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Weak Solutions via Two-Field Lagrange Multipliers for Boundary Value Problems in Mathematical Physics

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Abstract. A new variational approach for a boundary value problem in mathematical physics is proposed. By considering two-field Lagrange multipliers, we deliver a variational formulation consisting of a mixed variational problem which is equivalent with a saddle point problem. Thus, the unique solvability of the weak formulation we propose is governed by the saddle point theory. Alternative variational formulations and some of their connections are also discussed.

Keywords: partial differential equations, subdifferential inclusions, two-field Lagrange multipliers, weak solutions, saddle points.

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1 Introduction

In the present paper we draw the attention to a new approach to the weak solvability of the following boundary value problem:

Problem 1. Find $u : \overline{\Omega} \to \mathbb{R}$ and $\boldsymbol{D} : \overline{\Omega} \to \mathbb{R}^N$ such that

$$\begin{aligned} \nabla \cdot \boldsymbol{D}(\boldsymbol{x}) &= f_0(\boldsymbol{x}) && \text{in } \Omega, \\ -\boldsymbol{D}(\boldsymbol{x}) &\in \partial\varphi(\nabla u(\boldsymbol{x})) + \beta\,\nabla\, u(\boldsymbol{x}) && \text{in } \Omega, \qquad (1.1) \\ u(\boldsymbol{x}) &= 0 && \text{on } \Gamma_1, \\ \boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) &= f_2(\boldsymbol{x}) && \text{on } \Gamma_2, \\ \boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) &\in \partial\psi(u(\boldsymbol{x})) && \text{on } \Gamma_3, \qquad (1.2) \end{aligned}$$

where $\Omega \subset \mathbb{R}^N$ $(N > 1)$ is a bounded domain with smooth boundary Γ partitioned in three parts $\Gamma_1, \Gamma_2, \Gamma_3$ of positive measure, f_0, f_2 φ, ψ are given functions and β is a given parameter. As usual, by $\boldsymbol{\nu}$ we denote the unit outward normal vector defined almost everywhere on the boundary Γ and by $\cdot$ we denote the inner product on $\mathbb{R}^N$.

If $\varphi \equiv 0$ and $\Gamma_3 \equiv 0$ then we are driven to the classical boundary value problem in the electrostatic theory,

$$\begin{aligned} \nabla \cdot \boldsymbol{D}(\boldsymbol{x}) &= f_0(\boldsymbol{x}) && \text{in } \Omega, \\ \boldsymbol{D}(\boldsymbol{x}) &= -\beta\,\nabla\, u(\boldsymbol{x}) && \text{in } \Omega, \\ u(\boldsymbol{x}) &= 0 && \text{on } \Gamma_1, \\ \boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) &= f_2(\boldsymbol{x}) && \text{on } \Gamma_2; \end{aligned}$$

herein, the scalar function $u : \bar{\Omega} \to \mathbb{R}$ is the electrostatic potential and the vector function $\boldsymbol{D} : \bar{\Omega} \to \mathbb{R}$ is the electric field; see, e.g., Section 26, Chapter 8 in [10]. Thus, Problem 1 has a physical significance in the electricity theory; (1.1) is a generalized electric constitutive law and (1.2) describes a generalized electrically contact condition.

On the other hand, if $N = 2$, $\varphi \equiv 0$ and $\psi : \mathbb{R} \to [0, \infty)$ $\psi(r) = g|r|$ with $g > 0$, then (1.2) is equivalent with the well known Tresca's law

$$\left|\beta \frac{\partial\, u}{\partial\, \nu}(\boldsymbol{x})\right| \leq g, \quad \beta \frac{\partial\, u}{\partial\, \nu}(\boldsymbol{x}) = -g \frac{u(\boldsymbol{x})}{|u(\boldsymbol{x})|} \ \text{ if } u(\boldsymbol{x}) \neq 0 \ \text{ on } \Gamma_3.$$

As a result, Problem 1 reduces to the following boundary value problem:

$$\begin{aligned} -\beta \triangle\, u(\boldsymbol{x}) &= f_0(\boldsymbol{x}) && \text{in } \Omega, \\ u(\boldsymbol{x}) &= 0 && \text{on } \Gamma_1, \\ -\beta \frac{\partial\, u}{\partial\, \nu}(\boldsymbol{x}) &= f_2(\boldsymbol{x}) && \text{on } \Gamma_2, \\ \left|\beta \frac{\partial\, u}{\partial\, \nu}(\boldsymbol{x})\right| &\leq g, \quad \beta \frac{\partial\, u}{\partial\, \nu}(\boldsymbol{x}) = -g \frac{u(\boldsymbol{x})}{|u(\boldsymbol{x})|} && \text{if } u(\boldsymbol{x}) \neq 0, \qquad \text{on } \Gamma_3. \end{aligned}$$

In this context, Problem 1 is a frictional contact antiplane model for elastic materials, the unknown $u : \bar{\Omega} \to \mathbb{R}$ being the third component of the displacement field; see, e.g., Chapter 9 in [17] for a study in terms of variational inequalities of second kind.

Problem 1 was recently investigated in [5]. In [5], the study was governed by a single-field Lagrange multiplier. In contrast, in the present paper we weakly solve the boundary value problem Problem 1 by means of a two-field Lagrange multiplier $\overline{\lambda} = (\lambda_{\Omega}, \lambda_{\Gamma_3})$, λ_{Ω} being related to $\boldsymbol{D}$ in Ω and λ_{Γ_3} being related to $\boldsymbol{D}$ on Γ_3. Thus, the new variational formulation we propose allows to compute not only u in Ω but also $\boldsymbol{D}$ in Ω and $\boldsymbol{D}$ on Γ_3 as well.

In the present study we admit the following working hypotheses.

$(H1)$ The functionals $\varphi : \mathbb{R}^N \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ are seminorms. Moreover, there exist $M_{\varphi}, M_{\psi} > 0$ such that

- $|\varphi(\boldsymbol{w})| \leq M_{\varphi} \ \ \|w\|$ for all $\boldsymbol{w} \in \mathbb{R}^N$,
- $|\psi(s)| \leq M_{\psi}\,|s|$ for all $s \in \mathbb{R}$.

Here and everywhere below, $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^N$ and $|\cdot|$ denotes the absolute value of a real number.

$(H2)$ $f_0 \in L^2(\Omega)$, $f_2 \in L^2(\Gamma_2)$, $\beta > 0$.

The functional setting we adopt is governed by Hilbert spaces as follows.

- $X = \{v \in H^1(\Omega), \gamma\, v = 0 \text{ a.e. on } \Gamma_1\}$.
  Remind that $\gamma : H^1(\Omega) \to L^2(\Gamma)$ is the trace operator; we send the reader to, e.g., [13] for more details on its properties.
- $S = \{\widetilde{v} \in H^{\frac{1}{2}}(\Gamma) |$ there exists $v \in X$ such that $\widetilde{v} = \gamma\, v$ a.e. on $\Gamma\}$.
  For details on fractional spaces on the boundary, see, e.g., [11, 14, 16].
- S' is the dual of S.

For details on the theory of Hilbert spaces the reader can consult, e.g., [2, 4, 6, 15].

In the present work we focus on the weak solvability of Problem 1 via two-field Lagrange multipliers, under the hypotheses (H1) and (H2). The approach we propose leads us to variational systems which are mixed variational problems equivalent with saddle point problems. The weak solvability of problems in mathematical physics by using formulations with Lagrange multipliers allows to apply modern algorithms in order to efficiently approximate the weak solution; see, e.g., [9] where the primal-dual active strategy is applied.

The structure of this article is the following one. In Section 2 we provide some preliminary material in the saddle point theory, making the paper self-contained and easy to follow. In Section 3 we deliver a weak formulation of Problem 1 in terms of two-field Lagrange multipliers. Then, we discuss the weak solvability of Problem 1 via saddle point techniques. In Section 4 we draw attention to alternative weak formulations and some of their connections.

2 Preliminaries

Let $(\mathcal{X}, (\cdot,\cdot)_{\mathcal{X}}, \|\cdot\|_{\mathcal{X}})$, $(\mathcal{Y}, (\cdot,\cdot)_{\mathcal{Y}}, \|\cdot\|_{\mathcal{Y}})$ be two Hilbert spaces. We consider the following variational system.

Problem 2. Given $f \in \mathcal{X}$, find $u \in \mathcal{X}$ and $\lambda \in \Lambda \subseteq \mathcal{Y}$ such that

$$\hat{a}(u,v) + \hat{b}(v,\lambda) = (f,v)_{\mathcal{X}} \qquad \text{for all } v \in \mathcal{X}, \tag{2.1}$$

$$\hat{b}(u,\mu-\lambda) \leq 0 \qquad \text{for all } \mu \in \Lambda. \tag{2.2}$$

We assume that the following hypotheses hold true:

$(\widehat{a})$ $\hat{a} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is a symmetric, bilinear form such that: (1) there exists $M_{\hat{a}} > 0 : |\hat{a}(u,v)| \leq M_{\hat{a}} \|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}}$ for all $u, v \in \mathcal{X}$; (2) there exists $m_{\hat{a}} > 0 : \hat{a}(v,v) \geq m_{\hat{a}} \|v\|^2_{\mathcal{X}}$ for all $v \in \mathcal{X}$.

$(\widehat{b})$ $\hat{b} : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is a bilinear form such that: there exists $M_{\hat{b}} > 0$: $|\hat{b}(v,\mu)| \leq M_{\hat{b}} \|v\|_{\mathcal{X}} \|\mu\|_{\mathcal{Y}}$ for all $v \in \mathcal{X}$, $\mu \in \mathcal{Y}$.

$(\widehat{\Lambda})$ Λ is a bounded, closed, convex subset of $\mathcal{Y}$ that contains $0_{\mathcal{Y}}$.

According to the literature, see, e.g., [2], the bilinear form $\widehat{a}$ is continuous and $\mathcal{X}-$elliptic and the bilinear form $\widehat{b}$ is continuous.

We recall now a few tools in the saddle point theory that will be helpful in our study. The reader can consult, e.g., [1, 3, 7, 8] for details on the saddle point theory and its applications.

To start, we remind the definition of the saddle point.

Definition 1. [see, e.g., [7], page 166] Let A and B be two non-empty sets. A pair $(u,\lambda) \in A \times B$ is said to be a saddle point of a bifunctional $\mathcal{L} : A \times B \to \mathbb{R}$ if and only if

$$\mathcal{L}(u,\mu) \leq \mathcal{L}(u,\lambda) \leq \mathcal{L}(v,\lambda) \qquad \text{for all } v \in A,\ \mu \in B.$$

Let $\mathcal{L} : A \times B \to \mathbb{R}$ be a bifunctional fulfilling the following hypotheses:

$(\mathcal{L}_1)$ $v \to \mathcal{L}(v,\mu)$ *is convex and lower semi-continuous for all* $\mu \in B$,

$(\mathcal{L}_2)$ $\mu \to \mathcal{L}(v,\mu)$ *is concave and upper semi-continuous for all* $v \in A$,

(A) A *is bounded or* $\lim_{\|v\|_{\mathcal{X}} \to \infty,\, v \in A} \mathcal{L}(v,\mu_0) = \infty$ *for some* $\mu_0 \in B$,

(B) B *is bounded or* $\lim_{\|\mu\|_{\mathcal{Y}} \to \infty,\, \mu \in B} \inf_{v \in A} \mathcal{L}(v,\mu) = -\infty$.

The next theorem is an existence result.

Theorem 1. *[see, e.g., [7], page 176] Let* $A \subseteq \mathcal{X}$, $B \subseteq \mathcal{Y}$ *be non-empty, closed, convex subsets. If the bifunctional* $\mathcal{L} : A \times B \to \mathbb{R}$ *satisfies the hypotheses* $(\mathcal{L}_1)$, $(\mathcal{L}_2)$, (A) *and* (B), *then* $\mathcal{L}$ *has at least one saddle point.*

Proposition 1. *[see, e.g., [7], page 169] Let* $A \subseteq \mathcal{X}$, $B \subseteq \mathcal{Y}$ *be non-empty, closed, convex subsets. If the bifunctional* $\mathcal{L} : A \times B \to \mathbb{R}$ *satisfies the hypotheses* $(\mathcal{L}_1)$, $(\mathcal{L}_2)$, (A) *and* (B), *then the set* $A_0 \times B_0$ *of the saddle points of* $\mathcal{L}$ *is convex, where* $A_0 \subset A$ *and* $B_0 \subset B$. *Moreover,*

a) if $v \to \mathcal{L}(v,\mu)$ is strictly convex for all $\mu \in B$, *then* A_0 *contains at most one point;*

b) if $\mu \to \mathcal{L}(v,\mu)$ is strictly concave for all $v \in A$, *then* B_0 *contains at most one point.*

The results that will be presented below are standard, but we expose here for the convenience of the reader.

Let us associate to Problem 2 the bifunctional $\mathcal{L} : \mathcal{X} \times \Lambda \to \mathbb{R}$ defined as follows,

$$\mathcal{L}(v,\mu) = \frac{1}{2}\,\hat{a}(v,v) + \hat{b}(v,\mu) - (f,v)_{\mathcal{X}} \quad \text{for all } v \in \mathcal{X},\, \mu \in \Lambda. \tag{2.3}$$

Since $\hat{a}$ is a bilinear, symmetric, continuous and $\mathcal{X}-$elliptic form (so, it is a positive form), then $v \to \hat{a}(v,v)$ is strictly convex and lower semi-continuous; see, e.g., Proposition 1.30 in [18]. Using this result, we easily deduce that $\mathcal{L}$ is strictly convex and lower semi-continuous in the first argument. On the other hand, it is obvious that $\mathcal{L}$ is concave and upper semi-continuous in the second argument.

Lemma 1. *The pair $(u,\lambda) \in \mathcal{X} \times \Lambda$ is a solution of Problem 2 if and only if it is a saddle point of the bifunctional $\mathcal{L}$.*

Proof. Let $(u,\lambda) \in \mathcal{X} \times \Lambda$ be a solution of Problem 2. After summing relation (2.2) with $\frac{1}{2}\hat{a}(u,u) - (f,u)_{\mathcal{X}}$, we deduce that

$$\mathcal{L}(u,\mu) \le \mathcal{L}(u,\lambda) \quad \text{for all } \mu \in \Lambda.$$

By relations (2.3) and (2.1), due to the fact that $\hat{a}$ is a symmetric, bilinear and $\mathcal{X}-$elliptic form, we obtain

$$\mathcal{L}(u,\lambda) - \mathcal{L}(v,\lambda) = -\frac{1}{2}\hat{a}(u-v,u-v) \le 0.$$

Therefore, $(u,\lambda) \in \mathcal{X} \times \Lambda$ is a saddle point of the bifunctional $\mathcal{L}$.

We prove now the converse implication. Let $(u,\lambda) \in \mathcal{X} \times \Lambda$ be a saddle point of the bifunctional $\mathcal{L}$. Since

$$\mathcal{L}(u,\mu) \le \mathcal{L}(u,\lambda) \quad \text{for all } \mu \in \Lambda,$$

taking into account the definition of the bifunctional $\mathcal{L}$, it immediately results (2.2). Furthermore,

$$\mathcal{L}(u,\lambda) \le \mathcal{L}(w,\lambda) \text{ for all } w \in \mathcal{X},$$

drives us to

$$\frac{1}{2}\hat{a}(u,u) - \frac{1}{2}\hat{a}(w,w) + \hat{b}(u-w,\lambda) + (f,w-u)_{\mathcal{X}} \le 0 \text{ for all } w \in \mathcal{X}. \tag{2.4}$$

Setting $w = u + t\,v$ with $t > 0$ in this last relation, we have

$$-t\hat{a}(u,v) - \frac{t^2}{2}\hat{a}(v,v) - t\hat{b}(v,\lambda) + t(f,v)_{\mathcal{X}} \le 0 \text{ for all } v \in \mathcal{X}.$$

Dividing by $t > 0$ and then passing to the limit as $t \to 0$, we get

$$\hat{a}(u,v) + \hat{b}(v,\lambda) \ge (f,v)_{\mathcal{X}} \text{ for all } v \in \mathcal{X}. \tag{2.5}$$

Setting now $w = u - t\,v$ with $t > 0$ in relation (2.4) and then dividing by $t > 0$, we get

$$\hat{a}(u,u) - \frac{t}{2}\hat{a}(v,v) + \hat{b}(v,\lambda) - (f,v)_{\mathcal{X}} \leq 0 \ \text{ for all } \ v \in \mathcal{X}.$$

Passing now to the limit when $t \to 0$, we obtain

$$\hat{a}(u,v) + \hat{b}(v,\lambda) \leq (f,v)_{\mathcal{X}} \ \text{ for all } \ v \in \mathcal{X}. \tag{2.6}$$

Combining relations (2.5) and (2.6), we get (2.1). So, $(u,\lambda) \in \mathcal{X} \times \Lambda$ is a solution of Problem 2. □

Consequently, Problem 2 is equivalent with

$$\mathcal{L}(u,\mu) \leq \mathcal{L}(u,\lambda) \leq \mathcal{L}(v,\lambda) \quad \text{for all } v \in \mathcal{X},\ \mu \in \Lambda.$$

This fact allows us to say that Problem 2 is a saddle point problem.

Next we discuss the existence of at least one solution $(u,\lambda) \in \mathcal{X} \times \Lambda$ by using Theorem 1. Also, the uniqueness of $(u,\lambda) \in \mathcal{X} \times \Lambda$ will be investigated.

Theorem 2. *Assume that hypotheses* $(\widehat{a})$, $(\widehat{b})$ *and* $(\widehat{\Lambda})$ *hold true. Then, Problem 2 has a solution which is unique in its first component,* $u \in \mathcal{X}$. *If, in addition, there exists* $\alpha > 0$ *such that*

$$(isp) \qquad \inf_{\mu \in \mathcal{Y}, \mu \neq 0_{\mathcal{Y}}} \ \sup_{v \in \mathcal{X}, v \neq 0_{\mathcal{X}}} \frac{\hat{b}(v,\mu)}{\|v\|_{\mathcal{X}}\ \|\mu\|_{\mathcal{Y}}} \geq \alpha,$$

then the solution is unique in the second component too, $\lambda \in \Lambda$.

Proof. Let $\mathcal{L}$ be the bifunctional defined in (2.3). Following a technique from [8], we are going to apply Theorem 1 in order to conclude that the bifunctional $\mathcal{L}$ has at least one saddle point. As we mentioned before, the bifunctional $\mathcal{L}$ is strictly convex and lower semi-continuous in the first argument and concave and upper semi-continuous in the second one. Thus, $(\mathcal{L}_1)$ and $(\mathcal{L}_2)$ in Theorem 1 are fulfilled. On the other hand, as Λ is a bounded subset of $\mathcal{Y}$ and so condition (B) holds true, it remains to verify if

$$\lim_{\|v\|_{\mathcal{X}} \to \infty,\ v \in \mathcal{X}} \mathcal{L}(v,\mu_0) = \infty \ \textit{ for some } \ \mu_0 \in \Lambda. \tag{2.7}$$

Let $\mu_0 = 0_{\mathcal{Y}}$. We have

$$\mathcal{L}(v,0_{\mathcal{Y}}) = \frac{1}{2}\hat{a}(v,v) - (f,v)_{\mathcal{X}} \geq \frac{m_{\hat{a}}}{2}\,\|v\|^2_{\mathcal{X}} - \|f\|_{\mathcal{X}}\,\|v\|_{\mathcal{X}} \ \text{ for all } \ v \in \mathcal{X}.$$

Passing to the limit as $\|v\|_{\mathcal{X}} \to \infty$ in this last relation, we obtain (2.7). Therefore, Problem 2 has at least one solution $(u,\lambda) \in \mathcal{X} \times \Lambda$.

Let $(u_1,\lambda_1), (u_2,\lambda_2) \in \mathcal{X} \times \Lambda$ be two solutions of Problem 2. Since $v \to \mathcal{L}(v,\mu)$ is strictly convex for all $\mu \in \Lambda$, according to Proposition 1, we deduce that $u_1 = u_2$.

We investigate now the uniqueness in the second argument of the pair solution. We write

$$\hat{a}(u_1, v) + \hat{b}(v, \lambda_1) = (f, v)_{\mathcal{X}} \qquad \text{for all} \;\; v \in \mathcal{X}, \tag{2.8}$$

$$\hat{a}(u_2, v) + \hat{b}(v, \lambda_2) = (f, v)_{\mathcal{X}} \qquad \text{for all} \;\; v \in \mathcal{X}. \tag{2.9}$$

Combining relations (2.8) and (2.9), we obtain

$$\hat{b}(v, \lambda_1 - \lambda_2) = -\hat{a}(u_1 - u_2, v) \;\; \text{for all} \;\; v \in \mathcal{X}.$$

By the inf-sup property of the form $\hat{b}$, i.e. hypothesis (ips), we get

$$\alpha \, \|\lambda_1 - \lambda_2\|_{\mathcal{Y}} \leq 0.$$

Therefore, $\lambda_1 = \lambda_2$. □

3 Weak solutions

Let u and $\boldsymbol{D}$ be regular enough functions verifying Problem 1. By using a Green formula for Sobolev spaces, see, e.g., [18], page 90, the following relation holds true.

$$\int_{\Omega} \beta \, \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) \, dx + \int_{\Gamma_3} \boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) \, \gamma \, v(\boldsymbol{x}) \, d\Gamma + \int_{\Gamma_2} f_2(\boldsymbol{x}) \, \gamma \, v(\boldsymbol{x}) \, d\Gamma$$
$$- \int_{\Omega} (\boldsymbol{D}(\boldsymbol{x}) + \beta \, \nabla u(\boldsymbol{x})) \cdot \nabla v(\boldsymbol{x}) \, dx = \int_{\Omega} f_0(\boldsymbol{x}) \, v(\boldsymbol{x}) \, dx \qquad \text{for all } v \in X.$$

To proceed, we consider the following bilinear forms,

$$a : X \times X \to \mathbb{R}, \quad a(u, v) = \int_{\Omega} \beta \, \nabla \, u(\boldsymbol{x}) \cdot \nabla \, v(\boldsymbol{x}) \, dx,$$

$$b_1 : X \times X \to \mathbb{R}, \quad b_1(v, \mu) = (\mu, v)_X, \tag{3.1}$$

$$b_2 : X \times S' \to \mathbb{R}, \quad b_2(v, \xi) = \langle \xi, \gamma v \rangle. \tag{3.2}$$

Here and everywhere below, $\langle \; , \; \rangle$ denotes the duality pairing between S' and S. Subsequently, we define $f \in X$ by means of the Riesz's representation theorem,

$$(f, v)_X = \int_{\Omega} f_0 \, (\boldsymbol{x}) \, v(\boldsymbol{x}) \, dx - \int_{\Gamma_2} f_2 \, (\boldsymbol{x}) \gamma v \, (\boldsymbol{x}) d\Gamma \;\; \text{for all} \;\; v \in X.$$

Afterwards, we introduce a two-field Lagrange multiplier $\overline{\lambda} = (\lambda_{\Omega}, \lambda_{\Gamma_3}) \in X \times S'$ where λ_{Ω} and λ_{Γ_3} are defined as follows:

$$(\lambda_{\Omega}, z)_X \;\; = \;\; - \int_{\Omega} (\boldsymbol{D}(\boldsymbol{x}) + \beta \, \nabla u(\boldsymbol{x})) \cdot \nabla \, z(\boldsymbol{x}) \, dx \text{ for all } \; z \in X; \tag{3.3}$$

$$\langle \lambda_{\Gamma_3}, \widetilde{y} \rangle \;\; = \;\; \int_{\Gamma_3} \boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) \, \widetilde{y}(\boldsymbol{x}) \, d\Gamma \text{ for all } \; \widetilde{y} \in S. \tag{3.4}$$

Therefore, we can write the following variational equation:

$$a(u, v) + b_1(v, \lambda_{\Omega}) + b_2(v, \lambda_{\Gamma_3}) = (f, \, v)_X \;\; \text{for all} \;\; v \in X. \tag{3.5}$$

Let us define now a new form $\bar{b}$, by taking into account the forms b_1 and b_2:

$$\bar{b} : X \times (X \times S') \to \mathbb{R}, \quad \bar{b}(v, \bar{\lambda}) = b_1(v, \lambda_\Omega) + b_2(v, \lambda_{\Gamma_3}), \tag{3.6}$$

where $\bar{\lambda} = (\lambda_\Omega, \lambda_{\Gamma_3}) \in X \times S'$. Consequently, we can write (3.5) as follows,

$$a(u, v) + \bar{b}(v, \bar{\lambda}) = (f, v)_X \;\text{ for all }\; v \in X. \tag{3.7}$$

Next we introduce the set of Lagrange multipliers

$$\bar{\Lambda} = \Lambda_\varphi \times \Lambda_\psi, \tag{3.8}$$

where

$$\Lambda_\varphi = \left\{ \mu_\Omega \in X : (\mu_\Omega, v)_X \le \int_\Omega \varphi(\nabla v(\boldsymbol{x}))\, dx \;\text{ for all }\; v \in X \right\}, \tag{3.9}$$

$$\Lambda_\psi = \left\{ \mu_{\Gamma_3} \in S' : \langle \mu_{\Gamma_3}, \widetilde{w} \rangle \le \int_{\Gamma_3} \psi(\widetilde{w}(\boldsymbol{x}))\, d\Gamma \;\text{ for all }\; \widetilde{w} \in S \right\}. \tag{3.10}$$

Let us prove that $\lambda_\Omega \in \Lambda_\varphi$. Indeed, let $\boldsymbol{x} \in \Omega$. By (1.1) we can write for all $\boldsymbol{w} \in \mathbb{R}^N$,

$$\varphi(\boldsymbol{w}) - \varphi(\nabla u(\boldsymbol{x})) \ge -(\boldsymbol{D}(\boldsymbol{x}) + \beta \nabla u(\boldsymbol{x})) \cdot (\boldsymbol{w} - \nabla u(\boldsymbol{x})). \tag{3.11}$$

Next, we set $\boldsymbol{w} = \nabla v(\boldsymbol{x}) + \nabla u(\boldsymbol{x})$ in (3.11). Keeping in mind (3.3), the conclusion is immediately obtained.

Subsequently, we prove that $\lambda_{\Gamma_3} \in \Lambda_\psi$. To this end in view, let us fix an arbitrary $\boldsymbol{x} \in \Gamma_3$. Taking into account (1.2), we have,

$$\psi(r) - \psi(u(\boldsymbol{x})) \ge \boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x})(r - u(\boldsymbol{x})) \;\text{ for all } r \in \mathbb{R}. \tag{3.12}$$

By setting $r = w(\boldsymbol{x}) + u(\boldsymbol{x})$ in (3.12), keeping in mind (3.4) we easily obtain that $\lambda_{\Gamma_3} \in \Lambda_\psi$.

Furthermore, by (3.11) and the definitions of λ_Ω and Λ_φ we deduce that

$$b_1(u, \mu_\Omega - \lambda_\Omega) \le 0 \quad \text{ for all } \mu_\Omega \in \Lambda_\varphi. \tag{3.13}$$

Also, according to (3.12) and the definitions of λ_{Γ_3} and Λ_ψ, we have

$$b_2(u, \mu_{\Gamma_3} - \lambda_{\Gamma_3}) \le 0 \quad \text{ for all } \mu_{\Gamma_3} \in \Lambda_\psi. \tag{3.14}$$

Summing (3.13) and (3.14) we obtain

$$\bar{b}(u, \bar{\mu} - \bar{\lambda}) \le 0 \;\text{ for all } \bar{\mu} \in \bar{\Lambda}, \tag{3.15}$$

where $\bar{\mu} = (\mu_\Omega, \mu_{\Gamma_3})$ and $\bar{\lambda} = (\lambda_\Omega, \lambda_{\Gamma_3})$.

Thus, we can write the following variational formulation for Problem 1.

Problem 3. Find $u \in X$ and $\bar{\lambda} \in \bar{\Lambda} \subseteq X \times S'$ such that (3.7) and (3.15) hold true.

Any solution of Problem 3 is called weak solution of Problem 1.

We are going to prove the following existence and uniqueness result.

Theorem 3. *Under the hypotheses* $(H1)$ *and* $(H2)$*, Problem 1 has a weak solution* $(u,\overline{\lambda}) \in X \times \bar{\Lambda}$*, unique in its first component.*

Proof. We apply Theorem 2 by setting $\mathcal{X} = X$, $\mathcal{Y} = X \times S'$ $\Lambda = \bar{\Lambda}$, $\hat{a} = a$, $\hat{b} = \overline{b}$. Obviously, the hypotheses (â) and ($\hat{\text{b}}$) are verified. On the other hand, as Λ_φ defined in (3.9) and Λ_ψ defined in (3.10) are nonempty, closed, convex and bounded subsets, then the set $\bar{\Lambda} = \Lambda_\varphi \times \Lambda_\psi$ is a nonempty, closed, convex and bounded subset of $X \times S'$. □

Remark 1. Notice that

$$\|\mu_\Omega\|_X = \sup_{v_1 \in X,\ v_1 \neq 0_X} \frac{b_1(v_1,\mu_\Omega)}{\|v_1\|_X},$$
$$\|\mu_{\Gamma_3}\|_{S'} = \sup_{\gamma\, w \in S;\, \gamma\, w \neq 0_S} \frac{\langle \mu_{\Gamma_3}, \gamma\, w\rangle}{\|\gamma\, w\|_{H^{\frac{1}{2}}(\Gamma)}} \leq c_\ell \sup_{v_2 \in X,\, v_2 \neq 0_X} \frac{b_2(v_2,\mu_{\Gamma_3})}{\|v_2\|_X},$$

where $c_\ell > 0$. So, the forms b_1 and b_2 fulfill the inf-sup property.

However, the form $\overline{b}$ doesn't fulfill the inf-sup property. Indeed, let $\zeta_0 \in S'$, $\zeta_0 \neq 0_{S'}$. As $X \ni v \to \langle -\zeta_0, \gamma v\rangle$ is a linear and continuous form, then there exists a unique $\mu_0 \in X$ such that

$$\langle -\zeta_0, \gamma v\rangle = (\mu_0, v)_X \quad \text{for all } v \in X.$$

Let us define $\bar{\mu}_0 = (\mu_0, \zeta_0) \in X \times S'$. Clearly, $\|\bar{\mu}_0\|_{X\times S'} > 0$ because $\zeta_0 \neq 0_{S'}$. On the other hand,

$$\overline{b}(v,\bar{\mu}_0) = b_1(v,\mu_0) + b_2(v,\zeta_0) = (\mu_0, v)_X + \langle \zeta_0, \gamma v\rangle = 0 \quad \text{for all } v \in X.$$

Hence,

$$\sup_{v\in X,\, v\neq 0_X} \frac{\overline{b}(v,\bar{\mu}_0)}{\|v\|_X} = 0.$$

Since $\|\bar{\mu}_0\|_{X\times S'} > 0$,

$$\text{for all } \alpha > 0, \quad \sup_{v\in X,\, v\neq 0_X} \frac{\overline{b}(v,\bar{\mu}_0)}{\|v\|_X} < \alpha \|\bar{\mu}_0\|_{X\times S'}.$$

Thus,

$$\text{for all } \alpha > 0, \quad \sup_{v\in X,\, v\neq 0_X} \frac{\overline{b}(v,\bar{\mu}_0)}{\|v\|_X \|\bar{\mu}_0\|_{X\times S'}} < \alpha.$$

As a result,

$$\text{for all } \alpha > 0, \quad \inf_{\bar{\mu}\in X\times S',\, \bar{\mu}\neq 0_{X\times S'}} \sup_{v\in X,\, v\neq 0_X} \frac{\overline{b}(v,\bar{\mu})}{\|v\|_X \|\bar{\mu}\|_{X\times S'}} < \alpha.$$

Consequently, $\overline{b}$ doesn't fulfill the inf-sup property. The uniqueness of the solution of Problem 3 in its second component remains open.

4 Alternative variational formulations and some of their connections

By standard arguments in the calculus of variations, Problem 1 leads us to the following primal variational formulation.

Problem 4. Find $u_0 \in X$ such that

$$a(u_0, v - u_0) + J(v) - J(u_0) \geq (f, v - u_0)_X \quad \text{for all } v \in X, \tag{4.1}$$

where

$$a : X \times X \to \mathbb{R}, \quad a(u, v) = \int_\Omega \beta \, \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) \, dx, \tag{4.2}$$

$$J : X \to \mathbb{R}, \quad J(v) = \int_\Omega \varphi(\nabla v(\boldsymbol{x})) \, dx + \int_{\Gamma_3} \psi(\gamma \, v(\boldsymbol{x})) \, d\Gamma, \tag{4.3}$$

$$(f, v)_X = \int_\Omega f_0 \, (\boldsymbol{x}) \, v(\boldsymbol{x}) \, dx - \int_{\Gamma_2} f_2 \, (\boldsymbol{x}) \gamma v \, (\boldsymbol{x}) d\Gamma \quad \text{for all } v \in X. \tag{4.4}$$

According to the theory of variational inequalities of the second kind, see, e.g., [17, 18], as a is a bilinear, symmetric, continuous, X-elliptic form and J is a proper, convex and lower semicontinuous functional, then Problem 4 has a unique solution $u_0 \in X$.

Proposition 2. *If $u_0 \in X$ is the unique solution of Problem 4, then*

$$a(u_0, v) + J(v) \geq (f, v)_X \qquad \text{for all } v \in X. \tag{4.5}$$

Proof. Let us set successively $v = 0_X$ and $v = 2u_0$ in (4.1). As a result,

$$a(u_0, u_0) + J(u_0) = (f, u_0)_X.$$

Using this last identity and (4.1) we are driven to (4.5). □

On the other hand, according to [5], we can introduce the single-field Lagrange multiplier

$$(\lambda_1, z)_{X', X} = -\int_\Omega (\boldsymbol{D}(\boldsymbol{x}) + \beta \, \nabla u(\boldsymbol{x})) \cdot \nabla \, z(\boldsymbol{x}) \, dx + \int_{\Gamma_3} \boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) \, \gamma z(\boldsymbol{x}) \, d\Gamma;$$

herein, by X' we denote the dual of the space X.

Thus, Problem 1 drives us to the following weak formulation,

Problem 5. Find $u_1 \in X$ and $\lambda_1 \in \Lambda_1 \subseteq X'$ such that

$$a(u_1, v) + \widetilde{b}(v, \lambda_1) = (f, \, v)_X \qquad \text{for all } v \in X, \tag{4.6}$$
$$\widetilde{b}(u_1, \mu - \lambda_1) \leq 0 \qquad \text{for all } \mu \in \Lambda_1,$$

where

$$\widetilde{b} : X \times X' \to \mathbb{R}, \quad \widetilde{b}(v, \mu) = (\mu, v)_{X',X}, \tag{4.7}$$

$$\Lambda_1 = \{\mu \in X' : (\mu, v)_{X',X} \leq J(v) \text{ for all } v \in X\} \tag{4.8}$$

and a, f and J were defined in (4.2), (4.4) and (4.3), respectively. According to Theorem 2 in [5], Problem 5 has a unique solution $(u_1, \lambda_1) \in X \times X'$.

Proposition 3. *If* $(u_1, \lambda_1) \in X \times \Lambda_1$ *is the unique solution of Problem 5, then*

$$a(u_1, v) + J(v) \geq (f, v)_X \quad \text{for all } v \in X. \tag{4.9}$$

Proof. Keeping in mind (4.7) and (4.8), by (4.6) we immediately get (4.9). □

Finally, we pay attention to the following proposition.

Proposition 4. *If* $(u, \bar{\lambda}) \in X \times \bar{\Lambda}$ *is the unique solution of Problem 3 and* a, f *and* J *are that defined in (4.2), (4.4) and (4.3), then*

$$a(u, v) + J(v) \geq (f, v)_X \quad \text{for all } v \in X. \tag{4.10}$$

Proof. Keeping in mind (3.6), (3.1), (3.2) and (3.8), (3.9), (3.10), by (3.5) we immediately get (4.10). □

According to Propositions 2, 3, 4, the unique solution of Problem 4, the first component of the unique solution of Problem 5, as well as the first component of each pair solution of Problem 3, verify a common inequality.

In the particular case $N = 2$, $\varphi \equiv 0$ and $\psi : \mathbb{R} \to [0, \infty)$, $\psi(r) = g|r|$, where g is a positive constant, Problem 1 reduces to the following boundary value problem.

$$\begin{aligned}
-\beta \triangle u(\boldsymbol{x}) &= f_0(\boldsymbol{x}) && \text{in } \Omega, \\
u(\boldsymbol{x}) &= 0 && \text{on } \Gamma_1, \\
-\beta \frac{\partial u}{\partial \nu}(\boldsymbol{x}) &= f_2(\boldsymbol{x}) && \text{on } \Gamma_2, \\
\left|\beta \frac{\partial u}{\partial \nu}(\boldsymbol{x})\right| &\leq g, \quad \beta \frac{\partial u}{\partial \nu}(\boldsymbol{x}) = -g \frac{u(\boldsymbol{x})}{|u(\boldsymbol{x})|} && \text{if } u(\boldsymbol{x}) \neq 0, \qquad \text{on } \Gamma_3.
\end{aligned}$$

In this particular case, according to [12], the unique solution of Problem 4 coincides with the first component of the unique solution of Problem 5 as well as with the first component of each solution of Problem 3.

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