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Joint Discrete Approximation of Analytic Functions by Hurwitz Zeta-Functions

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Abstract. Let $H(D)$ be the space of analytic functions on the strip $D = \{\sigma + it \in \mathbb{C} : 1/2 < \sigma < 1\}$. In this paper, it is proved that there exists a closed non-empty set $F_{\alpha_1,\dots,\alpha_r} \subset H(D)$ such that every collection of the functions $(f_1, \dots, f_r) \in F_{\alpha_1,\dots,\alpha_r}$ is approximated by discrete shifts $(\zeta(s + ikh_1, \alpha_1), \dots, \zeta(s + ikh_r, \alpha_r))$, $h_j > 0$, $j = 1, \dots, r$, $k \in \mathbb{N} \cup \{0\}$, of Hurwitz zeta-functions with arbitrary parameters $\alpha_1, \dots, \alpha_r$.

Keywords: Hurwitz zeta-function, space of analytic functions, weak convergence, universality.

AMS Subject Classification: 11M35.

1 Introduction

Let $s = \sigma + it$ be a complex variable, and α, $0 < \alpha \leqslant 1$, be a fixed parameter. The Hurwitz zeta-function $\zeta(s, \alpha)$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s},$$

and can be continued analytically to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. For $\alpha = 1$, the function $\zeta(s, \alpha)$ becomes the Riemann zeta-function $\zeta(s)$, and, for $\alpha = \frac{1}{2}$, $\zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s)$. Further, the function $\zeta(s, \alpha)$, $\alpha \neq 1; \frac{1}{2}$, has no Euler's product, and this is reflected in its value distribution.

Suppose that $\mathfrak{a} = \{a_m : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ be a periodic sequence of complex numbers. A generalization of the function $\zeta(s, \alpha)$ is the periodic Hurwitz zeta-function

$$\zeta(s, \alpha; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m + \alpha)^s}, \quad \sigma > 1,$$

which also has the meromorphic continuation to the whole complex plane.

Analytic properties of the functions $\zeta(s, \alpha)$ and $\zeta(s, \alpha; \mathfrak{a})$, including the approximation of analytic functions, depend on the arithmetic nature of the parameter α. Let $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$. Denote by $H(D)$ the space of analytic functions on D endowed with the topology of uniform convergence on compacta. Approximation of all functions of the space $H(D)$ by shifts $\zeta(s + i\tau, \alpha)$ and $\zeta(s + i\tau, \alpha; \mathfrak{a})$, $\tau \in \mathbb{R}$, is called universality of the functions $\zeta(s, \alpha)$ and $\zeta(s, \alpha; \mathfrak{a})$, respectively. More precisely, the following results are known.

Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, and by $H(K)$ with $K \in \mathcal{K}$ the class of continuous functions on K that are analytic in the interior of K. Let meas$A$ stand for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Suppose that the number α is transcendental or rational $\neq 1$ or $1/2$, and $K \in \mathcal{K}$, $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau, \alpha) - f(s)| < \varepsilon \right\} > 0. \tag{1.1}$$

Different proofs of the latter inequality are given in [1, 10, 36] and [28].

The above theorem is of continuous type. Also, a similar result of discrete type is known. Denote by $\#A$ the cardinality of a set A, and let N run over the set $\mathbb{N}_0$. For α rational $\neq 1$ or $1/2$, let $h > 0$ be arbitrary, while, for transcendental α, let h be such that $\exp\{(2\pi l)/h\}$ is irrational for all $l \in \mathbb{N}$. Let K and $f(s)$ be as above, then, for every $\varepsilon > 0$,

$$\liminf_{N \to \infty} \frac{1}{N + 1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{s \in K} |\zeta(s + ikh, \alpha) - f(s)| < \varepsilon \right\} > 0.$$

For the proof, see [1, 28, 35].

Universality results for the function $\zeta(s, \alpha)$ also follows from the Mishou theorem on the joint universality of the Riemann and Hurwitz zeta-functions

[33] and other results of a such type [5,6,7,18,20]. More general, shifts $\zeta(s+i\varphi(k),\alpha)$ with a certain function $\varphi(k)$ were used in [19]. The shifts $\zeta(s+ih\gamma_k,\alpha)$, where $0<\gamma_1<\gamma_2<\cdots\leqslant\gamma_k\leqslant\gamma_{k+1}\leqslant\cdots$ is a sequence of imaginary parts of non-trivial zeros of the Riemann zeta-function were applied in [3,23,26] and [32]. Analogical universality theorems for the function $\zeta(s,\alpha;\mathfrak{a})$ were proved in [11,29,31], and follow from joint universality theorems for periodic zeta-functions (see, for example, [12,14,17,22,25,30]).

Universality of the functions $\zeta(s,\alpha)$ and $\zeta(s,\alpha;\mathfrak{a})$ with algebraic irrational parameter α is a very complicated and open problem.

In [13,15], for universality of $\zeta(s,\alpha)$, the linear independence over the field of rational numbers for the sets

$$\{\log(m+\alpha):m\in\mathbb{N}_0\}\ \text{ and }\ \{(\log(m+\alpha):m\in\mathbb{N}_0),2\pi/h\}$$

was required. This requirement is weaker than the transcendence of α, however, examples of such α are not known. In the joint case, the above sets were generalized [13,16] by

$$\{(\log(m+\alpha_1):m\in\mathbb{N}_0),\dots,(\log(m+\alpha_1):m\in\mathbb{N}_0)\}$$

and

$$\{(h_1\log(m+\alpha_1):m\in\mathbb{N}_0),\dots,(h_r\log(m+\alpha_r):m\in\mathbb{N}_0),2\pi\}.$$

There are known several results of approximation of analytic functions by shifts of the functions $\zeta(s,\alpha)$ and $\zeta(s,\alpha;\mathfrak{a})$ with algebraic irrational parameter α, however, the set of approximated functions is not identified. The first results of such a kind has been obtained in [2]. Suppose that $0<\alpha<1$ is arbitrary. Then there exists a closed non-empty subset $F_\alpha\subset H(D)$ such that, for every compact $K\subset D$, $f(s)\in F_\alpha$ and $\varepsilon>0$, inequality (1.1) holds. The analogical statements for the functions $\zeta(s,\alpha;\mathfrak{a})$ and the Lerch zeta-function are given in [9] and [21], respectively. Generalizations of [2] for the Mishou theorem were obtained in [24]. In [8], the following joint approximation theorem for Hurwitz zeta-functions has been proved.

Theorem 1. *Suppose that the numbers $0<\alpha_j<1$, $\alpha_j\neq1/2$, $j=1,\dots,r$, are arbitrary. Then there exists a closed non-empty set $F_{\alpha_1,\dots,\alpha_r}\subset H^r(D)$ such that, for every compact sets $K_1,\dots,K_r\subset D$, $(f_1,\dots,f_r)\in F_{\alpha_1,\dots,\alpha_r}$ and $\varepsilon>0$,*

$$\liminf_{T\to\infty}\frac{1}{T}\mathrm{meas}\Big\{\tau\in[0,T]:\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|\zeta(s+i\tau,\alpha_j)-f_j(s)|<\varepsilon\Big\}>0.$$

Moreover, the limit

$$\lim_{T\to\infty}\frac{1}{T}\mathrm{meas}\Big\{\tau\in[0,T]:\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|\zeta(s+i\tau,\alpha_j)-f_j(s)|<\varepsilon\Big\}>0$$

exists for all but at most countably many $\varepsilon>0$.

The aim of this paper is a discrete version of Theorem 1. For brevity, let $\underline{\alpha}=(\alpha_1,\dots,\alpha_r)$ and $\underline{h}=(h_1,\dots,h_r)$.

Theorem 2. *Suppose that the numbers $0 < \alpha_j < 1$, $\alpha_j \neq 1/2$ and positive numbers h_j, $j = 1, \dots, r$, are arbitrary. Then there exists a closed non-empty set $F_{\underline{\alpha},\underline{h}} \subset H^r(D)$ such that, for every compact sets $K_1, \dots, K_r \subset D$, $(f_1, \dots, f_r) \in F_{\underline{\alpha},\underline{h}}$ and $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s + ikh_j, \alpha_j) - f_j(s)| < \varepsilon \right\} > 0.$$

Moreover, "$\liminf$*" can be replaced by "*$\lim$*" for all but at most countably many $\varepsilon > 0$.*

It will be proved that the set $F_{\underline{\alpha},\underline{h}}$ is the support of a certain $H^r(D)$-valued random element.

2 Probabilistic results

Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of the space $\mathbb{X}$, and, for $A \in \mathcal{B}(H^r(D))$, define

$$P_{N,\underline{\alpha},\underline{h}}(A) = \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \underline{\zeta}(s + ik\underline{h}, \underline{\alpha}) \in A \right\},$$

where

$$\underline{\zeta}(s + ik\underline{h}, \underline{\alpha}) = (\zeta(s + ikh_1, \alpha_1), \dots, \zeta(s + ikh_r, \alpha_r)).$$

In this section, we deal with weak convergence of $P_{N,\underline{\alpha},\underline{h}}$ as $N \to \infty$.

We start with definition of one probability space. Define

$$\Omega = \prod_{m\in\mathbb{N}_0} \gamma_m,$$

where $\gamma_m = \{s \in \mathbb{C} : |s| = 1\}$ for all $m \in \mathbb{N}_0$. By the Tikhonov theorem, with the product topology and pointwise multiplication, the torus Ω is a compact topological Abelian group. Therefore, $\Omega^r = \Omega_1 \times \cdots \times \Omega_r$, where $\Omega_j = \Omega$ for all $j = 1, \dots, r$, again is a compact topological Abelian group. Thus, on $(\Omega^r, \mathcal{B}(\Omega^r))$, the probability Haar measure m_H can be defined, and we have the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H)$. Denote by $\omega_j(m)$ the mth component of an element $\omega_j \in \Omega_j$, $j = 1, \dots, r$, $m \in \mathbb{N}$. Characters of the group Ω^r are of the form

$$\prod_{j=1}^{r} \prod_{m\in\mathbb{N}_0}^{*} \omega_j^{k_{jm}}(m),$$

where the sign "$*$" shows that only a finite number of integers k_{jm} are distinct from zero. Therefore, putting $\underline{k} = \{k_{jm} : k_{jm} \in \mathbb{Z}, m \in \mathbb{N}_0\}$, $j =, \dots, r$, we have that the Fourier transform $g(\underline{k}_1, \dots, \underline{k}_r)$ of a probability measure μ on $(\Omega^r, \mathcal{B}(\Omega^r))$ is given by

$$g(\underline{k}_1, \dots, \underline{k}_r) = \int_{\Omega^r} \left(\prod_{j=1}^{r} \prod_{m\in\mathbb{N}_0}^{*} \omega_j^{k_{jm}}(m) \right) \mathrm{d}\mu. \tag{2.1}$$

Define two collections

$$A(\underline{\alpha},\underline{h}) = \Big\{(\underline{k}_1,\dots,\underline{k}_r) : \exp\Big\{-i\sum_{j=1}^{r} h_j \sum_{m\in\mathbb{N}_0}^{*} k_{jm}\log(m+\alpha_j)\Big\} = 1\Big\},$$

$$B(\underline{\alpha},\underline{h}) = \Big\{(\underline{k}_1,\dots,\underline{k}_r) : \exp\Big\{-i\sum_{j=1}^{r} h_j \sum_{m\in\mathbb{N}_0}^{*} k_{jm}\log(m+\alpha_j)\Big\} \neq 1\Big\}.$$

Let $Q_{\underline{\alpha},\underline{h}}$ be the probability measure on $(\Omega^r, \mathcal{B}(\Omega^r))$ having the Fourier transform

$$g_{\underline{\alpha},\underline{h}}(\underline{k}_1,\dots,\underline{k}_r) = \begin{cases} 1 & \text{if } (\underline{k}_1,\dots,\underline{k}_r) \in A(\underline{\alpha},\underline{h}), \\ 0 & \text{if } (\underline{k}_1,\dots,\underline{k}_r) \in B(\underline{\alpha},\underline{h}). \end{cases}$$

For $A \in \mathcal{B}(\Omega^r)$, define

$$Q_{N,\underline{\alpha},\underline{h}}(A) = \frac{1}{N+1}\#\Big\{0 \leqslant k \leqslant N : \Big(\big((m+\alpha_1)^{-ikh_1} : m\in\mathbb{N}_0\big),\ \dots,\ \big((m+\alpha_r)^{-ikh_r} : m\in\mathbb{N}_0\big)\Big) \in A\Big\}.$$

Lemma 1. *$Q_{N,\underline{\alpha},\underline{h}}$ converges weakly to the measure $Q_{\underline{\alpha},\underline{h}}$ as $N\to\infty$.*

Proof. In view of (2.1), the Fourier transform $g_{N,\underline{\alpha},\underline{h}}(\underline{k}_1,\dots,\underline{k}_r)$ of $Q_{N,\underline{\alpha},\underline{h}}$ is given by

$$\begin{aligned} &g_{N,\underline{\alpha},\underline{h}}(\underline{k}_1,\dots,\underline{k}_r) \\ &= \int_{\Omega^r}\Big(\prod_{j=1}^{r}\prod_{m\in\mathbb{N}_0}^{*}\omega_j^{k_{jm}}(m)\Big)\,\mathrm{d}Q_{N,\underline{\alpha},\underline{h}} = \frac{1}{N+1}\sum_{k=0}^{N}\prod_{j=1}^{r}\prod_{m\in\mathbb{N}_0}^{*}(m+\alpha_j)^{-ikh_jk_{jm}} \\ &= \frac{1}{N+1}\sum_{k=0}^{N}\exp\Big\{-ik\sum_{j=1}^{r}h_j\sum_{m\in\mathbb{N}_0}^{*}k_{jm}\log(m+\alpha_j)\Big\}. \end{aligned}$$

Thus, $g_{N,\underline{\alpha},\underline{h}}(\underline{k}_1,\dots,\underline{k}_r) = 1$ for $(\underline{k}_1,\dots,\underline{k}_r) \in A(\underline{\alpha},\underline{h})$. If $(\underline{k}_1,\dots,\underline{k}_r) \in B(\underline{\alpha},\underline{h})$, then by the sum formula of geometric progression, we have

$$\begin{aligned} &g_{N,\underline{\alpha},\underline{h}}(\underline{k}_1,\dots,\underline{k}_r) \\ &\quad = \frac{1-\exp\Big\{-i(N+1)\sum_{j=1}^{r}h_j\sum_{m\in\mathbb{N}_0}^{*}k_{jm}\log(m+\alpha_j)\Big\}}{(N+1)\Big(1-\exp\Big\{-i\sum_{j=1}^{r}h_j\sum_{m\in\mathbb{N}_0}^{*}k_{jm}\log(m+\alpha_j)\Big\}\Big)}. \end{aligned}$$

Therefore,

$$\lim_{N\to\infty} g_{N,\underline{\alpha},\underline{h}}(\underline{k}_1,\dots,\underline{k}_r) = \begin{cases} 1 & \text{if } (\underline{k}_1,\dots,\underline{k}_r) \in A(\underline{\alpha},\underline{h}), \\ 0 & \text{if } (\underline{k}_1,\dots,\underline{k}_r) \in B(\underline{\alpha},\underline{h}), \end{cases}$$

This together with a continuity theorem for probability measures on compact groups proves the lemma. □

Now, let $\theta > 1/2$ be a fixed number, and, for $m \in \mathbb{N}_0$, $n \in \mathbb{N}$,

$$v_n(m, \alpha_j) = \exp\left\{ -\left(\frac{m+\alpha_j}{n+\alpha_j}\right)^\theta \right\}, \quad j = 1, \dots, r.$$

Define $\underline{\zeta}_n(s, \underline{\alpha}) = (\zeta_n(s, \alpha_1), \dots, \zeta_n(s, \alpha_r))$, where

$$\zeta_n(s, \alpha_j) = \sum_{m=0}^{\infty} \frac{v_n(m, \alpha_j)}{(m+\alpha_j)^s}, \quad j = 1, \dots, r.$$

In view of the definition $v_n(m, \alpha_j)$, the latter Dirichlet series are absolutely convergent for $\sigma > 1/2$. For $A \in \mathcal{B}(H^r(D))$, define

$$V_{N,n,\underline{\alpha},\underline{h}}(A) = \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \underline{\zeta}_n(s + ik\underline{h}, \underline{\alpha}) \in A \right\}.$$

To obtain the weak convergence for $V_{N,n,\underline{\alpha},\underline{h}}$ as $N \to \infty$, introduce the mapping $u_{n,\underline{\alpha}} : \Omega^r \to H^r(D)$ given by

$$u_{n,\underline{\alpha}}(\omega) = \underline{\zeta}_n(s, \underline{\alpha}, \omega), \quad \omega = (\omega_1, \dots, \omega_r) \in \Omega^r,$$

where $\underline{\zeta}_n(s, \underline{\alpha}, \omega) = (\zeta_n(s, \alpha_1, \omega_1), \dots, \zeta_n(s, \alpha_r, \omega_r))$ with

$$\zeta_n(s, \alpha_j, \omega_j) = \sum_{m=0}^{\infty} \frac{\omega_j(m) v_n(m, \alpha_j)}{(m+\alpha_j)^s}, \quad j = 1, \dots, r.$$

Obviously, the latter series also are absolutely convergent for $\sigma > 1/2$. Therefore, the mapping u_n is continuous, hence, it is $(\mathcal{B}(\Omega^r), \mathcal{B}(H^r(D)))$-measurable. Thus, the measure $Q_{\underline{\alpha},\underline{h}}$ defines the unique probability measure $V_{\underline{\alpha},\underline{h}}$ on $(H^r(D), \mathcal{B}(H^r(D)))$ by the formula

$$V_{\underline{\alpha},\underline{h}}(A) = Q_{\underline{\alpha},\underline{h}}(u_{n,\underline{\alpha}}^{-1} A), \quad A \in \mathcal{B}(H^r(D)).$$

Moreover, the definitions of $V_{N,n,\underline{\alpha},\underline{h}}$ and $Q_{N,\underline{\alpha},\underline{h}}$ imply the equality

$$V_{N,n,\underline{\alpha},\underline{h}}(A) = Q_{N,\underline{\alpha},\underline{h}}(u_{n,\underline{\alpha}}^{-1} A), \quad A \in \mathcal{B}(H^r(D)).$$

All these remarks together with Lemma 1 and the property of preservation of weak convergence under continuous mappings lead to the following limit lemma.

Lemma 2. *$V_{N,n,\underline{\alpha},\underline{h}}$ converges weakly to $V_{n,\underline{\alpha},\underline{h}}$ as $N \to \infty$.*

To obtain a limit theorem for $P_{N,\underline{\alpha},\underline{h}}$, we need the estimation a distance between $\underline{\zeta}_n(s, \underline{\alpha})$ and $\underline{\zeta}(s, \underline{\alpha})$. Let $g_1, g_2 \in H(D)$. Recall that

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|},$$

where $\{K_l : l \in \mathbb{N}\}$ is a certain sequence of compact subsets of the strip D, is a metric on $H(D)$ inducing its topology of uniform convergence on compacta. Let $\underline{g}_1 = (g_{11}, \dots, g_{1r})$, $\underline{g}_2 = (g_{21}, \dots, g_{2r}) \in H^r(D)$. Then

$$\underline{\rho}(\underline{g}_1, \underline{g}_2) = \max_{1 \leqslant j \leqslant r} \rho(g_{1j}, g_{2j})$$

is a metric on $H^r(D)$ that induces the product topology.

Let θ be the same parameter as in definition of $v_n(m, \alpha_j)$, and

$$l_n(s, \alpha) = \frac{s}{\theta} \Gamma\left(\frac{s}{\theta}\right) (n + \alpha)^{-s},$$

where $\Gamma(s)$ is the Euler gamma-function. Then the following integral representation is known [28].

Lemma 3. *For $s \in D$,*

$$\zeta(s, \alpha) = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} \zeta(s + z, \alpha) l_n(z, \alpha) \frac{\mathrm{d}z}{z}.$$

We will use some mean square results of discrete type. For the proof of them, the next lemma connecting the continuous and discrete mean squares is useful.

Lemma 4. *Suppose that T, $T_0 \geqslant \delta > 0$ are real numbers, $\mathcal{T} \neq \varnothing$ is a finite set lying in the interval $[T_0 + \delta/2, T_0 + T - \delta/2]$, and*

$$N_\delta(x) = \sum_{t \in \mathcal{T}, |t - x| < \delta} 1.$$

Let $S(x)$ be a complex valued function continuous in $[T_0, T_0 + T]$ and have a continuous derivative in $(T_0, T_0 + T)$. Then

$$\sum_{t \in \mathcal{T}} N_\delta^{-1}(t) |S(t)|^2 \leqslant \frac{1}{\delta} \int_{T_0}^{T_0 + T} |S(x)|^2 \, \mathrm{d}x + \left(\int_{T_0}^{T_0 + T} |S(x)|^2 \, \mathrm{d}x \int_{T_0}^{T_0 + T} |S'(x)|^2 \, \mathrm{d}x \right)^{1/2}.$$

The lemma is called the Gallagher lemma, its proof is given in [34, Lemma 1.4].

Lemma 5. *Suppose that $0 < \alpha \leqslant 1$, $1/2 < \sigma < 1$ and $h > 0$ are fixed numbers. Then, for every $t \in \mathbb{R}$,*

$$\sum_{k=0}^{N} \left|\zeta(\sigma + ikh + it, \alpha)\right|^2 \ll_{\alpha, \sigma, h} N(1 + |t|).$$

Proof. It is well known that

$$\int_0^T \left|\zeta(\sigma + it, \alpha)\right|^2 \ll_{\alpha, \sigma} T, \quad \int_0^T \left|\zeta'(\sigma + it, \alpha)\right|^2 \ll_{\alpha, \sigma} T.$$

Therefore, an application of Lemma 4 with $\delta = h$ gives the estimate of the lemma. □

The next lemma is very important for the proof of weak convergence for $P_{N, \underline{\alpha}, \underline{h}}$.

Lemma 6. *For arbitrary* $0 < \alpha_j \leqslant 1$ *and* $h_j > 0$, $j = 1, \dots, r$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \underline{\rho}\left(\underline{\zeta}(s+ik\underline{h}, \underline{\alpha}), \underline{\zeta}_n(s+ik\underline{h}, \underline{\alpha})\right) = 0.$$

Proof. The definition of the metric $\underline{\rho}$ implies that it suffices to show the equality

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \rho\left(\zeta(s+ikh, \alpha), \zeta_n(s+ikh, \alpha)\right) = 0$$

for arbitrary $0 < \alpha \leqslant 1$ and $h > 0$. On the other hand, the latter equality is implied by

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \sup_{s\in K} |\zeta(s+ikh, \alpha) - \zeta_n(s+ikh, \alpha)| = 0$$

for every compact subset $K \subset D$.

Thus, let $K \subset D$ be an arbitrary compact set. There exists $\varepsilon > 0$ such that all points of the set K lie in the strip $\{s \in \mathbb{C} : 1/2 + 2\varepsilon \leqslant \sigma \leqslant 1 - \varepsilon\}$. Let $s = \sigma + it \in K$, and $\theta_1 = \sigma - 1/2 - \varepsilon > 0$. Then, in view of Lemma 3 and the residue theorem,

$$\zeta_n(s, \alpha) - \zeta(s, \alpha) = \frac{1}{2\pi i} \int_{-\theta_1 - i\infty}^{-\theta_1 + i\infty} \zeta(s+z, \alpha) l_n(z, \alpha) \frac{\mathrm{d}z}{z} + R_n(s, \alpha),$$

where

$$R_n(s, \alpha) = \operatorname*{Res}_{z=1} \zeta(s+z, \alpha) l_n(z, \alpha) \frac{1}{z} = \frac{l_n(1-s, \alpha)}{1-s}.$$

Hence, for $s \in K$,

$$\begin{aligned} \zeta_n(s+ikh, \alpha) - \zeta(s+ikh, \alpha) &\ll \sup_{s\in K} |R_n(s+ikh, \alpha)| \\ &+ \int_{-\infty}^{\infty} |\zeta(1/2+\varepsilon+ikh+i\tau, \alpha)| \sup_{s\in K} \left| \frac{l_n(1/2+\varepsilon-s+i\tau, \alpha)}{1/2+\varepsilon-s+i\tau} \right| \mathrm{d}\tau. \end{aligned}$$

Therefore,

$$\frac{1}{N+1} \sum_{k=0}^{N} \sup_{s\in K} |\zeta(s+ikh, \alpha) - \zeta_n(s+ikh, \alpha)| \ll I_1 + I_2, \tag{2.2}$$

where

$$I_1 = \int_{-\infty}^{\infty} \left(\frac{1}{N+1} \sum_{k=0}^{N} \left| \zeta\left(\frac{1}{2}+\varepsilon+ikh+i\tau, \alpha\right) \right| \right) \sup_{s\in K} \left| \frac{l_n(1/2+\varepsilon-s+i\tau, \alpha)}{1/2+\varepsilon-s+i\tau} \right| \mathrm{d}\tau$$

and

$$I_2 = \frac{1}{N+1} \sum_{k=0}^{N} \sup_{s\in K} |R_n(s+ikh, \alpha)|.$$

The crucial role in the estimation of $l_n(s,\alpha)$ is played by the gamma-function. It is well known that there exists $c>0$ such that, uniformly in $\sigma_1 \leqslant \sigma \leqslant \sigma_2$,

$$\Gamma(\sigma+it) \ll \exp\{-c|t|\}. \tag{2.3}$$

This estimate leads, for $\sigma+it \in K$, to

$$\begin{aligned}\frac{l_n(1/2+\varepsilon-\sigma-it+i\tau,\alpha)}{1/2+\varepsilon-\sigma-it+i\tau} &\ll \frac{(n+\alpha)^{1/2+\varepsilon-\sigma}}{\theta}\exp\{-(c/\theta)|\tau-t|\}\\ &\ll_{\theta,K} (n+\alpha)^{-\varepsilon}\exp\{-(c/\theta)|\tau|\}.\end{aligned}$$

Therefore, in view of Lemma 5,

$$\begin{aligned}I_1 \ll_{\theta,K}(n+\alpha)^{-\varepsilon}\int_{-\infty}^{\infty}\left(\frac{1}{N}\sum_{k=0}^{N}\left|\zeta(1/2+\varepsilon+ikh+i\tau,\alpha)\right|^2\right)^{1/2}\\ \times\exp\{-(c/\theta)|\tau|\}\,\mathrm{d}\tau \ll_{\theta,K,\varepsilon,h} (n+\alpha)^{-\varepsilon}.\end{aligned} \tag{2.4}$$

By estimate (2.3) again, we find that, for $s\in K$,

$$\begin{aligned}\frac{l_n(1-s-ikh,\alpha)}{1-s-ikh} &\ll_\theta (n+\alpha)^{1-\sigma}\exp\{-(s/\theta)|kh-t|\}\\ &\ll_{\theta,K} (n+\alpha)^{1/2-2\varepsilon}\exp\{-((ch)/\theta)k\}.\end{aligned}$$

Therefore,

$$I_2 \ll_{\theta,K} (n+\alpha)^{1/2-2\varepsilon}\frac{1}{N}\sum_{k=0}^{N}\exp\{-((ch)/\theta)k\} \ll_{\theta,K,h}(n+\alpha)^{1/2-2\varepsilon}\frac{\log N}{N}.$$

This, together with (2.4) and (2.2) proves the lemma. □

Now, we define the marginal measures of $V_{n,\underline{\alpha},\underline{h}}$. For $A\in\mathcal{B}(\Omega_j)$, $j=1,\dots,r$ define

$$Q_{N,\alpha_j,h_j}(A) = \frac{1}{N+1}\#\left\{0\leqslant k\leqslant N : \left((m+\alpha_j)^{-ikh_j} : m\in\mathbb{N}_0\right)\in A\right\}.$$

Then by Lemma 1 of [27], Q_{N,α_j,h_j} converges weakly to a certain probability measure Q_{α_j,h_j} on $(\Omega_j,\mathcal{B}(\Omega_j))$ as $N\to\infty$, $j=1,\dots,r$. Let the mapping $u_{n,\alpha_j}:\Omega_j\to H(D)$ be given by $u_{n,\alpha_j}(\omega_j)=\zeta_n(s,\alpha_j,\omega_j)$. Define

$$V_{n,\alpha_j,h_j}(A) = Q_{\alpha_j,h_j}u_{n,\alpha_j}^{-1}(A) = Q_{\alpha_j,h_j}(u_{n,\alpha_j}^{-1}A),\quad A\in\mathcal{B}(H(D)),\ j=1,\dots,r.$$

Then in [27, Lemma 4], the following statement has been obtained.

Lemma 7. *For all $0<\alpha_j\leqslant 1$ and $h_j>0$, $j=1,\dots,r$, the family of probability measures $\{V_{n,\alpha_j,h_j} : n\in\mathbb{N}\}$ is tight, i.e., for every $\varepsilon>0$, there exists a compact set $K_j=K_j(\varepsilon)\subset H(D)$ such that $V_{n,\alpha_j,h_j}(K_j)>1-\varepsilon$ for all $n\in\mathbb{N}$.*

We apply Lemma 7 for the family of probability measures $\{V_{n,\underline{\alpha},\underline{h}} : n\in\mathbb{N}\}$.

Lemma 8. *The family* $\{V_{n,\underline{\alpha},\underline{h}} : n \in \mathbb{N}\}$ *is tight.*

Proof. Let $\varepsilon > 0$ be an arbitrary number. By Lemma 7, there exist compact sets $K_1, \dots, K_r \subset H(D)$ such that

$$V_{n,\alpha_j,h_j}(K_j) > 1 - \varepsilon/r \tag{2.5}$$

for all $n \in \mathbb{N}$. Let $K = K_1 \times \cdots \times K_r$. Then K is a compact set in $H^r(D)$. Denoting

$$(H(D) \setminus K_j)_r = (\underbrace{H(D) \times \cdots \times H(D)}_{j-1} \times (H(D) \setminus K_j) \times H(D) \times \cdots \times H(D)),$$

by (2.5), we have

$$V_{n,\underline{\alpha},\underline{h}}(H^r(D) \setminus K) = V_{n,\underline{\alpha},\underline{h}} \left(\bigcup_{j=1}^{r} (H(D) \setminus K_j) \right)_r \leqslant \sum_{j=1}^{r} V_{n,\alpha_j,h_j}(H(D) \setminus K_j) \leqslant \varepsilon$$

for all $n \in \mathbb{N}$. Thus, $V_{n,\underline{\alpha},\underline{h}}(K) \geqslant 1 - \varepsilon$ for all $n \in \mathbb{N}$. $\square$

Now we are in position to prove a limit theorem for $P_{N,\underline{\alpha},\underline{h}}$.

Theorem 3. *On* $(H^r(D), \mathcal{B}(H^r(D)))$, *there exists a probability measure* $P_{\underline{\alpha},\underline{h}}$ *such that* $P_{N,\underline{\alpha},\underline{h}}$ *converges weakly to* $P_{\underline{\alpha},\underline{h}}$ *as* $N \to \infty$.

Proof. Let ξ_N be a random variable defined on a certain probability space with measure μ and having the distribution

$$\mu\{\xi_N = k\} = 1/(N+1), \quad k = 0, 1, \dots, N.$$

On the mentioned probability space, define the $H^r(D)$-valued random elements

$$X_{N,n,\underline{\alpha},\underline{h}} = X_{N,n,\underline{\alpha},\underline{h}}(s) = \zeta_n(s + i\xi_N \underline{h}, \underline{\alpha}), \; X_{N,\underline{\alpha},\underline{h}} = X_{N,\underline{\alpha},\underline{h}}(s) = \zeta(s + i\xi_N \underline{h}, \underline{\alpha}).$$

Moreover, let $Y_{n,\underline{\alpha},\underline{h}}$ be the $H^r(D)$-valued random element having the distribution $V_{n,\underline{\alpha},\underline{h}}$. Then, in view of Lemma 2,

$$X_{N,n,\underline{\alpha},\underline{h}} \xrightarrow[N\to\infty]{\mathcal{D}} Y_{n,\underline{\alpha},\underline{h}}, \tag{2.6}$$

where $\xrightarrow{\mathcal{D}}$ means the convergence in distribution.

By the Prokhorov theorem, see, for example, [4], every tight family of probability measures is relatively compact. Thus, in view of Lemma 8, the family $\{V_{n,\underline{\alpha},\underline{h}}\}$ is relatively compact. Therefore, there exists a subsequence $\{V_{n_l,\underline{\alpha},\underline{h}}\}$ weakly convergent to a certain probability measure $P_{\underline{\alpha},\underline{h}}$ as $l \to \infty$. Hence,

$$Y_{n_l,\underline{\alpha},\underline{h}} \xrightarrow[l\to\infty]{\mathcal{D}} P_{\underline{\alpha},\underline{h}}. \tag{2.7}$$

Moreover, Lemma 6 implies that, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \mu \left\{ \underline{\rho} \left(X_{N,\underline{\alpha},\underline{h}}, X_{N,n,\underline{\alpha},\underline{h}} \right) \geqslant \varepsilon \right\}$$

$$\leqslant \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{\varepsilon(N+1)} \sum_{k=0}^{N} \underline{\rho} \Big(\underline{\zeta}(s+ik\underline{h}, \underline{\alpha}), \underline{\zeta}_n(s+ik\underline{h}, \underline{\alpha}) \Big) = 0.$$

This, (2.6) and (2.7) together with Theorem 4.2 of [4] show that

$$X_{N,\underline{\alpha},\underline{h}} \xrightarrow[N\to\infty]{\mathcal{D}} P_{\underline{\alpha},\underline{h}}.$$

Since the latter relation is equivalent to weak convergence of $P_{N,\underline{\alpha},\underline{h}}$ to $P_{\underline{\alpha},\underline{h}}$ as $N \to \infty$, the theorem is proved. $\square$

3 Proof of approximation

Denote by $F_{\underline{\alpha},\underline{h}}$ the support of the limit measure $P_{\underline{\alpha},\underline{h}}$ in Theorem 3. Thus $F_{\underline{\alpha},\underline{h}} \subset H^r(D)$ is a minimal closed set such that $P_{\underline{\alpha},\underline{h}}(F_{\underline{\alpha},\underline{h}}) = 1$. The set $F_{\underline{\alpha},\underline{h}}$ consists of all elements $\underline{g} \in H^r(D)$ such that, for every open neighbourhood G of $\underline{g}$, the equality $P_{\underline{\alpha},\underline{h}}(G) > 1$ is satisfied. Obviously, $F_{\underline{\alpha},\underline{h}} \neq \varnothing$.

Proof. (Proof of Theorem 2).

1. Let $(f_1(s), \dots, f_r(s)) \in F_{\underline{\alpha},\underline{h}}$. Define the set

$$G_\varepsilon = \Big\{ (g_1, \dots, g_r) \in H^r(D) : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |g_j(s) - f_j(s)| < \varepsilon \Big\}.$$

Then G_ε is an open neighbourhood of an element of the support of the measure $P_{\underline{\alpha},\underline{h}}$, therefore $P_{\underline{\alpha},\underline{h}}(G_\varepsilon) > 0$. Hence, by Theorem 3 and equivalent of weak convergence of probability measures in terms of open sets,

$$\liminf_{N\to\infty} P_{N,\underline{\alpha},\underline{h}}(G_\varepsilon) \geqslant P_{\underline{\alpha},\underline{h}}(G_\varepsilon) > 0.$$

This, the definitions of $P_{N,\underline{\alpha},\underline{h}}$ and G_ε prove the first assertion of the theorem.

2. The boundary of the set G_ε lies in the set

$$\Big\{ (g_1, \dots, g_r) \in H^r(D) : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |g_j(s) - f_j(s)| = \varepsilon \Big\}.$$

Therefore, these boundaries do not intersect for different ε. Hence, the set G_ε is a continuity set of the measure $P_{\underline{\alpha},\underline{h}}$ for all but at most countably many $\varepsilon > 0$. Therefore, Theorem 3 together with equivalent of weak convergence of probability measures in terms of continuity sets implies that

$$\lim_{N\to\infty} P_{N,\underline{\alpha},\underline{h}}(G_\varepsilon) = P_{\underline{\alpha},\underline{h}}(G_\varepsilon) > 0$$

for all but at most countably many $\varepsilon > 0$, and the second assertion of the theorem is proved. $\square$

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