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An Effective Computational Approach Based on Hermite Wavelet Galerkin for Solving Parabolic Volterra Partial Integro Differential Equations and its Convergence Analysis

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Abstract. In this research article Hermite wavelet based Galerkin method is developed for the numerical solution of Volterra integro-differential equations in one-dimension with initial and boundary conditions. These equations include the partial differential of an unknown function and the integral term containing the unknown function which is the memory of the problem. Wavelet analysis is a recently developed mathematical tool in applied mathematics. For this purpose, Hermit wavelet Galerkin method has proven a very powerful numerical technique for the stable and accurate solution of giving boundary value problem. The theorem of convergence analysis and compare some numerical examples with the use of the proposed method and the exact solutions shows the efficiency and high accuracy of the proposed method. Several figures are plotted to establish the error analysis of the approach presented.

Keywords: Volterra partial integro-differential equation, Hermite wavelet, Galerkin method.

AMS Subject Classification: 65R20; 45K99.

1 Introduction

The orthogonal basis functions can be truncated into series for the solutions of the physical or mathematical problems. Mainly, there are three classes for orthogonal basis functions. Firstly, the piecewise constant functions including the block pulse and Walsh functions and related ones. Secondly, the family of sine-cosine functions. Lastly, the family of orthogonal polynomials like Hermite,

Laguerre, and Chebyshev. The construction of wavelets based on orthogonal polynomials is more recent. Since it is easy to generate an orthonormal basis of $L^2(\mathbb{R})$ with help of orthogonal polynomials, so wavelets based on orthogonal polynomial work well. There are a few orthogonal wavelet methods that can provide a tangible level of accuracy which are considered by researchers. The Hermite polynomial-based wavelet techniques can offer a better solution because they reduce the computational cost at a tangible level and provide a better rate of accuracy [22].

Wavelets have found their way into many different fields of science and engineering. They have been used in a broad scope of technology disciplines; particularly, signal analysis for waveform representation and segmentations, time-frequency analysis and fast algorithm for easy execution [13, 14, 15]. In general results in theoretical problems are as partial differential equations, integral and integro-differential equations. Partial differential equations provide an important tool for modeling the numerous problems in various fields of engineering and physics and systems included the space and time variables. When we must consider the effects of the "memory" of the system, the modeling leads to the integral term containing the unknown function.Therefore, a partial integro-differential equation (PIDE) is obtained, which combined the partial differentiations and integral terms.

Integro-differential equations are normally hard to solve analytically so it is needed to get an efficient approximate solution. This necessitates either discretization of partial differential equations leading to numerical solutions or their qualitative study which is concerned with a deduction of important properties of the solutions without actually solving them. Partial differential equations have many different types such as elliptic, hyperbolic and parabolic in one or multi-dimension. The main purpose of this theme is to draw an approximate solution for a parabolic Volterra partial integro-differential equation (VPIDE). This paper deals with the following nonlinear partial integro-differential equations for the one-dimension (1D):

$$u_t(x,t) = \mu\nabla^2 u + \int_0^t k(x,t,s,u(x,s))ds + f(x,t), \qquad x \in I \subset \mathbb{R},\ t \in J, \quad (1.1)$$

where $\mu > 0$, $u_t = \partial u/\partial t$, $\nabla^2 u = \partial^2 u/\partial x^2$. The kernel k, function f and the unknown real function $u(x,t)$ are continuous on $\Omega = \{(x,t) \in \mathbb{R}^2 : x \in I, t \in J\}$ and $I \times S \times \mathbb{R}$, respectively, with $I = [a,b]$, $J = [0,T]$ and $S = \{(t,s) \in J \times J : s \leq t\}$. We assume that the function k is continuously differentiable and therefore satisfies a Lipschitz condition with respect to its last variable, i.e., there exists a constant $L \geq 0$ such that for all $u, v \in \mathbb{R}$

$$|k(x,t,s,u) - k(x,t,s,v)| \leq L\,|u - v|$$

along with the initial value (IV) condition

$$u(x,0) = u^0(x), \qquad x \in I, \quad (1.2)$$

and the boundary value (BV) conditions

$$u(a,t) = h(t),\ u(b,t) = l(t), \quad t \in J. \quad (1.3)$$

This class of equations is applied in many different field of engineering and physics science, e.g. geophysics [27], reaction-diffusion problems [24], biofluids flow in fractured biomaterials [28], nuclear reactor dynamics [17], financial modelling [1], pricing of derivatives in finance [19] and electricity swaptions [7].

Numerical solution of partial integro-differential equations has recently drawn much attention of researchers. There are various methods for solving several classes of PIDEs, finite element methods [26], finite difference methods [6], spline collocation methods [25], wavelet collocation method [13].

The Galerkin method is very powerful tools for solving many kinds of differential and integral equations arising in several areas of scientific discipline [3,18]. Collocation method involves numerical operators acting on point values (collocation points) in the physical space. Generally, wavelet collocation methods are created by choosing a wavelet and some kind of grid structure which will be computationally adapted. The treatment of nonlinearities in wavelet collocation method is a straight forward task due to collocation nature of algorithm as compared to wavelet-Galerkin method. Moreover, implementation are more practical with Galerkin methods. The wavelet Galerkin method is one of the best known methods for obtaining numerical solutions to integro-differential equations [16]. The approximations of PIDE with wavelet basis are more attentive, because the reality of orthogonality of compactly supported wavelets. The concepts of multiresolution analysis based have given great momentum to make wavelet best approximations of ODE's and PDE's [11, 20].

The basis functions may be global polynomials, piecewise polynomials, trigonometric polynomials or other functions. The basis functions used for the spectral method are mutually orthogonal. Using orthogonal basis functions tends to reduce the effects of rounding errors that occur when computing the approximation. Hermite wavelets have been widely applied in numerical solution of differential equations [22], boundary value problems [2], singular initial value problems [21], integral equations [12], integro-differential equations [9].

In this paper, we apply discrete Hermite wavelet Galerkin method (HWGM) to solve VPIDE with IV and BV conditions using global wavelet basis functions. The properties of Hermite wavelets together with the Galerkin method are utilized to evaluate the unknown coefficients and then a numerical solution of the partial integro-differential equations is obtained. This paper is organized as follows. In Section 2, we give a brief introduction to the Hermite wavelets. Based on Hermit wavelet, we develop the fully discrete scheme and numerical algorithms with Galerkin method to solve PIDE in Section 3. In Section 4, we discuss the convergence analysis of the proposed method. Finally, in Section 5, numerical experiments are solved to verify the accuracy and efficiency of the proposed approach. This paper ends with a conclusion.

2 Hermite wavelet

Wavelets constitute a family of functions constructed from dilation and translation of a single function called mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following

family of continuous wavelets as:

$$\psi_{a,b}(x) = |a|^{-\frac{1}{2}}\psi(\frac{x-b}{a}), \qquad a \neq 0, \qquad a, b \in \mathbb{R}.$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-j}, b = kb_0a_0^{-j}, a_0 > 1, b_0 > 0$ and j, k are positive integers, we have the following family of discrete wavelets:

$$\psi_{j,k}(x) = |a_0|^{-\frac{j}{2}}\psi(a_0^j x - kb_0),$$

where $\psi_{j,k}(x)$ forms a wavelet basis for $L^2(\mathbb{R})$. In particular, when $a_0 = 2$ and $b_0 = 1$, then $\psi_{j,k}(x)$ form an orthonormal basis. Hermit wavelets are defined as [2, 21]:

$$\psi_{k,m} = \begin{cases} 2^{\frac{j}{2}}\tilde{H}_m(2^j x - 2k + 1), & \frac{k-1}{2^{j-1}} \leq x \leq \frac{k}{2^{j-1}}, \\ 0, & \text{else}, \end{cases} \tag{2.1}$$

$$\tilde{H}_m = \sqrt{\frac{2}{\pi}}H_m(x),$$

where $m = 0, 1, ..., M-1$. In Equation (2.1), the coefficients are used for orthonormality. Here $H_m(x)$ are the second Hermite polynomials of degree m with respect to weight function $w(x) = (1-x)^\alpha(1+x)^\beta$ for $\alpha = \beta = \frac{1}{2}$ then $w(x) = \sqrt{1-x^2}$ on the real line $\mathbb{R}$ and satisfies the following formula $H_0(x) = 1, H_1(x) = 2x$,

$$H_{m+2}(x) = 2xH_{m+1}(x) - 2(m+1)H_m(x).$$

For $j = 1, k = 1$ in Equation (2.1) the Hermite wavelets are given by

$$\begin{aligned}
\psi_{1,0}(x) =& \frac{2}{\sqrt{\pi}}, \\
\psi_{1,1}(x) =& \frac{2}{\sqrt{\pi}}(4x-2), \\
\psi_{1,2}(x) =& \frac{2}{\sqrt{\pi}}(16x^2 - 16x + 2), \\
\psi_{1,3}(x) =& \frac{2}{\sqrt{\pi}}(64x^3 - 96x^2 + 36x - 2), \\
\psi_{1,4}(x) =& \frac{2}{\sqrt{\pi}}(256x^4 - 512x^3 + 320x^2 - 64x + 2),
\end{aligned}$$

and so on.

2.1 Function approximation

Here we approximating the solution function $u(x)$ of VPIDE by using Hermite wavelet basis as follows:

$$u(x) = \sum_{k=1}^{\infty}\sum_{m=0}^{\infty} c_{k,m}\psi_{k,m}(x), \tag{2.2}$$

where $\psi_{k,m}$ are given in Equation (2.1) and

$$c_{k,m} = \langle u(x), \psi_{k,m}(x) \rangle .$$

We approximate $u(x)$ by truncating the series represented, which is involving unknown parameter as follows:

$$u^n(x) = \sum_{k=1}^{2^{j-1}} \sum_{m=0}^{M-1} c_{k,m} \psi_{k,m}(x) = C^T \Psi(x), \tag{2.3}$$

where C and $\Psi(x)$ are $2^{j-1}M \times 1$ matrix and $c_{k,m}$ are unknown coefficients to be determined:

$$\Psi(x) = [\psi_{1,0}, ..., \psi_{1,M-1}, \psi_{2,0}, ..., \psi_{2,M-1}, \psi_{2^{j-1},0}, ..., \psi_{2^{j-1},M-1}]^T,$$
$$C = [c_{1,0}, ..., c_{1,M-1}, c_{2,0}, ..., c_{2,M-1}, c_{2^{j-1},0}, ..., c_{2^{j-1},M-1}]^T.$$

3 Numerical algorithms

3.1 Discretization in time

Now, the forward Euler method is applied for discretize time derivatives to the θ-weighted scheme in Equation (1.1). Let $t_n = n\Delta t$ with Δt being the time step. If $u^n(x)$ be the approximation to the value of $u^n = u(x, t_n)$ at the time point $t = t_n$ and $f^n = f(x, t_n)$ for $n = 0, 1, ..., N$, $N = \left[\frac{T}{k}\right], k \in \mathbb{N}$. We will get:

$$u_t(x, t_n) \approx \frac{u^{n+1}(x) - u^n(x)}{\Delta t}, \tag{3.1}$$

and

$$\nabla^2 u(x, t) \approx [\theta \nabla^2 u^{n+1} + (1 - \theta)\nabla^2 u^n], \tag{3.2}$$

substituting Equations (3.1) and (3.2) into Equation (1.1)

$$\begin{aligned} u^{n+1} - u^n =& \Delta t \mu \theta \nabla^2 u^{n+1} + \Delta t \mu (1 - \theta)\nabla^2 u^n \\ &+ \Delta t \int_0^{t_n} k(x, t_n, s, u(x, s))ds + \Delta t f^n. \end{aligned} \tag{3.3}$$

3.2 Discretization in space

We discretize the spatial-derivative in Equation (1.1) by the described HWGM based numerical method in Section 2. We consider the uniform grid $0 \le x_i \le 1$, let $\Delta x = 1/i$ denote the special step. We denote a grid point (x_i, t_n) by $t_n = n\Delta t$, $x_i = i\Delta x$, u_i^n is an approximation to the value of $u(x, t)$ with $i = 0, ..., n$, $n \ge 0$.

According to the numerical integration composite trapezoidal rule we have:

$$\begin{aligned} \int_0^{t_n} k(x, t_n, s, u(x, s))ds \approx \frac{t_n}{2n} [k(x, t_n, s_0, u(x, s_0)) \\ + k(x, t_n, s_n, u(x, s_n)) + 2 \sum_{i=1}^{n-1} k(x, t_n, s_i, u(x, s_i))]. \end{aligned} \tag{3.4}$$

We emphasize that the chosen nodes $s_i,\ i = 0, ..., n$, in the integration composite trapezoidal rule are coincided with t_n in Equation (3.3). So, we substitute Equation (3.4) into Equation (3.3) and obtain:

$$u^{n+1} - u^n \approx \Delta t\mu\theta\nabla^2 u^{n+1} + \Delta t\mu(1-\theta)\nabla^2 u^n + \Delta t\frac{t_n}{2n}\Big[k(x,t_n,t_0,u(x,t_0)) + k(x,t_n,t_n,u(x,t_n)) + 2\sum_{i=1}^{n-1} k(x,t_n,t_i,u(x,t_i))\Big] + \Delta t f_n,$$

where $t_0 = 0,\ u(x,t_0) = u^0(x)$. Now we approximate $u^n(x)$ by HWGM as follows:

$$\widehat{u}^{n+1} \approx \widehat{u}^n + \Delta t\left[\mu\theta\nabla^2\widehat{u}^{n+1} + \mu(1-\theta)\nabla^2\widehat{u}^n + f_n\right] + \Delta t\frac{t_n}{2n}\left[k(x,t_n,\widehat{u}^0) + k(x,t_n,\widehat{u}^n) + 2\sum_{i=1}^{n-1} k(x,t_i,\widehat{u}^i)\right], \tag{3.5}$$

$\hat{u}^{n+1}$ can be computed by the obtained recursive Equation (3.5) for $n = 0, 1, ...$ So, as the first step we can be obtained by the initial condition and

$$\widehat{u}^n(x) = \sum_{k=1}^{2^{j-1}}\sum_{m=0}^{M-1} c_{k,m}\psi_{k,m}(x) = C^T\Psi(x), \tag{3.6}$$

$$f_n(x) = \Psi^T F, \quad k(x,t,s,u(x,s)) = \Psi^T K\Psi. \tag{3.7}$$

We discrete $\nabla^2\widehat{u}^n$ as follows:

$$\nabla^2\hat{u}^n(x) = \sum_{k=1}^{2^{j-1}}\sum_{m=0}^{M-1} c_{k,m}\frac{\partial^2}{\partial x^2}\psi_{k,m}(x), \tag{3.8}$$

which can be rewritten as

$$[\nabla^2\hat{u}^n] = D[c_k]^n,$$

where

$$D = [\psi''_1, \psi''_{-3}, ..., \psi''_{2^j-1}]^T. \tag{3.9}$$

and the substitution of Equations (3.6)–(3.9) into the Equation (3.5) gives

$$\hat{u}^{n+1} \approx \hat{u}^n + \mu_1 D\hat{u}^{n+1} + \mu_2 D\hat{u}^n + F + \mu_3 K,$$

where $\mu_1 = \Delta t\mu\theta$, $\mu_2 = \Delta t\mu(1-\theta)$ and $\mu_3 = \Delta t\frac{t_n}{2n}$.

3.3 Galerkin wavelet approximation

For numerical solving of Equation (1.1) we should choose a finite dimension family of functions which the exact solution can be estimated by them. The methods which apply this strategy are called projection methods, because the exact solution of an equation is projected on the space with finite dimensions. Consider the projection approximation $u^n(x)$ is used to solve the PIDE problem

of form $L(u) = f$ with wavelet base function $\psi_i(x), i = 0, 1, , M-1$. The Galerkin idea is to require the residual:

$$R_n(x) = L(u^n(x)) - f(x),$$

since $u^n(x)$ is not an exact solution of PIDE problem, R_n will never be exactly zero. All the same, we can minimize it with regard to a set of weight functions. The method minimizes the residual with respect to the set of approximating functions by requiring $\langle R_n, \psi_i \rangle = 0$ using the inner product definition

$$\int_{-1}^{1} w(x) R_n(x) \psi_i(x) dx = 0, \quad i = 0, 1, .., M-1$$

with $w(x)$ is the weight function corresponds to the base functions $\psi_i(x)$. The Galerkin method deals with the solution of the approximate problem Equations (1.1)–(1.3) in the form of a finite series and substituting Equations (3.6)–(3.8) in Equation (3.5) leads to:

$$\begin{aligned} \Psi^T(x) C^{n+1} \approx & \Psi^T(x) C^n + \Delta t(\mu\theta \Psi^T(x) D C^{n+1} \\ & + \mu(1-\theta)\Psi^T(x) D C^n + \Psi^T(x) F) + \Delta t \frac{t_n}{2n} (\Psi^T K \Psi). \end{aligned} \tag{3.10}$$

We can now give a computational procedure for computing Equation (3.10) using wavelet compression

Algorithm 1. (i) $c^0 \leftarrow$ compute initial guess in wavelet basis

(ii) $R_0 \leftarrow trunc \left\langle c^0, \psi \right\rangle$

(iii) for $i = 0$ to $n+1$ do

(iv) $R_1 \leftarrow \left\langle Dc^i, \psi \right\rangle$

(v) end for

(vi) $R_2 \leftarrow \langle F, \psi \rangle$

(vii) $R_3 \leftarrow \langle K, \psi \rangle$

(viii) for $i = 0$ to $n+1$ do

(ix) $c^{i+1} \leftarrow R_0 + \mu_1 R_1 + \mu_2 R_1 + R_2 + \mu_3 R_3$

(x) end for

(xi) $c^{n+1} \leftarrow trunc \left\langle c^{n+1}, \psi \right\rangle$

(xii) $\hat{u}^{n+1} \leftarrow assign \left\{ c^1, c^2, .., c^{n+1} \right\}$

In the wavelet Galerkin method, to determine the coefficients $\left\{ c^1, c^2, .., c^{n+1} \right\}$, we replace the expansion Equation (3.6) with $\hat{u}^{n+1}$ and taking the inner product $\langle ., \psi_i \rangle$ on the domain Ω upon both sides.

In general, this leads to implicit methods where a linear system has to be solved for each time step. For the differential operator in one dimension, the matrices to be inverted in each time-step are banded and can be factored in $O(N)$ operations. If the differential operator is nonlocal, however, standard Galerkin discretizations of $\widehat{u}^n$ with N degrees of freedom entail dense stiffness matrices and hence at least $O(N^2)$ complexity per time step for the numerical solution of Equation (1.1). We reduce this complexity by a wavelet-based matrix compression. The basic idea behind this compression is to represent the Galerkin approximation $\widehat{u}^n$ of Equation (1.1) in a wavelet basis.

4 Convergence analysis

Let $u^n(x,t)$ be the exact solution of VPIDE Equation (1.1), that is, the solution of giving equations at the n-th time level. Also, we assume that u_h is the approximate solution in Sobolev space $H^m(\Omega)$ of m derivative order. For existence and uniqueness of a weakly solution based on assumptions of minimal regularity, see [10, 29]. The convergence of the expansion of the Hermite wavelets in Equation (2.2) and the estimate of the error of the truncated series in Equation (2.3) are analyzed by the following theorem.

Lemma 1. *(see [23])Let $v \in H_0^1(\Omega)$ and projection operator $P_h : L^2(\Omega) \to V_h$ that V_h be set of wavelet polynomial functions. Then, there exists a constant $C > 0$, independent of h, k, such that*

$$\|P_h v - v\|_{L^2(\Omega)} + h\|P_h v - v\|_{H^1(\Omega)} \leq Ch^2(\|v\|_{H^2} + \|v\|_{H^2}).$$

We get for basis function $\psi \in V_h$:

$$\langle P_h v, \psi\rangle = \langle v, \psi\rangle\,,$$

or

$$\|P_h v\|_{H^1} \leq C\|v\|_{H^1}.$$

Theorem 1. *Let u_h be an approximate solution of VPIDE (1.1) with the initial value condition $u(x,0) = u^0(x)$ and $u^0 \in H_0^1$, $f \in H^2(L^2(\Omega))$, $\Delta x = h = \frac{1}{i}$, $\Delta t = k = \frac{t}{n}$. Then, there exists a constant $C > 0$, independent of h, k, such that*

$$\|u^n - u_h\|_{L^2} \leq C(u_0, f, u, u_t, u_{tt})(h + k).$$

Proof.

If $u(x,t) = u^n = u(t_n)$ be exact solution and u_h be an approximate solution of VPIDE (1.1) by Galerkin method with basis function v_h and $Au = \nabla^2(u) = \frac{\partial}{\partial x}(\frac{\partial u}{\partial x})$ so:

$$\langle u_t^n, v_h\rangle \approx A\,\langle u^n, v_h\rangle + \int_0^{t_n} \langle k(t_n, s, u(s)), v_h\rangle\, ds + \langle f^n, v_h\rangle,$$

that $v_h = P_h u^n$ so $e^n = u^n - u_h$ and $E^n = u^n - P_h u_h$,

$$\left\langle \frac{\partial u_h}{\partial t}, v_h \right\rangle \approx A\,\langle u_h, v_h\rangle + \int_0^{t} \langle k(t, s, u_h), v_h\rangle\, ds + \langle f, v_h\rangle.$$

The integral term can be approximated by unusual quadrature approximation, that is, a kind of the product trapezoidal integration rule as follows:

$$\left\langle \frac{\partial u_h}{\partial t}, v_h \right\rangle \approx A\left\langle u_h, v_h\right\rangle + \frac{t}{n}\sum_{i=1}^{n-1}\left\langle k(t_n, s_i, u(s_i)), v_h\right\rangle + \left\langle f, v_h\right\rangle.$$

Now, the forward Euler method is applied for the time derivatives in the equation:

$$\Delta_k u \approx u^{n+1} - u^n,$$

and

$$\|u^n - u_h\| = (u^n - u_h) - (u^n - P_h u_h) = e^n - E^n,$$

so,

$$\begin{aligned}
&\left\langle \Delta_k e^n, e^n\right\rangle + A\left\langle e^n, e^n\right\rangle = \left\langle \Delta_k e^n, E^n\right\rangle + A\left\langle e^n, E^n\right\rangle + \left\langle \Delta_k u^n - u_t^n, P_h u^n - u_h\right\rangle \\
&+ \left[A_h\left\langle u_h, P_h u^n - u_h\right\rangle - A\left\langle u_h, P_h u^n - u_h\right\rangle\right] \\
&+ \left[\int_0^{t_n}\left\langle k(t_n, s, u(s)), u_h - P_h u^n\right\rangle ds - k\sum_{j=0}^{n-1}\left\langle k(t_n, t_j, u^j), u_h - P_h u^n\right\rangle\right] \\
&+ k\sum_{j=0}^{n-1}\left\langle k(t_n, t_j, e^j), u_h - P_h u^n\right\rangle,
\end{aligned}$$

we put

$$\begin{aligned}
&I_1 = \left\langle \Delta_k e^n, E^n\right\rangle, \quad I_2 = A\left\langle e^n, E^n\right\rangle, \quad I_3 = \left\langle \Delta_k u^n - u_t^n, P_h u^n - u_h\right\rangle, \\
&I_4 = \left[A_h\left\langle u_h, P_h u^n - u_h\right\rangle - A\left\langle u_h, P_h u^n - u_h\right\rangle\right], \\
&I_5 = \left[\int_0^{t_n}\left\langle k(t_n, s, u(s)), u_h - P_h u^n\right\rangle ds - k\sum_{j=0}^{n-1}\left\langle k(t_n, t_j, u^j), u_h - P_h u^n\right\rangle\right], \\
&I_6 = k\sum_{j=0}^{n-1}\left\langle k(t_n, t_j, e^j), u_h - P_h u^n\right\rangle,
\end{aligned}$$

we have:

$$\left\langle \Delta_k e^n, e^n\right\rangle + A\left\langle e^n, e^n\right\rangle \le I_1 + I_2 + I_3 + I_4 + I_5 + I_6,$$

with Holder unequal can write:

$$I_1 \le Ch\left\|u^n\right\|^2_{H^1(\Omega)} + C\left\|\Delta_k e^n\right\|^2_{L^2(\Omega)},$$

and

$$I_2 \le Ch\left\|u^n\right\|^2_{H^1(\Omega)} + C\left\|\Delta_k e^n\right\|^2_{H^1(\Omega)},$$

for I_3 explain:

$$\Delta_k u^n - \frac{\partial u^n}{\partial t} = \frac{-1}{k}\int_{t_{n-1}}^{t_n}(s - t_{n-1})u_{ss}(s)ds,$$

with use lemma have:

$$I_3 \leq Ck^2 \left\|u_{tt}\right\|_{L^2}^2 + Ch^2 \left\|u^n\right\|_{H^1}^2 + C \left\|e^n\right\|_{L^2}^2,$$

given that:

$$|A_h(u_h, v_h) - A(u_h, v_h)| \leq Ch \left\|\nabla u_h\right\| \left\|\nabla v_h\right\|,$$

with use lemma can be obtained:

$$I_4 \leq Ch\|u_h\|_{H^1}\|P_h u^n - u_h\|_{H^1} \leq Ch \left\|e^n\right\|_{H^1}^2 + Ch^3 \left\|u^n\right\|_{H^1}^2$$
$$+ Ch \left\|f^n\right\|_{L^2}^2 + Chk \int_0^T \|u_t\|_{L^2}^2 ds + Chk \int_0^T \|u\|_{L^2}^2 ds + Ch \left\|u^0\right\|_{L^2}^2,$$

for I_5 by quadrature trapezoidal integration rule, that is:

$$\int_0^{t_n} f(s)ds \simeq \sum_{j=0}^{n-1} w_{nj} f(jk) + \mathrm{O}(k),$$

for weight function $w_{nj} = k$ can write:

$$\int_0^{t_n} f(s)ds \simeq k \sum_{j=0}^{n-1} f_j = I_n,$$

and

$$\left| I_n - \int_0^{t_n} f(s)ds \right| \leq k \int_0^{t_n} |f'(\tau)|\, d\tau,$$

so we will:

$$I_5 \leq k \int_{t_{n-1}}^{t_n} [\|u\|_{H^1} + \|u_s\|_{L^2}]ds \|E^n\|_{H^1} + k \int_{t_{n-1}}^{t_n} [\|u\|_{H^1} + \|u_s\|_{L^2}]ds \|e^n\|_{H^1}$$
$$\leq Ck \int_{t_{n-1}}^{t_n} [\|u\|_{H^1} + \|u_s\|_{L^2}]^2 ds + Ck \left(\|e^n\|_{H^1}^2 + \|E^n\|_{H^1}^2 \right)$$
$$\leq Ck[\|u\|_{L^2} + \|u_t\|_{L^2}]^2 + Ck \left(h^2 \|u^n\|_{H^1}^2 + \|e^n\|_{H^1}^2 \right),$$

for I_6 have :

$$\|e^n\|_{H^1}^2 \leq Ch^2 \left(\left\|u^0\right\|_{H^1}^2 + \|u^n\|_{H^1}^2 + \|u\|_{L^2}^2 + \|u_t\|_{L^2}^2 + \|u_{tt}\|_{L^2}^2 + \|f^n\|_{L^2}^2 \right)$$
$$+ Ck \sum_{j=0}^{n-1} \left\langle k(t_n, t_j, e^j), u_h - P_h u^n \right\rangle,$$

then:

$$\|e^n\|_{H^1}^2 \leq Ch^2 \left(\left\|u^0\right\|_{H^1}^2 + \|u^n\|_{H^1}^2 + \|u\|_{L^2}^2 + \|u_t\|_{L^2}^2 + \|u_{tt}\|_{L^2}^2 + \|f^n\|_{L^2}^2 \right).$$

So we will have the following boundaries:

$$\|u^n - u_h\|_{L^2} \leq Ck\|u_t\|_{L^2} + C\left(\|e^n\|_{H^1}^2 \right)^{\frac{1}{2}}.$$

The proof is completed. □

5 Numerical experiments

The numerical experiments are implemented in Maple 2018 software. The computer programmes are accomplished on a PC with Intel Core i7 central processing unit 2.60 GHz with 8 GB RAM. In order to show the error, we introduce the following notation

$$e^n(x,t) = |u(x,t) - u^n(x,t)|,$$

where $u(x,t)$ and $u^n(x,t)$ are the exact solution and the approximation solution obtained by the presented method, respectively. We are going to use the error norms defined by

$$L_2 = \left[\frac{1}{2^{j-1}M} \sum_{k=1}^{2^{j-1}M} \left(u(x,t) - u^n(x,t)\right)^2 \right]^{\frac{1}{2}},$$

With illustrative examples, to show the accuracy and efficiency of the described method, we present numerical examples, then we tabulated the absolute error and L_2 error norm with HWGM in Tables 1–3 and also we compared the present method results by radial basis functions [4], Legendre collocatiom method [5] and Euler method [8] for VPID equations. Figures show a comparison of exact and approximate solutions of the examples. We used $\theta = 1/2$, $x \in [0,1]$, $t \in [0,1]$, $j = 1$, $M = 5$ and $M = 6$.

Example 1. In this example, we investigate the VPIDE with $k(x,t,s,u) = (2x-1)\exp(s-t)u^2$, $\mu = 1$ and initial and boundary conditions $u(x,0) = x(1-x)$, $u(1,t) = u(0,t) = 0$ and $f(x,t) = (x^2(x-1))/2$ be chosen such that the exact solution in this example is $u(x,t) = x(1-x)\exp(-xt)$. In Figure 1, the exact solution, numerical solution are plotted by using HWGM with $\Delta t = 0.01$ and $M = 5$. The graph of exact and approximated solutions for $t = 1$ is illustrated in Figure 2. The comparison among the present method besides the solutions of radial basis functions [4] and Euler method [8] are shown in Table 1.

Example 2. Consider the kernel $k(x,t,s,u) = -u$ with the initial and boundary conditions $u(x,0) = (1-x^6)\sin x$, $u(0,t) = u(1,t) = 0$, and $\mu = 1$, and $f(x,t) = (1-x^6)\cos(x) + 12x^5)\cos(x+t) + (1+30x^4-x^6)\sin(x+t)$ be chosen such that the exact solution in this example is $u(x,t) = (1-x^6)\sin(x+t)$. In Figure 3, the

Table 1. Absolute errors and L_2 error norm by HWGM with other method at $\Delta t = 0.01, \Delta t = 0.001$ for Example 1.

	(M=5)		**(M=6)**		Method [4]	Method [8]
x,t	$\Delta t = 0.01$	$\Delta t = 0.001$	$\Delta t = 0.01$	$\Delta t = 0.001$	$N = 40$	$N = M = 40$
0.1	3.1(−6)	4.4(−7)	3.2(−9)	4.7(−9)	7.3(−6)	
0.2	2.2(−6)	6.0(−7)	5.1(−9)	3.6(−9)	1.2(−5)	
0.3	2.9(−7)	2.9(−7)	1.3(−9)	4.9(−11)	1.6(−5)	
0.4	5.7(−7)	4.2(−7)	4.2(−9)	7.7(−11)	1.9(−5)	
0.5	4.2(−7)	4.0(−8)	7.7(−11)	2.8(−13)	2.1(−5)	
0.6	3.9(−9)	2.1(−8)	6.4(−11)	2.9(−11)	2.2(−5)	
0.7	2.7(−7)	3.2(−9)	3.7(−11)	6.1(−13)	2.3(−5)	
0.8	5.6(−9)	7.1(−9)	3.7(−11)	5.4(−11)	2.4(−5)	
0.9	3.2(−9)	4.6(−9)	4.3(−9)	8.5(−11)	2.4(−5)	
L_2	3.3(−6)	4.5(−8)	6.3(−9)	2.8(−11)	5.8(−5)	3.5(−4)
CPU time	4.29	7.12	10.11	15.23	98.52	

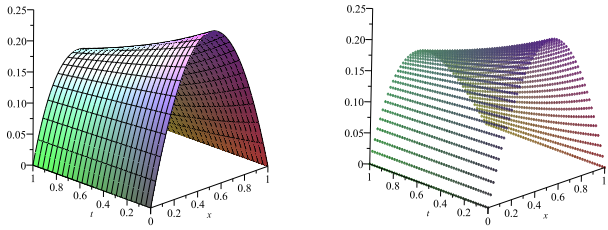


Figure 1. Comparison of exact solution (a) with HWGM solution (b) at $M = 5$ and $\Delta t = 0.01$ for Example 1.

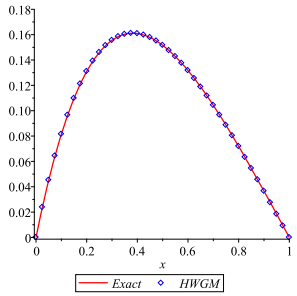


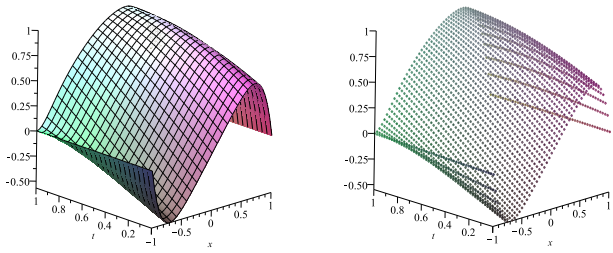
Figure 2. Exact and HWGM solutions at $t = 1$, $M = 5$ and $\Delta t = 0.01$ for Example 1.

exact solution, numerical solution are plotted by using HWGM with $\Delta t = 0.01$ and $M = 5$. The graph of exact and approximated solutions for $t = 1$ is

illustrated in Figure 4. The comparison among the present method besides the solutions of radial basis functions [4] and Legendre collocation method [5] are shown in Table 2.

Table 2. Absolute errors and L_2 error norm by HWGM with other method at $\Delta t = 0.01, \Delta t = 0.001$ for Example 2.

	(M=5)		**(M=6)**		Method [4]	Method [5]
x,t	$\Delta t = 0.01$	$\Delta t = 0.001$	$\Delta t = 0.01$	$\Delta t = 0.001$	$N = 40$	$N = M = 14$
0.1	2.5(−7)	5.7(−7)	6.2(−9)	1.1(−11)	2.1(−9)	
0.2	3.2(−7)	2.4(−7)	7.1(−9)	1.6(−9)	2.1(−9)	
0.3	2.6(−7)	2.5(−9)	1.6(−9)	2.4(−11)	2.3(−9)	
0.4	2.5(−7)	8.3(−7)	8.8(−11)	2.7(−12)	2.3(−9)	
0.5	4.7(−8)	4.8(−8)	5.2(−11)	5.8(−12)	2.3(−9)	
0.6	7.4(−8)	4.7(−9)	4.6(−11)	2.7(−13)	2.3(−9)	
0.7	5.7(−9)	3.4(−9)	4.1(−9)	3.1(−13)	2.4(−9)	
0.8	7.9(−9)	8.1(−9)	3.0(−11)	5.7(−11)	2.4(−9)	
0.9	4.4(−9)	3.0(−9)	4.1(−9)	2.2(−11)	2.5(−9)	
L_2	4.4(−7)	6.2(−8)	1.1(−9)	5.5(−12)	4.2(−9)	6.0(−10)
CPU time	5.01	6.61	8.02	19.01	79.35	



a) b)

Figure 3. Comparison of exact solution (a) with HWGM solution (b) at $M = 5$ and $\Delta t = 0.01$ for Example 2.

Example 3. As a last example, we consider $\mu = 1$, $k(x,t,s,u) = 3tx^2s^2e^u$ and initial condition $u(x,0) = 0$ and boundary conditions $u(0,t) = 0$, $u(1,t) = t^3/3$ and $f(x,t)$ be chosen such that the exact solution in this example is $u(x,t) = x^3t^3/3$. In Figure 5, the exact solution, numerical solution are plotted by using HWGM with $\Delta t = 0.01$ and $M = 5$. The comparison among the present method besides the solutions of radial basis functions [4] are shown in Table 3. The graph of exact and approximated solutions for $t = 1$ is illustrated in Figure 6.

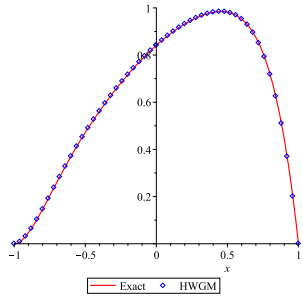


Figure 4. Exact and HWGM solutions at $t = 1$, $M = 5$ and $\Delta t = 0.01$ for Example 2.

Table 3. Absolute errors and L_2 error norm by HWGM with other method at $\Delta t = 0.01, \Delta t = 0.001$ for Example 3.

	(M=5)		**(M=6)**		Method [4]
x,t	$\Delta t = 0.01$	$\Delta t = 0.001$	$\Delta t = 0.01$	$\Delta t = 0.001$	$N = 40$
0.1	6.4(−6)	2.4(−7)	5.2(−11)	1.2(−13)	2.7(−6)
0.2	2.4(−6)	6.4(−7)	5.8(−9)	8.6(−11)	9.6(−6)
0.3	8.9(−6)	8.9(−7)	1.6(−9)	4.3(−11)	2.0(−5)
0.4	3.7(−7)	4.8(−9)	1.2(−9)	3.7(−9)	3.5(−5)
0.5	6.6(−7)	1.0(−9)	7.0(−9)	2.3(−9)	5.3(−5)
0.6	3.2(−7)	4.1(−8)	6.3(−11)	9.9(−11)	7.6(−5)
0.7	2.0(−7)	3.6(−9)	1.7(−11)	3.1(−13)	1.0(−4)
0.8	5.2(−9)	3.1(−7)	3.6(−11)	5.8(−11)	1.3(−4)
0.9	1.2(−9)	4.1(−9)	4.7(−10)	2.5(−9)	1.6(−4)
L_2	3.0(−7)	5.1(−8)	2.0(−11)	1.5(−13)	1.5(−4)
CPU time	5.12	8.43	11.46	17.50	92.06

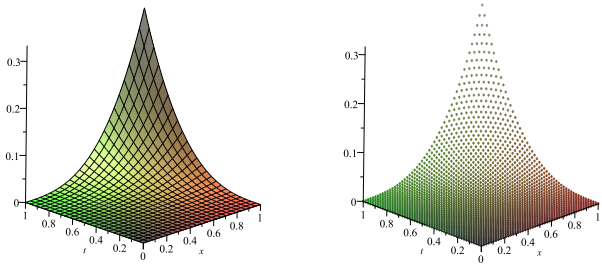


Figure 5. Comparison of exact solution (a) with HWGM solution (b) at $M = 5$ and $\Delta t = 0.01$ for Example 3.

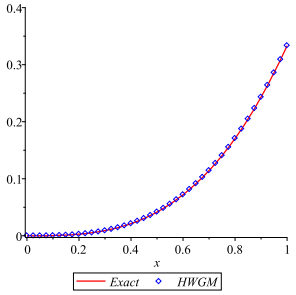


Figure 6. Exact and HWGM solutions at $t = 1$, $M = 5$ and $\Delta t = 0.01$ for Example 3.

6 Conclusions

In this paper, the Hermit wavelets based numerical method has been used for solving the nonlinear parabolic Volterra integro-differential equations in one-dimension. Hermit wavelet Galerkin method has proven a very powerful numerical technique for the stable and accurate solution of giving boundary value problem. Obviously, the Legendre collocation method [5] can only solve linear problems. In fact, a nonlinear kernel with respect to $u(x,t)$ leads to a nonlinear system of equations, which needs Newton's method or other efficient methods to solve the obtained heavy nonlinear systems. Clearly, the mentioned Legendre collocation method described in [5] is not sufficiently flexible to solve Examples 1–3. Also, comparison between the finite difference method [8] with the presented method signifies both methods are involved only with linear algebraic techniques. To exemplify the potency of the method, some examples were solved based on the suggested algorithm. Also, the convergence of the method was given. The results show that the proposed method is practically reliable and consistent in comparison with other mentioned methods, the rate of convergence improved.

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