{12}a_{21}}, \\ k_{01} &= \frac{\gamma_1(b_1 a_{22} + a_{21}) - \gamma_2 a_{12} b_1}{b_1 a_{11} a_{22} + a_{11}a_{21} + a_{12}a_{21}}, \; k_{11} = \frac{-\gamma_1 a_{22} + \gamma_2(a_{11} + a_{12})}{b_1 a_{11} a_{22} + a_{11}a_{21} + a_{12}a_{21}}, \end{aligned} \tag{2.3}$$

for $b_1 a_{11} a_{22} + a_{11}a_{21} + a_{12}a_{21} \neq 0$.

Let now $y \in C^1[0,b]$ be a solution of problem (1.1)–(1.2) such that $D^\alpha_{\text{Cap}} y \in C[0,b]$. By substituting (2.2) into (1.1) we obtain that $z = D^\alpha_{\text{Cap}} y$ is a solution to an integral equation of the form

$$\begin{aligned} z(t) = f(t) &- h(t)\big[(J^\alpha z)(b_1)k_{00} + k_{01} + ((J^\alpha z)(b_1)k_{10} + k_{11})t + (J^\alpha z)(t)\big] \\ &- \int_0^t L_\kappa(t,s)\big[(J^\alpha z)(b_1)k_{00} + k_{01} + ((J^\alpha z)(b_1)k_{10} + k_{11})s + (J^\alpha z)(s)\big]ds \end{aligned}$$

or

$$\begin{aligned} z(t) = f(t) &- h(t)\big[(J^\alpha z)(b_1)k_{00} + k_{01} + ((J^\alpha z)(b_1)k_{10} + k_{11})t + (J^\alpha z)(t)\big] \\ &- \int_0^t L_\kappa(t,s)\Big[(J^\alpha z)(b_1)k_{00} + k_{01} + ((J^\alpha z)(b_1)k_{10} + k_{11})s\Big]ds \\ &- \int_0^t L_\kappa(t,s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1} z(\tau)d\tau ds. \end{aligned} \tag{2.4}$$

By changing the order of integration in the last integral on the right-hand side of equation (2.4) we find that, for $t \in [0,b]$,

$$\int_0^t L_\kappa(t,s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1} z(\tau)d\tau ds = \frac{1}{\Gamma(\alpha)}\int_0^t z(s)\int_s^t L_\kappa(t,\tau)(\tau-s)^{\alpha-1}d\tau ds.$$

Using the change of variables $\tau = (t-s)\sigma + s$, we get

$$\frac{1}{\Gamma(\alpha)} \int_0^t z(s) \int_s^t L_\kappa(t,\tau)(\tau-s)^{\alpha-1} d\tau ds$$
$$= \frac{1}{\Gamma(\alpha)} \int_0^t z(s)(t-s)^\alpha \int_0^1 L_\kappa(t,(t-s)\sigma+s)\sigma^{\alpha-1} d\sigma ds, \quad t \in [0,b].$$

Thus, we can rewrite (2.4) in the form

$$z = Tz + g, \tag{2.5}$$

where, for $t \in [0,b]$,

$$\begin{aligned}(Tz)(t) =& -h(t)\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-x)^{\alpha-1}z(x)dx + \frac{k_{00}+k_{10}t}{\Gamma(\alpha)}\int_0^{b_1}(b_1-x)^{\alpha-1}z(x)dx\right] \\ &- \frac{k_{00}}{\Gamma(\alpha)}\int_0^t L_\kappa(t,s)ds \int_0^{b_1}(b_1-x)^{\alpha-1}z(x)dx - \frac{k_{10}}{\Gamma(\alpha)}\int_0^t sL_\kappa(t,s)ds \\ &\times \int_0^{b_1}(b_1-x)^{\alpha-1}z(x)dx - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^\alpha \left(\int_0^1 L_\kappa(t,(t-s)\sigma+s)\sigma^{\alpha-1}d\sigma\right) z(s)ds,\end{aligned} \tag{2.6}$$

and

$$g(t) = f(t) - h(t)[k_{01}+k_{11}t] - k_{01}\int_0^t L_\kappa(t,s)ds - k_{11}\int_0^t sL_\kappa(t,s)ds, \tag{2.7}$$

with L_κ defined by (1.3) and $k_{00}, k_{01}, k_{10}, k_{11}$ given by (2.3).

Conversely, it can be shown that if $z \in C[0,b]$ is a solution to (2.5), then y determined by formula (2.2) belongs to $C^1[0,b]$ and is a solution to (1.1)–(1.2). In this sense equation (2.5) is equivalent to problem (1.1)–(1.2).

In order to study the regularity properties of the exact solution of problem (1.1)–(1.2), we first introduce the weighted space $C^{q,\nu}(0,b]$ of smooth functions on $(0,b]$, an adaptation of a more general weighted space of functions introduced by Vainikko in [24] (see also [5]).

For given $b \in \mathbb{R}$, $b > 0$, $q \in \mathbb{N}$ and $\nu \in \mathbb{R}$, $\nu < 1$, by $C^{q,\nu}(0,b]$ we denote the set of continuous functions $y : [0,b] \to \mathbb{R}$ which are q times continuously differentiable in $(0,b]$ such that for all $t \in (0,b]$ and $i = 1,\dots,q$, the following estimate holds:

$$|y^{(i)}(t)| \le c \begin{cases} 1, & \text{if } i < 1-\nu, \\ 1 + |\log t|, & \text{if } i = 1-\nu, \\ t^{1-\nu-i}, & \text{if } i > 1-\nu. \end{cases}$$

Here $c = c(y)$ is a positive constant. The set $C^{q,\nu}(0,b]$ becomes a Banach space if it is equipped with the norm

$$\|y\|_{C^{q,\nu}(0,b]} = \|y\|_\infty + \sum_{i=1}^q \sup_{0<t\le b} \omega_{i-1+\nu}(t)|y^{(i)}(t)|, \quad y \in C^{q,\nu}(0,b],$$

where, for $t > 0$, $\lambda \in \mathbb{R}$,

$$\omega_\lambda(t) = \begin{cases} 1, & \text{if } \lambda < 0, \\ 1/(1+|\log t|), & \text{if } \lambda = 0, \\ t^\lambda, & \text{if } \lambda > 0. \end{cases}$$

For example, the function $y(t) = t^{\frac{5}{2}}$, $t \in [0,b]$, belongs to $C^{q,-\frac{3}{2}}(0,b]$ with arbitrary $q \in \mathbb{N}$. Note also that

$$C^n[0,b] \subset C^{n,\nu}(0,b] \subset C^{m,\mu}(0,b] \subset C[0,b], \quad n \geq m \geq 1, \quad \nu \leq \mu < 1.$$

Next two lemmas follow from the corresponding results of [5].

Lemma 1. *If $y_1, y_2 \in C^{q,\nu}(0,b]$, $q \in \mathbb{N}$, $\nu < 1$, then $y_1 y_2 \in C^{q,\nu}(0,b]$ and*

$$\|y_1 y_2\|_{C^{q,\nu}(0,b]} \leq c \, \|y_1\|_{C^{q,\nu}(0,b]} \, \|y_2\|_{C^{q,\nu}(0,b]} \, ,$$

with a constant c which is independent of y_1 and y_2.

Lemma 2. *Let $\eta \in (-\infty, 1)$ and $U \in C(\Delta)$. Then operators S_1 and S_2 defined by*

$$(S_1 y)(t) = \int_0^t (t-s)^{-\eta} U(t,s) y(s) ds, \quad t \in [0,b],$$
$$(S_2 y)(t) = \int_0^t [1 + \log(t-s)] U(t,s) y(s) ds, \quad t \in [0,b],$$

are both compact as operators from $L^\infty(0,b)$ into $C[0,b]$, thus also from $C[0,b]$ into $C[0,b]$ and from $L^\infty(0,b)$ into $L^\infty(0,b)$. If, in addition, $U \in C^q(\Delta)$, $q \in \mathbb{N}$, then S_1 is compact as an operator from $C^{q,\nu}(0,b]$ into $C^{q,\nu}(0,b]$, where $\eta \leq \nu < 1$, and S_2 is compact as an operator from $C^{q,\nu}(0,b]$ into $C^{q,\nu}(0,b]$ for $0 \leq \nu < 1$.

For Banach spaces E and F, by $\mathcal{L}(E,F)$ we denote the Banach space of linear bounded operators $A : E \to F$ with the norm $\|A\|_{\mathcal{L}(E,F)} = \sup\{\|Ax\|_F : x \in E, \|x\|_E \leq 1\}$. In our discussions below we also need the following results from the classical theory of compact operators, see, e.g., [2].

Lemma 3. *Let E, F and G be normed spaces and let $A : E \to F$ and $B : F \to G$ be bounded linear operators. Then the product $BA : E \to G$ is a compact operator if one of the two operators A or B is compact.*

Lemma 4. *(Fredholm alternative) Let E be a Banach space and let $A \in \mathcal{L}(E,E)$ be a compact operator. Then the equation $z = Az + f$, $f \in E$, has a unique solution $z \in E$ if and only if the homogeneous equation $z = Az$ has only the trivial solution $z = 0$.*

The existence, uniqueness and regularity properties of the solution to (1.1)–(1.2) can be characterized by the following result.

Theorem 1. *(i) Assume that* $\alpha \in (1,2)$*,* $\kappa \in [0,1)$*,* $h, f \in C[0,b]$ *and* L_κ *is defined by* (1.3)*, where* $K \in C(\Delta)$*. Moreover, let* $b_1a_{11}a_{22}+a_{11}a_{21}+a_{12}a_{21} \neq 0$ *and assume that the problem* (1.1)–(1.2) *with* $f = 0$ *and* $\gamma_1 = \gamma_2 = 0$ *has in* $C[0,b]$ *only the trivial solution* $y = 0$*. Then problem* (1.1)–(1.2) *possesses a unique solution* $y \in C^1[0,b]$ *such that* $D^\alpha_{Cap} y \in C[0,b]$*.*
(ii) Let the assumptions of (i) hold and let $K \in C^q(\Delta)$*,* $h, f \in C^{q,\mu}(0,b]$*,* $q \in \mathbb{N}$*,* $\mu \in (-\infty, 1)$*. Then* y*, the solution of problem* (1.1)–(1.2)*, and its derivative* $D^\alpha_{Cap} y$ *belong to* $C^{q,\nu}(0,b]$*, where*

$$\nu = \begin{cases} \max\{\kappa, \mu\}, & \text{if} \quad K \neq 0, \\ \max\{1-\alpha, \mu\}, & \text{if} \quad K = 0 \ (K \text{ vanishes identically}). \end{cases} \tag{2.8}$$

Proof. We begin by considering equation $z - Tz = g$ (see (2.5)), where operator T and right-hand side g are defined by (2.6) and (2.7), respectively. We rewrite the function g in the form $g = g_1 - g_2$, where

$$g_1(t) = f(t) - h(t)[k_{01} + k_{11}t], \quad g_2(t) = \int_0^t L_\kappa(t,s)(k_{01} + k_{11}s)ds, \quad t \in [0,b].$$

The operator T can be written in the form

$$T = -H_2(J^\alpha + H_1) - MH_1 - B/\Gamma(\alpha).$$

Here H_1, H_2, M and B are defined by the following formulas:

$$(H_1z)(t) = (J^\alpha z)(b_1)k_{00} + (J^\alpha z)(b_1)k_{10}t, \quad (H_2z)(t) = h(t)z(t),$$
$$(Mz)(t) = \int_0^t L_\kappa(t,s)z(s)ds,$$
$$(Bz)(t) = \int_0^t (t-s)^\alpha \left(\int_0^1 L_\kappa(t,(t-s)\sigma + s)\sigma^{\alpha-1}d\sigma \right) z(s)ds,$$

with $t \in [0,b]$ and $z \in C[0,b]$. We are now ready to prove (i) and (ii). Our aim is to use Lemma 4 (Fredholm alternative).

Proof of (i). First, we show that T is compact as an operator from $C[0,b]$ into $C[0,b]$. Indeed, if $0 < \kappa < 1$, then we can write

$$(Bz)(t) = \int_0^t (t-s)^{\alpha-\kappa} F(t,s)z(s)ds, \quad t \in [0,b],$$

where $\alpha - \kappa > 0$ and

$$F(t,s) = \int_0^1 K(t,(t-s)\sigma + s)(1-\sigma)^{-\kappa}\sigma^{\alpha-1}d\sigma, \quad (t,s) \in \Delta.$$

Due to $K \in C(\Delta)$ also $F \in C(\Delta)$. Hence, by Lemma 2, $B : C[0,b] \to C[0,b]$ is compact. A similar approach shows that $B : C[0,b] \to C[0,b]$ is compact also for $\kappa = 0$. Moreover, Lemma 2 implies that J^α is compact as an operator from $C[0,b]$ into $C[0,b]$. Clearly, $H_1 : C[0,b] \to C[0,b]$ is compact. We also see that

H_2 and M are bounded as operators from $C[0,b]$ into $C[0,b]$. All this together with Lemma 3 yields that $T : C[0,b] \to C[0,b]$ is compact.

Since $f,h \in C[0,b]$ and $K \in C(\Delta)$, it is easy to see that $g_1, g_2 \in C[0,b]$. Thus, $g = g_1 - g_2$ is a continuous function on $[0,b]$. Also note that, if $f = 0$ and $\gamma_1 = \gamma_2 = 0$, then $k_{01} = k_{11} = 0$ (see (2.3)) and hence $g = 0$. This together with the assumption that the problem (1.1)–(1.2) with $f = 0$ and $\gamma_1 = \gamma_2 = 0$ possesses in $C[0,b]$ only the trivial solution $y = 0$ implies that equation $z = Tz$ has in $C[0,b]$ only the trivial solution $z = 0$. By Lemma 4 we obtain that the equation $z = Tz + g$ possesses a unique solution $z \in C[0,b]$. Now, with the help of (2.2), (1.5)–(1.7) we obtain that problem (1.1)–(1.2) has a unique solution $y \in C^1[0,b]$ such that $D^\alpha_{\mathrm{Cap}} y = z \in C[0,b]$.

Proof of (ii). Observe first that the right-hand side of equation $z - Tz = g$ belongs to $C^{q,\nu}(0,b]$. Indeed, $g_1 \in C^{q,\mu}(0,b] \subset C^{q,\nu}(0,b]$, because $f,h \in C^{q,\mu}(0,b]$ and $\mu \le \nu < 1$ (see (2.8)). If K vanishes identically, then it follows from (1.3) that L_κ ($\kappa \in [0,1)$) vanishes identically and $g_2(t) = 0$ for any $t \in [0,b]$. Therefore, $g_2 \in C^{q,\nu}(0,b]$ for $K = 0$. If $K \neq 0$, then it follows from Lemma 2 that $g_2 \in C^{q,\nu}(0,b]$. Consequently, $g = g_1 - g_2$ belongs to $C^{q,\nu}(0,b]$.

Next, we show that T is a compact operator from $C^{q,\nu}(0,b]$ into $C^{q,\nu}(0,b]$. Since $1 - \alpha \le \nu$, it follows from Lemma 2 that J^α is a compact operator from $C^{q,\nu}(0,b]$ into $C^{q,\nu}(0,b]$. Also, $H_1 : C^{q,\nu}(0,b] \to C^{q,\nu}(0,b]$ is a compact operator. Furthermore, H_2 and M are bounded as operators from $C^{q,\nu}(0,b]$ into $C^{q,\nu}(0,b]$ (see Lemmas 1 and 2). Consequently, by using Lemma 3 we obtain that operators $H_2(H_1 + J^\alpha)$ and MH_1 are compact operators from $C^{q,\nu}(0,b]$ into $C^{q,\nu}(0,b]$. As $\kappa - \alpha < 1 - \alpha \le \nu$, we see with the help of Lemma 2 that operator B is compact from $C^{q,\nu}(0,b]$ into $C^{q,\nu}(0,b]$. Thus, T is compact as an operator from $C^{q,\nu}(0,b]$ into $C^{q,\nu}(0,b]$.

Since the homogeneous equation $z = Tz$ has in $C^{q,\nu}(0,b] \subset C[0,b]$ only the trivial solution, it follows from Lemma 4 that equation $z = Tz + g$ has a unique solution $z \in C^{q,\nu}(0,b]$. With the help of relation (2.2) and Lemma 2 we see that the problem (1.1)–(1.2) possesses a unique solution $y \in C^{q,\nu}(0,b]$ such that $D^\alpha_{\mathrm{Cap}} y = z \in C^{q,\nu}(0,b]$. □

3 Numerical method

Let $N \in \mathbb{N}$, we introduce a partition (a graded grid) $\Pi_N = \{t_0, \dots, t_N\}$ of the interval $[0,b]$ with the grid points

$$t_j = b\left(\frac{j}{N}\right)^r, \quad j = 0, 1, \dots, N, \tag{3.1}$$

where the so-called grading exponent r belongs to $[1,\infty)$. If $r = 1$, then the grid points (3.1) are distributed uniformly; for $r > 1$ the points (3.1) are more densely clustered near the left endpoint of the interval $[0,b]$.

For a given integer $k \in \mathbb{N}_0$, by $S_k^{(-1)}(\Pi_N)$ we denote the space of piecewise polynomial functions

$$S_k^{(-1)}(\Pi_N) = \{v : \quad v|_{[t_{j-1},t_j]} \in \pi_k,\ j = 1, \dots, N\}.$$

Here, $v|_{[t_{j-1},t_j]}$ is the restriction of function $v:[0,b]\to\mathbb{R}$ onto the subinterval $[t_{j-1},t_j]\subset[0,b]$ and π_k denotes the set of polynomials of degree not exceeding k. Observe that the elements of the space $S_k^{(-1)}(\Pi_N)$ may have jump discontinuities at the interior points $t_1,\dots,t_{N-1}$ of Π_N.

In every interval $[t_{j-1},t_j]$, $j=1,\dots,N$, we define $m\in\mathbb{N}$ collocation points

$$t_{jk}=t_{j-1}+\eta_k(t_j-t_{j-1}),\quad k=1,\dots,m,\quad j=1,\dots,N, \tag{3.2}$$

where $\eta_1,\dots,\eta_m$ are some fixed collocation parameters which do not depend on j and N and satisfy inequality

$$0\le\eta_1<\eta_2<\ldots<\eta_m\le 1.$$

We find approximations $z_N\in S_{m-1}^{(-1)}(\Pi_N)$ $(m,N\in\mathbb{N})$ to the exact solution z of equation (2.5) by collocation conditions

$$z_N(t_{jk})=(Tz_N)(t_{jk})+g(t_{jk}),\quad k=1,\dots,m,\quad j=1,\dots,N, \tag{3.3}$$

with $\{t_{jk}\}$ defined by (3.2). Having found an approximation z_N we use (2.2) to determine the approximation y_N to y, the solution of problem (1.1)–(1.2), in the following way:

$$y_N(\tau)=(J^\alpha z_N)(b_1)k_{00}+k_{01}+[(J^\alpha z_N)(b_1)k_{10}+k_{11}]\tau+(J^\alpha z_N)(\tau), \tag{3.4}$$

where $\tau\in[0,b]$ and $z_N\in S_{m-1}^{(-1)}(\Pi_N)$ is determined by (3.3).

For given $N,m\in\mathbb{N}$ we define the interpolation operator $\mathcal{P}_N=\mathcal{P}_{N,m}:C[0,b]\to S_{m-1}^{(-1)}(\Pi_N)$ by

$$\mathcal{P}_N v\in S_{m-1}^{(-1)}(\Pi_N),\ \ (\mathcal{P}_N v)(t_{jk})=v(t_{jk}),\ \ j{=}1,\dots,N,\ \ k{=}1,\dots,m, \tag{3.5}$$

for any continuous function $v\in C[0,b]$. If $\eta_1=0$, then by $(\mathcal{P}_N v)(t_{j1})$ we denote the right limit $\lim_{t\to t_{j-1},t>t_{j-1}}(\mathcal{P}_N v)(t)$. If $\eta_m=1$, then $(\mathcal{P}_N v)(t_{jm})$ denotes the left limit $\lim_{t\to t_j,t<t_j}(\mathcal{P}_N v)(t)$. By using operator $\mathcal{P}_N$ conditions (3.3) for finding $z_N\in S_{m-1}^{(-1)}(\Pi_N)$ take the form

$$z_N=\mathcal{P}_N T z_N+\mathcal{P}_N g. \tag{3.6}$$

The collocation conditions (3.3) lead to a system of linear equations to uniquely determine $z_n\in S_{m-1}^{(-1)}(\Pi_N)$. The exact form of the system of equations is determined by the choice of a basis in the space $S_{m-1}^{(-1)}(\Pi_N)$. If $\eta_1>0$ or $\eta_m<1$, then we can use the Lagrange fundamental polynomial representation

$$z_N(\tau)=\sum_{\lambda=1}^{N}\sum_{\mu=1}^{m}c_{\lambda\mu}l_{\lambda\mu}(\tau),\quad \tau\in[0,b], \tag{3.7}$$

where $l_{\lambda\mu}(\tau)=0$, if $\tau\notin[t_{\lambda-1},t_\lambda]$, and

$$l_{\lambda\mu}(\tau)=\prod_{i=1,i\ne\mu}^{m}\frac{\tau-t_{\lambda i}}{t_{\lambda\mu}-t_{\lambda i}}\ \text{for}\ \tau\in[t_{\lambda-1},t_\lambda],\quad \mu=1,\dots,m,\quad \lambda=1,\dots,N.$$

Then $z_N \in S^{(-1)}_{m-1}(\varPi_N)$ and $z_N(t_{jk}) = c_{jk}$ for every $k = 1,\dots,m$, $j = 1,\dots,N$. To determine approximation z_N in the form (3.7) we have to solve a system of linear algebraic equations with respect to $\{c_{jk}\}$:

$$c_{jk} = \sum_{\lambda=1}^{N}\sum_{\mu=1}^{m}(Tl_{\lambda\mu})(t_{jk})c_{\lambda\mu} + g(t_{jk}), \quad k = 1,\dots,m,\ j = 1,\dots,N. \tag{3.8}$$

Having found $\{c_{jk}\}$ from (3.8) we get that

$$\begin{aligned} y_N(\tau) = &\sum_{\lambda=1}^{N}\sum_{\mu=1}^{m} c_{\lambda\mu}(J^\alpha l_{\lambda\mu})(b_1)k_{00} + k_{01} + \left[\sum_{\lambda=1}^{N}\sum_{\mu=1}^{m} c_{\lambda\mu}(J^\alpha l_{\lambda\mu})(b_1)k_{10} + k_{11}\right]\tau \\ &+ \sum_{\lambda=1}^{N}\sum_{\mu=1}^{m} c_{\lambda\mu}(J^\alpha l_{\lambda\mu})(\tau), \quad \tau \in [0,b]. \end{aligned} \tag{3.9}$$

It follows from (1.6) and (1.7) that the function y_N defined by (3.9) is continuous on $[0,b]$.

4 Convergence analysis

In this section, we study the convergence and convergence order of our method. Throughout this section assume that $\mathcal{P}_N : C[0,b] \to S^{(-1)}_{m-1}(\varPi_N)$ is defined by (3.5). Lemmas 5 and 6 below follow from the results of [5] (see also [24]).

Lemma 5. *The operators $\mathcal{P}_N$, $N \in \mathbb{N}$, belong to the space $\mathcal{L}(C[0,b], L^\infty(0,b))$ and $\|\mathcal{P}_N\|_{\mathcal{L}(C[0,b],L^\infty(0,b))} \le c$, with a positive constant c which is independent of N. Moreover, for every $u \in C[0,b]$ we have*

$$\|u - \mathcal{P}_N u\|_{L^\infty(0,b)} \to 0 \quad as \quad N \to \infty.$$

Lemma 6. *Let $A : L^\infty(0,b) \to C[0,b]$ be a linear compact operator. Then*

$$\|A - \mathcal{P}_N A\|_{\mathcal{L}(L^\infty(0,b),L^\infty(0,b))} \to 0 \quad as \quad N \to \infty.$$

The following theorem gives the conditions for the convergence of the method proposed in the previous section.

Theorem 2. *Let the assumptions introduced in the part (i) of Theorem 1 be fulfilled. Let $m, N \in \mathbb{N}$ and assume that the collocation points* (3.2) *with arbitrary parameters $\eta_1,\dots,\eta_m$ satisfying $0 \le \eta_1 < \dots < \eta_m \le 1$ and grid points* (3.1) *are used. Then problem* (1.1)–(1.2) *possesses a unique solution $y \in C^1[0,b]$ such that $D^\alpha_{Cap} y \in C[0,b]$. There exists an integer $N_0 > 0$ such that, for $N \ge N_0$, Equation* (3.6) *possesses a unique solution $z_N \in S^{(-1)}_{m-1}(\varPi_N)$, determining by* (3.9) *a unique approximation $y_N \in C[0,b]$ to y, the solution of* (1.1)–(1.2), *and*

$$\|y - y_N\|_\infty \to 0 \quad as \quad N \to \infty. \tag{4.1}$$

Proof. Due to Theorem 1, we only need to prove the convergence (4.1). Let T and g be defined by (2.6) and (2.7), respectively. We showed in the proof of Theorem 1 that T is compact as an operator from $C[0,b]$ into $C[0,b]$. In a similar way it can be shown that T is compact from $L^\infty(0,b)$ into $C[0,b]$, thus also from $L^\infty(0,b)$ into $L^\infty(0,b)$. Furthermore, $g \in C[0,b] \subset L^\infty(0,b)$ and the homogeneous equation $z = Tz$ has in $C[0,b]$ only the solution $z = 0$. Since T belongs to $\mathcal{L}(L^\infty(0,b), C[0,b])$, equation $z = Tz$ possesses also in $L^\infty(0,b)$ only the trivial solution. By Lemma 4, equation $z = Tz + g$ possesses a unique solution $z \in L^\infty(0,b)$. In other words, operator $I - T$ is invertible in $L^\infty(0,b)$ and its inverse is bounded: $(I-T)^{-1} \in \mathcal{L}(L^\infty(0,b), L^\infty(0,b))$. From Lemma 6 and the boundedness of $(I-T)^{-1}$ in $L^\infty(0,b)$ we obtain that for all sufficiently large N, operator $I - \mathcal{P}_N T$ is invertible in $L^\infty(0,b)$ and

$$\left\|(I - \mathcal{P}_N T)^{-1}\right\|_{\mathcal{L}(L^\infty(0,b),L^\infty(0,b))} \le c, \quad N \ge N_0, \tag{4.2}$$

where c is a constant not depending on N. Thus, for $N \ge N_0$, Equation (3.6) has a unique solution $z_N \in S_{m-1}^{(-1)}(\mathit{\Pi}_N)$. Now, for z_N and z, the solution of equation $z = Tz + g$ (hence $\mathcal{P}_N z = \mathcal{P}_N Tz + \mathcal{P}_N g$), we see that

$$(I - \mathcal{P}_N T)(z - z_N) = z - z_N - \mathcal{P}_N Tz + \mathcal{P}_N T z_N = z - \mathcal{P}_N z, \quad N \ge N_0.$$

Therefore, by (4.2),

$$\|z - z_N\|_{L^\infty(0,b)} \le c \,\|z - \mathcal{P}_N z\|_{L^\infty(0,b)}\,, \quad N \ge N_0,$$

where c is a positive constant not depending on N. By (2.2) and (3.4), we have for $t \in [0,b]$ that

$$\begin{aligned} |y(t) - y_N(t)| &= |[(J^\alpha z)(b_1)k_{00} + k_{01} + [(J^\alpha z)(b_1)k_{10} + k_{11}]t + (J^\alpha z)(t)] \\ &\quad - [(J^\alpha z_N)(b_1)k_{00} + k_{01} + [(J^\alpha z_N)(b_1)k_{10} + k_{11}]t + (J^\alpha z_N)(t)]| \\ &= |(J^\alpha(z - z_N))(b_1)k_{00} + (J^\alpha(z - z_N))(b_1)k_{10}t + (J^\alpha(z - z_N))(t)|. \end{aligned} \tag{4.3}$$

Thus,

$$\|y - y_N\|_\infty \le c_1 \|z - z_N\|_\infty \le c_2 \|z - \mathcal{P}_N z\|_\infty\,,$$

where c_1 ja c_2 are some positive constants not depending on N. Using Lemma 5, we see that the convergence (4.1) holds. □

The convergence behaviour of our method is described by Theorem 3 below. Before presenting this theorem, we first introduce a result from [13].

Lemma 7. *Let* $u \in C^{m+1,\nu}(0,b]$, $m \in \mathbb{N}$, $\nu \in (-\infty, 1)$, $N \in \mathbb{N}$, $r \in [1,\infty)$ *and* J^1 *be defined by* (1.4). *Assume that the collocation points* (3.2) *with grid points* (3.1) *and parameters* $\eta_1, \dots, \eta_m$ *satisfying* $0 \le \eta_1 < \dots < \eta_m \le 1$ *are used. Moreover, assume that* $\eta_1, \dots, \eta_m$ *are such that a quadrature approximation*

$$\int_0^1 F(x)dx \approx \sum_{k=1}^m w_k F(\eta_k) \quad (0 \le \eta_1 < \dots < \eta_m \le 1) \tag{4.4}$$

with appropriate weights $\{w_k\}$ *is exact for all polynomials* F *of degree* m.

Then

$$\left\|J^1(\mathcal{P}_N u - u)\right\|_\infty \le cE_N(m,\nu,r),$$

where c is a positive constant not depending on N and

$$E_N(m,\nu,r) = \begin{cases} N^{-m-1} & \textit{for} \quad m < 2-\nu, \;\; r \ge 1, \\ N^{-m-1}(1+\log N)^2 & \textit{for} \quad m = 2-\nu, \;\; r = 1, \\ N^{-m-1}(1+\log N) & \textit{for} \quad m = 2-\nu, \;\; r > 1, \\ N^{-r(2-\nu)} & \textit{for} \quad m > 2-\nu, \; 1 \le r < \dfrac{m+1}{2-\nu}, \\ N^{-m-1} & \textit{for} \quad m > 2-\nu, \;\; r \ge (m+1)/(2-\nu). \end{cases} \tag{4.5}$$

Theorem 3. *Let $m, N \in \mathbb{N}$, $r \ge 1$ and let the assumptions of part (ii) of Theorem 1 be fulfilled with $K \in C^{m+1}(\Delta)$, $h, f \in C^{m+1,\mu}(0,b]$, $\mu \in (-\infty, 1)$. Assume that the collocation points (3.2) with grid points (3.1) and parameters $\eta_1, \ldots, \eta_m$ satisfying $0 \le \eta_1 < \ldots < \eta_m \le 1$ are used. Moreover, assume that $\eta_1, \ldots, \eta_m$ are such that a quadrature approximation (4.4) with appropriate weights $\{w_k\}$ is exact for all polynomials of degree m.*

Then problem (1.1)–(1.2) has a unique solution $y \in C^1[0,b]$ such that y and $D^\alpha_{Cap} y$ belong to $C^{m+1,\nu}(0,b]$. There exists an integer $N_0 > 0$ such that, for $N \ge N_0$, Equation (3.6) possesses a unique solution $z_N \in S^{(-1)}_{m-1}(\Pi_N)$, determining by (3.9) a unique approximation y_N to y, the solution of (1.1)–(1.2), and the following error estimate holds:

$$\|y - y_N\|_\infty \le cE_N(m,\nu,r). \tag{4.6}$$

Here $E_N(m,\nu,r)$ is defined by (4.5), ν is given by the formula (2.8), r is the grading exponent in (3.1) and c is a positive constant not depending on N.

Proof. It follows from part (ii) (with $q = m+1$) of Theorem 1 that problem (1.1)–(1.2) has a unique solution $y \in C^1[0,b]$ such that $y, D^\alpha_{\text{Cap}} y \in C^{m+1,\nu}(0,b]$. From the proof of Theorem 2 we know that there exists an integer $N_0 > 0$ such that for $N \ge N_0$, Equation (3.6) has a unique solution $z_N \in S^{(-1)}_{m-1}(\Pi_N)$ and the estimate (4.2) holds. Denote

$$\hat{z}_N = Tz_N + g, \quad N \ge N_0, \tag{4.7}$$

where T and g are defined by (2.6) and (2.7), respectively. It follows from (3.6) and (4.7) that $\mathcal{P}_N \hat{z}_N = z_N$. Using this and (4.7) we obtain an equation with respect to $\hat{z}_N$:

$$\hat{z}_N = T\mathcal{P}_N \hat{z}_N + g, \quad N \ge N_0. \tag{4.8}$$

With the help of $z = Tz + g$ and (4.8) we get for every $N \ge N_0$ that

$$(I - T\mathcal{P}_N)(\hat{z}_N - z) = T(\mathcal{P}_N z - z). \tag{4.9}$$

Due to the existence of the inverse $(I - \mathcal{P}_N T)^{-1} \in \mathcal{L}(L^\infty(0,b), L^\infty(0,b))$ for $N \ge N_0$, there exists also the inverse $(I - T\mathcal{P}_N)^{-1} \in \mathcal{L}(L^\infty(0,b), L^\infty(0,b))$ and

$$(I - T\mathcal{P}_N)^{-1} = I + T(I - \mathcal{P}_N T)^{-1}\mathcal{P}_N, \quad N \ge N_0. \tag{4.10}$$

Using (4.9), (4.10), (4.2) and Lemma 5, we get

$$\|\hat{z}_N - z\|_\infty = \left\|(I - T\mathcal{P}_N)^{-1} T(\mathcal{P}_N z - z)\right\|_\infty \le c \, \|T(\mathcal{P}_N z - z)\|_\infty, \; N \ge N_0, \quad (4.11)$$

where c is a positive constant independent of N. Further, on the basis of the definition of operator T (see (2.6)), we have

$$\|T(\mathcal{P}_N z - z)\|_\infty \le c \, \|J^\alpha(\mathcal{P}_N z - z)\|_\infty, \quad N \ge N_0, \quad (4.12)$$

where c is a positive constant not depending on N. It follows from (4.12) and (4.11) that

$$\|\hat{z}_N - z\|_\infty \le c_1 \, \|J^\alpha(\mathcal{P}_N z - z)\|_\infty, \quad N \ge N_0, \quad (4.13)$$

where c_1 is a positive constant not depending on N. Since $z_N = \mathcal{P}_N \hat{z}_N$, we have $z_N - z = \mathcal{P}_N \hat{z}_N - z = \mathcal{P}_N(\hat{z}_N - z) + \mathcal{P}_N z - z$. This, together with (4.3) leads to the following estimate:

$$\begin{aligned} |y_N(t) - y(t)| &= |(J^\alpha(z_N - z))(b_1)k_{00} + t(J^\alpha(z_N - z))(b_1)k_{10} + (J^\alpha(z_N - z))(t)| \\ &\le |(J^\alpha \mathcal{P}_N(\hat{z}_N - z))(b_1)k_{00} + t(J^\alpha \mathcal{P}_N(\hat{z}_N - z))(b_1)k_{10} + (J^\alpha \mathcal{P}_N(\hat{z}_N - z))(t)| \\ &\quad + |(J^\alpha(\mathcal{P}_N z - z))(b_1))k_{00} + t(J^\alpha(\mathcal{P}_N z - z))(b_1)k_{10} + (J^\alpha(\mathcal{P}_N z - z))(t)|, \end{aligned}$$

with $t \in [0, b]$. Thus, it follows from Lemma 5 and inequality (4.13) that

$$\|y_N - y\|_\infty \le c_1 \, \|\hat{z}_N - z\|_\infty + c_2 \, \|J^\alpha(\mathcal{P}_N z - z)\|_\infty \le c_3 \, \|J^\alpha(\mathcal{P}_N z - z)\|_\infty, \quad (4.14)$$

where c_1, c_2 and c_3 are some positive constants not depending on $N \ge N_0$.

Since $\alpha \in (1, 2)$ we get from (4.14), (1.5) and the boundedness of operator $J^{\alpha-1} : C[0, b] \to C[0, b]$ that

$$\|y_N - y\|_\infty \le c_1 \left\|J^{\alpha-1} J^1(\mathcal{P}_N z - z)\right\|_\infty \le c_2 \left\|J^1(\mathcal{P}_N z - z)\right\|_\infty, \quad N \ge N_0,$$

where c_1, c_2 are positive constants independent of N. With the help of Lemma 7 we now see that the inequality (4.6) holds. □

5 Numerical experiments

In this section, we present some numerical examples to illustrate our theoretical results. In the examples below y is the exact solution of the underlying problem and y_N ($N \in \mathbb{N}$) is its approximation found using the method described in Section 3. In particular, approximations $z_N \in S_{m-1}^{(-1)}(\Pi_N)$ ($N \in \mathbb{N}$) to the solution z of Equation (2.5) (with the corresponding data given in Examples 1–3 below) are found by (3.3) using grid points (3.1) and collocation points (3.2), where

$$\eta_1 = \frac{3 - \sqrt{3}}{6}, \quad \eta_2 = 1 - \eta_1 \quad (\text{if } m = 2), \quad (5.1)$$

$$\eta_1 = \frac{5 - \sqrt{15}}{10}, \quad \eta_2 = \frac{1}{2}, \quad \eta_3 = 1 - \eta_1 \quad (\text{if } m = 3) \quad (5.2)$$

are collocation parameters that satisfy the conditions of Theorem 3. Note that (5.1) and (5.2) are actually the knots of the m-point Gaussian quadrature approximation (4.4) for $m = 2$ and $m = 3$, respectively (see, e.g., [6]). Further, the function (approximation) z_N is defined by (3.7), where the coefficients $\{c_{\lambda\mu}\}$ are determined by (3.8). Finally, the approximate solution y_N of (1.1)–(1.2) is found using (3.4).

We present in the tables below some results of numerical experiments for different values of parameters m, N and r. The errors ε_N are calculated as follows:

$$\varepsilon_N = \max_{j=1,\dots,N} \max_{k=0,\dots,10} |y(\tau_{jk}) - y_N(\tau_{jk})|,$$

where

$$\tau_{jk} = t_{j-1} + k(t_j - t_{j-1})/10, \quad k = 0, \dots, 10, \quad j = 1, \dots, N,$$

with the gridpoints t_j defined by (3.1). The ratios $\Theta_N = \varepsilon_{\frac{N}{2}}/\varepsilon_N$, characterising the observed convergence rate, are also presented.

To perform the numerical experiments, we wrote the code in Python. For calculating integrals, we used the *numpy* library.

Example 1. Consider the following problem:

$$(D^{\frac{21}{20}}_{\text{Cap}} y)(t) + h(t)y(t) + \int_0^t (t-s)^{-\frac{3}{4}} K(t,s)y(s)ds = f(t), \quad t \in [0,1], \quad (5.3)$$

$$y(0) + y\left(\frac{1}{10}\right) = \left(\frac{1}{10}\right)^{\frac{11}{10}}, \quad y'(0) + y\left(\frac{1}{10}\right) = \left(\frac{1}{10}\right)^{\frac{11}{10}}, \quad (5.4)$$

with

$$h(t) = t, \quad K(t,s) = ts, \quad f(t) = \frac{\Gamma\left(\frac{21}{10}\right)}{\Gamma\left(\frac{21}{20}\right)} t^{\frac{1}{20}} + t^{\frac{21}{10}} + \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{31}{10}\right)}{\Gamma\left(\frac{67}{20}\right)} t^{\frac{67}{20}},$$

where $t \in [0,1]$ and $s \in [0,t]$. We see that (5.3)–(5.4) is a problem of the form (1.1)–(1.2) with $\alpha = \frac{21}{20}$, $\kappa = \frac{3}{4}$, $b_1 = \frac{1}{10}$, $b = 1$, $a_{11} = a_{12} = a_{21} = a_{22} = 1$, $\gamma_1 = \gamma_2 = \left(\frac{1}{10}\right)^{\frac{11}{10}}$ and that $y(t) = t^{\frac{11}{10}}$, $t \in [0,1]$ is its exact solution. Clearly, $f, h \in C^{q,\mu}(0,1]$ with $\mu = \frac{19}{20}$ and arbitrary $q \in \mathbb{N}$. Therefore, by (2.8),

$$\nu = \max\{\kappa, \mu\} = 19/20.$$

In the case $m = 3$ it follows from the error estimate (4.6) with $\nu = \frac{19}{20}$ that, for sufficiently large N,

$$\varepsilon_N \le c \begin{cases} N^{-r\left(\frac{21}{20}\right)}, & \text{if } 1 \le r < 80/21, \\ N^{-4}, & \text{if } r \ge 80/21, \end{cases} \quad (5.5)$$

where c is a positive constant not depending on N. Due to (5.5), the ratios Θ_N for $r = 1$, $r = 2$, $r = 3$ and $r = 4$ ought to be approximately $2^{\frac{21}{20}} \approx 2.07$, $2^{\frac{42}{20}} \approx 4.29$, $2^{\frac{63}{20}} \approx 8.88$ and $2^4 = 16$, respectively. These values are given in the last row of Table 1. We see that the numerical results are in accordance with the theoretical estimates given by Theorem 3.

Table 1. Numerical results for problem (5.3)–(5.4) with $m = 3$.

	$r=1$		$r=2$		$r=3$		$r=4$	
N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N
4	4.85E-04		8.77E-05		4.30E-05		4.73E-05	
8	2.01E-04	2.42	2.03E-05	4.31	4.61E-06	9.34	4.18E-06	11.33
16	9.55E-05	2.10	4.07E-06	5.00	4.33E-07	10.64	1.77E-07	23.65
32	4.37E-05	2.19	8.42E-07	4.83	3.66E-08	11.83	1.07E-08	16.49
64	2.00E-05	2.19	1.73E-07	4.87	3.64E-09	10.05	6.64E-10	16.12
128	9.11E-06	2.19	3.77E-08	4.58	3.58E-10	10.17	4.13E-11	16.10
256	4.15E-06	2.19	8.80E-09	4.29	3.81E-11	9.39	2.61E-12	15.84
		2.07		4.29		8.88		16.00

Example 2. Consider the following problem:

$$(D^{\frac{3}{2}}_{\mathrm{Cap}}y)(t) + h(t)y(t) + \int_0^t [1+\log(t-s)]y(s)ds = f(t), \quad t\in[0,1], \tag{5.6}$$

$$y(0)=1, \quad y'(0)=1, \tag{5.7}$$

with $h(t)=1$, $f(t)=t^{1/10}$, $t\in[0,1]$.

We see that (5.6)–(5.7) is a problem of the form (1.1)–(1.2) with $\alpha = \frac{3}{2}$, $\kappa = 0$, $K = 1$, $b = 1$, $a_{11} = a_{21} = \gamma_1 = \gamma_2 = 1$, $a_{12} = a_{22} = 0$. It is easy to see that $h, f \in C^{q,\mu}(0,1]$ with $\mu = \frac{9}{10}$ and arbitrary $q \in \mathbb{N}$. Therefore, by (2.8),

$$\nu = \max\{\kappa, \mu\} = 9/10.$$

Here, the exact solution is not known. For the numerical tests we use approximation y_N obtained with $m = 3$, $r = 4$ and $N = 1024$, i.e., $y(x) \approx y_{1024}(x)$ $(0 \le x \le 1)$.

In the case $m = 2$ it follows from the error estimate (4.6) with $\nu = \frac{9}{10}$ that, for sufficiently large N,

$$\varepsilon_N \le c\begin{cases} N^{-r\left(\frac{11}{10}\right)}, & \text{if } 1 \le r < 30/11,\\ N^{-3}, & \text{if } r \ge 30/11,\end{cases} \tag{5.8}$$

where c is a positive constant not depending on N. Due to (5.8), the ratios Θ_N for $r = 1$, $r = 2$, $r = 3$ and $r = 4$ ought to be approximately $2^{\frac{11}{10}} \approx 2.14$, $2^{\frac{22}{10}} \approx 4.59$, $2^3 = 8$ and $2^3 = 8$, respectively. These values are given in the last row of Table 2.

In the case $m = 3$, it follows from (4.6) with $\nu = \frac{9}{10}$ that, for sufficiently large N,

$$\varepsilon_N \le c\begin{cases} N^{-r\left(\frac{11}{10}\right)}, & \text{if } 1 \le r < 40/11,\\ N^{-4}, & \text{if } r \ge 40/11,\end{cases} \tag{5.9}$$

where c is a positive constant not depending on N. Due to (5.9) the ratios Θ_N for $r = 1$, $r = 2$, $r = 3$ and $r = 4$ ought to be approximately $2^{\frac{11}{10}} \approx 2.14$, $2^{\frac{22}{10}} \approx 4.59$, $2^{\frac{33}{10}} \approx 9.85$ and $2^4 = 16$, respectively. These values are given in the last row of Table 3. As we can see from Tables 2 and 3, the numerical results are in accordance with our theoretical estimates.

Table 2. Numerical results for problem (5.6) – (5.7) with $m = 2$.

	$r=1$		$r=2$		$r=3$		$r=4$	
N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N
4	1.69E-03		7.58E-04		1.40E-03		2.61E-03	
8	6.96E-04	2.42	1.14E-04	6.65	1.61E-04	8.71	3.44E-04	7.58
16	3.03E-04	2.30	1.96E-05	5.81	1.57E-05	10.21	3.50E-05	9.81
32	1.36E-04	2.23	3.80E-06	5.16	1.45E-06	10.84	3.19E-06	10.97
64	6.21E-05	2.19	7.85E-07	4.83	1.32E-07	11.03	2.78E-07	11.48
128	2.86E-05	2.17	1.68E-07	4.69	1.19E-08	11.03	2.39E-08	11.64
256	1.33E-05	2.16	3.62E-08	4.63	1.09E-09	10.97	2.05E-09	11.65
		2.14		4.59		8.00		8.00

Table 3. Numerical results for problem (5.6)– (5.7) with $m = 3$.

	$r=1$		$r=2$		$r=3$		$r=4$	
N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N
4	6.85E-04		1.43E-04		8.06E-05		1.42E-04	
8	3.00E-04	2.28	2.95E-05	4.85	6.55E-06	12.30	8.57E-06	16.57
16	1.35E-04	2.22	6.33E-06	4.66	5.65E-07	11.61	4.65E-07	18.42
32	6.18E-05	2.19	1.37E-06	4.62	5.16E-08	10.94	2.36E-08	19.68
64	2.85E-05	2.17	2.98E-07	4.60	4.94E-09	10.44	1.14E-09	20.72
128	1.32E-05	2.16	6.48E-08	4.60	4.87E-10	10.14	5.33E-11	21.41
256	6.15E-06	2.15	1.41E-08	4.60	4.88E-11	9.98	2.44E-12	21.82
		2.14		4.59		9.85		16.00

Example 3. Consider the following problem:

$$(D^{\frac{11}{10}}_{\mathrm{Cap}}y)(t) + h(t)y(t) = f(t), \quad t \in [0,1], \tag{5.10}$$

$$y(0) + \frac{1}{2}y\left(\frac{1}{2}\right) = \frac{1}{128}, \quad y'(0) + \frac{1}{4}y\left(\frac{1}{2}\right) = \frac{1}{256}, \tag{5.11}$$

with

$$h(t) = t^{\frac{25}{10}}, \quad f(t) = \frac{\Gamma(7)}{\Gamma\left(\frac{59}{10}\right)}t^{\frac{49}{10}} + t^{\frac{85}{10}}, \quad t \in [0,1].$$

We see that (5.10)–(5.11) is a problem of the form (1.1)–(1.2) where $K = 0$, $\alpha = \frac{11}{10}$, $b = 1$, $b_1 = \frac{1}{2}$, $a_{11} = a_{21} = 1$, $a_{12} = \frac{1}{2}$, $a_{22} = \frac{1}{4}$, $\gamma_1 = \frac{1}{128}$, $\gamma_2 = \frac{1}{256}$ and that

$$y(t) = t^6, \quad t \in [0,1],$$

is its exact solution. Clearly, $f, h \in C^{q,\mu}(0,1]$ with $\mu = -\frac{3}{2}$ and arbitrary $q \in \mathbb{N}$. Therefore, by (2.8) we have

$$\nu = \max\{1-\alpha, \mu\} = -1/10.$$

In the case $m = 2$ it follows from the error estimate (4.6) with $\nu = -\frac{1}{10}$ that, for sufficiently large N,

$$\varepsilon_N \le cN^{-3} \qquad \text{if } r \ge 1, \tag{5.12}$$

Table 4. Numerical results for problem (5.10)–(5.11) with $m = 2$.

N	$r = 1$		$r = 2$		$r = 3$	
	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N
4	9.31E-03		4.08E-02		9.42E-02	
8	1.26E-03	7.39	7.79E-03	5.24	2.09E-02	4.51
16	1.57E-04	8.00	1.19E-03	6.56	3.45E-03	6.05
32	1.90E-05	8.29	1.50E-04	7.90	4.97E-04	6.93
64	2.25E-06	8.43	1.90E-05	7.89	6.24E-05	7.97
128	2.65E-07	8.50	2.25E-06	8.47	7.70E-06	8.10
256	3.10E-08	8.54	2.62E-07	8.59	9.32E-07	8.25
		8.00		8.00		8.00

where c is a positive constant not depending on N. Due to (5.12), the ratios Θ_N for every $r \geq 1$ ought to be $2^3 = 8$. These values are given in the last row of Table 4. We again see that the numerical results are in accordance with our theoretical estimates.

6 Conclusions

We have introduced and analysed a high order numerical method for solving boundary value problems for linear fractional weakly singular integro-differential equations with a Caputo fractional derivative. By reformulating the problem as an integral equation of the second kind, the regularity properties of the solution of the original problem and its Caputo derivative were studied. Using the obtained knowledge about the behaviour of the solution an algorithm was constructed for finding approximate solutions of the original problem. We showed that despite the lack of regularity of the solution, the proposed method is of optimal order. The performed numerical experiments were in accordance with theoretical results.

Acknowledgements

The authors would like to thank the reviewers for their insightful comments. This work was supported by the Estonian Research Council grant (PRG864).

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