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A Semi-Analytic Method for Solving Singularly Perturbed Twin-Layer Problems with a Turning Point

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Abstract. This computational study investigates a class of singularly perturbed second-order boundary-value problems having dual (twin) boundary layers and simple turning points. It is well-known that the classical discretization methods fail to resolve sharp gradients arising in solving singularly perturbed differential equations as the perturbation (diffusion) parameter decreases, i.e., $\varepsilon \to 0^+$. To this end, this paper proposes a semi-analytic hybrid method consisting of a numerical procedure based on finite differences and an asymptotic method called the Successive Complementary Expansion Method (SCEM) to approximate the solution of such problems. Two numerical experiments are provided to demonstrate the method's implementation and to evaluate its computational performance. Several comparisons with the numerical results existing in the literature are also made. The numerical observations reveal that the hybrid method leads to good solution profiles and achieves this in only a few iterations.

Keywords: asymptotic expansion, dual-layers, finite differences, singular perturbation, turning point.

AMS Subject Classification: 34B05; 65L11.

1 Introduction

Differential equations in which the highest-order derivative term(s) are controlled by (a) small positive parameter(s), $0 < \varepsilon \ll 1$, are called singularly perturbed differential equations (SPDEs). As the perturbation parameter tends to zero, i.e., $\varepsilon \to 0^+$, the solution of the problem exhibits rapid changes because the order of the differential equation decreases. The thin region resulting from the existence of the small parameter is called the inner region or boundary layer. The region where the solution exhibits mild changes is called the outer region [10].

One encounters singularly perturbed problems frequently in natural and engineering sciences, such as power systems [36], electromagnetics [5], fluid mechanics [6, 9], quantum field theory [8], celestial mechanics [29], chemical kinetics [38], financial mathematics [12], etc. In studies by Kumar and Mittal [25] and by Kadalbajoo and Gupta [20], the classical methods used for solving singularly perturbed problems are analyzed in detail. One can also refer to the excellent books by Cousteix and Mauss [10], Hinch [15], Kevorkian and Cole [22], Verhulst [39], and Nayfeh [33] for more on the theoretical and applied considerations regarding SPDEs.

Numerous studies are available in the literature based on various numerical methods and techniques for solving singularly perturbed boundary-value problems with a turning point, such as the finite elements [37], finite differences [40], collocation methods [1, 30], reproducing kernel methods [13], and initial value techniques [39]. For a detailed review of the asymptotic and numerical methods developed in four decades (1970–2011) for solving singularly perturbed turning point and interior-layer problems, the interested readers are referred to the study by Sharma et al. [35].

A brief literature review is provided in the following lines and paragraphs. An exponentially fitted first-order scheme is constructed by Vulanovic and Farrell [40] for solving dual-layer problems in the following form

$$\varepsilon y''(x) + x^k a(x) y'(x) - b(x)y(x) = f(x), \ \ x \in [s, 1],$$

with suitable Dirichlet boundary conditions, where $k \in \mathbb{N}$, $s = -1$ or $s = 0$. Here, and in the remaining part of this study, unless otherwise stated, the terms y' and y'' denote the first- and second-order spatial derivatives with respect to x, respectively.

The SPDEs in the form of

$$\varepsilon y''(x) + a(x) y'(x) - b(x)y(x) = f(x), \ \ x \in [-1, 1], \tag{1.1}$$

with suitable Dirichlet boundary conditions were studied by Natesan and Ramanujam [32]. They proposed a numerical method combining exponentially-fitted differences with classical numerical methods. For solving problems having the form of Equation (1.1), Natesan and Ramanujam [31] used an initial-value technique originally developed for solving non-turning point problems. A numerical method based on cubic splines was employed on a non-uniform mesh by Kadalbajoo and Patidar [19] for solving SPDEs in the form of Equation (1.1).

In [30], the authors proposed a finite difference scheme for solving turning point problems on a piecewise uniform Shishkin mesh.

Kadalbajoo and Gupta [18] proposed a parameter-uniform B-spline collocation method for solving Equation (1.1). Kadalbajoo et al. [17] considered problem (1.1) by proposing a B-spline collocation method with artificial (numerical) viscosity. A reproducing kernel method was employed for solving Equation (1.1) by Geng and Qian [13]. Munyakazi and Patidar developed a fitted-mesh finite difference method with Richardson extrapolation in [28]. For solving Equation (1.1), a fitted-operator scheme was constructed by Phaneendra et al. in [34] using nonsymmetric finite differences for the first-order derivatives. Becher and Roos employed a finite difference scheme with Richardson extrapolation for solving the model problem on a piecewise-uniform Shishkin mesh [2]. Kumar [24] proposed a parameter-uniform B-spline collocation method for solving one-dimensional stationary turning point problems exhibiting interior layers.

In this paper, we develop an efficient and straightforward hybrid method for solving problems in the form of Equation (1.1). This hybrid method consists of an asymptotic method introduced in [27] and a numerical method based on a sixth-order finite-difference scheme with a four-stage Lobatto IIIa formula [23]. We modify the Successive Complementary Expansion Method (SCEM) employed for solving several types of SPDEs before (see [7] and the references therein) for solving dual-layer problems. In order to apply the method, we first divide the problem domain (interval) into an appropriate number of subintervals and then utilize a stretching (local) variable transformation. Later, we employ the numerical procedure for solving the initial/boundary-value problems arising from the SCEM process.

In this present method, there is no need for any matching procedure as opposed to the well-known classical approach called the Method of Matched Asymptotic Expansions. Beyond that, the SCEM yields approximations that satisfy the boundary conditions exactly but not asymptotically. For more on the SCEM, the interested readers are referred to [6, 9]. On the other hand, one can refer to the materials in [4, 26] for further details on asymptotic series, expansions, and approximations.

In the next section, we review the singularly perturbed turning point problems and give some theorems on the existence and uniqueness of the solutions. Section 3 contains the necessary definitions for asymptotic approximations and a description of the hybrid method. In Section 4, we examine two test examples in detail to show the implementation of the hybrid method. Finally, in the last section, some comments on the results are made, and possible future works are discussed.

2 Singularly perturbed turning point problems

Consider the following singularly perturbed second-order ordinary differential equation:

$$\varepsilon y''(x) + a(x)y'(x) - b(x)y(x) = f(x), \;\; x \in \Omega = (-1, 1), \tag{2.1}$$

where $0 < \varepsilon \ll 1$ is a positive small parameter. The Dirichlet boundary conditions associated with Equation (2.1) are given as follows:

$$y(-1) = \alpha, \;\; y(1) = \beta, \tag{2.2}$$

where α, $\beta \in \mathbb{R}$. The functions $a(x)$, $b(x)$, and $f(x)$ are assumed to be sufficiently smooth. The points satisfying the condition $a(x) = 0$ are called the turning points of Equation (2.1). In this paper, we assume that Equation (2.1) has only one turning point, x_0, and call it the simple turning point.

The behavior of the solution, $y(x)$, near the turning point, i.e., $x = x_0$, depends both on ε and the parameter γ, which is defined as [17]:

$$\gamma = b(x_0)/a'(x_0).$$

If $\gamma < 0$, then the solution of Equations (2.1)–(2.2) is smooth near the turning point x_0, and if $\gamma \geqslant 0$, then there is an interior layer, where the solution exhibits rapid changes around it. In the following parts of this study, we only consider the case $\gamma < 0$, i.e., it is assumed that the solution to Equations (2.1)–(2.2) does not possess interior-layer(s).

Certain assumptions should be made on the boundary-value problem given by (2.1)–(2.2) so that it has unique solution having dual boundary layers of exponential type at the endpoints of the closure of the domain $\Omega = (-1, 1)$, $\overline{\Omega} = [-1, 1]$:

- $a(x_0) = 0, \;\; a'(x_0) \leqslant 0$ (the problem has a turning point at $x = x_0$, no interior-layer),
- $|a(x)| \geqslant a_0 > 0, \;\; 0 < \delta \leqslant |x| \leqslant 1$ for some δ,
- $b(x) \geqslant b_0 > 0, \;\; \forall x \in \overline{\Omega}$ (the maximum principle),
- $|a'(x)| \geqslant \frac{|a'(x_0)|}{2}, \;\; \forall x \in \overline{\Omega}$ (uniqueness of the turning point at $x = x_0$).

We divide the interval into four sub-intervals as follows:

$$\overline{\Omega}_{-1^-} = [-1, -\delta/2] \,, \; \overline{\Omega}_{-1^+} = [-\delta/2, 0] \,, \; \overline{\Omega}_{1^-} = [0, \delta/2] \,, \; \overline{\Omega}_{1^+} = [\delta/2, 1] \,,$$

where $\overline{\Omega} = \overline{\Omega}_{-1^-} \cup \overline{\Omega}_{-1^+} \cup \overline{\Omega}_{1^-} \cup \overline{\Omega}_{1^+}$. Since the symmetry property of the turning point problems considered in this study leads to symmetric approximations (and naturally to symmetric errors) under the following theorems and lemmas, we only consider sub-intervals $\overline{\Omega}_{-1^-} \cup \overline{\Omega}_{-1^+}$.

Remark 1. The reason we divide the interval into four parts (intervals) is that the model problem given by Equations (2.1)–(2.2) has a single (simple) turning point and behaves symmetrically over the intervals on either side of that point. Therefore, the closure of the problem domain, $\overline{\Omega}$, is divided into two sub-intervals yielding a boundary value problem for each interval.

Lemma 1 [Minimum Principle]. *Let $y(x) \in C^2(\Omega)$ with $y(-1) \geqslant 0$, and $y(1) \geqslant 0$ such that $Ly(x) \leqslant 0$, $\forall x \in \Omega$. Under these conditions, $y(x) \geqslant 0$, $\forall x \in \overline{\Omega}$.*

Proof. In contrast, let us assume that for some $\hat{x} \in \overline{\Omega}$, $y(\hat{x}) = \min_{x\in\Omega} y(x) < 0$. It means that $y'(\hat{x}) = 0$ and $y''(\hat{x}) \geqslant 0$; thus, one finds

$$Ly(\hat{x}) = \varepsilon y''(\hat{x}) + a(\hat{x})y'(\hat{x}) - b(\hat{x})y(\hat{x}) > 0,$$

which is a contradiction. As a result, one can obtain $y(x) \geqslant 0$, $\forall x \in \overline{\Omega}$. □

Lemma 2. *If $y(x)$ is the solution to the problem* (2.1)–(2.2)*, then there exists a constant $C \in \mathbb{R}^+$ such that:*

$$|y| \leqslant C\left[\max\{|a|, |b|\} + |f|/b_0\right], \quad \forall x \in \Omega,$$

where $|g| = \max_{x\in\Omega} |g(x)|$.

A proof for Lemma 2 can be found in [24].

The following theorem states that the solution $y(x)$ is smooth away from the endpoints of the problem domain and the near vicinity of the turning point $x = x_0$.

Theorem 1. *Let $y(x)$ be the solution to the problem* (2.1)–(2.2)*, and $a(x)$, $b(x)$, and $f(x)$ belong to the set $C^m(\Omega)$, $m \geqslant 0$, then the bounds*

$$|y^{(i)}(x)| \leqslant C\left[1 + \varepsilon^{-i}\exp\left(-a_0\frac{x+1}{\varepsilon}\right)\right], \quad i = 1, 2, \ldots, m, m+1, \quad x \in \Omega_{-1^-},$$

$$|y^{(i)}(x)| \leqslant C\left[1 + \varepsilon^{-i}\exp\left(-a_0\frac{1-x}{\varepsilon}\right)\right], \quad i = 1, 2, \ldots, m, m+1, \quad x \in \Omega_{1^+}$$

are valid for any $\delta > 0$. Here, a_0 and C are positive real constants independent of ε and x.

For the proof of Theorem 1, the interested readers are referred to [21].

Theorem 2. *Let $\gamma = \frac{b(x_0)}{a'(x_0)} < 0$ and $y(x)$ be the solution to the problem* (2.1)–(2.2) *satisfying the existence and uniqueness conditions given above. If $a(x)$, $b(x)$, $f(x) \in C^m(\Omega)$ and $m \geq 0$, then*

$$|y^{(i)}(x)| \leqslant C, \; i = 1, 2, \ldots, m, \quad \forall x \in \Omega_{-1^+} \cup \Omega_{1^-}$$

for sufficiently small $\delta > 0$.

For the proof of Theorem 2, the interested readers are referred to [3].

The theory mentioned above also works for the following singularly perturbed turning point boundary-value problems for which the solution exhibits dual boundary layers:

$$Lu(x) \equiv \varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x), \quad 0 < x < 1, \tag{2.3}$$

subject to the Dirichlet boundary conditions given as

$$u(0) = \alpha, \quad u(1) = \beta. \tag{2.4}$$

The theoretical considerations given for problem (2.3)–(2.4) can be obtained in the same manner as shown for problem (2.1)–(2.2). Under the above-mentioned theoretical considerations, we describe the hybrid method in the following section.

3 The semi-analytic hybrid method

The SCEM approximation, in its regular form, can be given as follows [10]:

$$y_n^{scem}(x,\overline{x},\varepsilon)=\sum_{i=0}^{n}\delta_i(\varepsilon)\left[y_i(x)+\Psi_i(\overline{x})\right],$$

where $\delta_i(\varepsilon)$ are asymptotic sequences and $\Psi_i(\overline{x})$ are the complementary functions depending on the stretching (zoom) variable $\overline{x}$. The functions $y_i(x)$ are the outer approximations which only depend on x. If the functions $y_i(x)$ and $\Psi_i(\overline{x})$ also depend on the perturbation parameter, ε, then the uniformly valid SCEM approximation is called the generalized SCEM approximation and is given as follows:

$$y_{ng}^{scem}(x,\overline{x},\varepsilon)=\sum_{i=0}^{n}\delta_i(\varepsilon)\left[y_i(x,\varepsilon)+\Psi_i(\overline{x},\varepsilon)\right]. \tag{3.1}$$

If only one-term SCEM approximation is desired, then one seeks a uniformly valid SCEM approximation in the following form:

$$y_0^{scem}(x,\overline{x},\varepsilon)=y_0(x,\varepsilon)+\Psi_0(\overline{x},\varepsilon). \tag{3.2}$$

Remark 2. To improve the accuracy of (generalized) SCEM approximations, Equation (3.2) can be iterated using Equation (3.1) [10]. It means that successive complementary terms are added to the approximation given by Equation (3.2).

Then, in the light of Remark 2, the second SCEM approximation, $y_1^{scem}(x,\overline{x},\varepsilon)$, can be given in the following form:

$$y_1^{scem}(x,\overline{x},\varepsilon)=y_0(x,\varepsilon)+\Psi_0(\overline{x},\varepsilon)+\varepsilon\left(y_1(x,\varepsilon)+\Psi_1(\overline{x},\varepsilon)\right).$$

Cousteix and Mauss [10] established some error estimates for the first and second SCEM approximations as $|y-y_0^{scem}|<\varepsilon K_0$ and $|y-y_1^{scem}|<\varepsilon^2 K_1$, respectively, where K_0 and K_1 are positive constants independent of ε. As a consequence, the SCEM yields uniformly valid approximations [9, 11].

Remark 3. Although the number of iterations can be increased until the desired precision is achieved depending on the perturbation parameter, the first two SCEM approximations appear to yield quite good solution profiles and are adopted in this study (see also [7]).

The second complement (numerical part) of the hybrid method is from the well-known family of numerical methods for solving differential equations, the so-called Runge–Kutta (R–K) methods. This numerical complement is employed for solving the ordinary differential equations (ODEs) for complementary functions arising from the SCEM process. Some basic definitions are given in the following lines.

Definition 1. Let $b_i, a_{ij} \in \mathbb{R}$ and $c_i = \sum_{j=1}^{i-1} a_{ij}$ with $i = 2, 3, \ldots, s$. An s-stage R–K method is given as follows:

$$u_1 = u_0 + h \sum_{i=1}^{s} b_i k_i, \tag{3.3}$$

where

$$k_i = f\Big(x_0 + c_i h,\ u_0 + h \sum_{j=1}^{s-1} a_{ij} k_j\Big). \tag{3.4}$$

Remark 4. Notice that the terms $y(x_0) = u_0$ given in (3.3)–(3.4) refers to the initial condition associated with the differential equation under consideration. The point x_0 should not be confused with the turning point, here.

Definition 2. Let $c_1, c_2, \ldots, c_s \in \mathbb{R}$, and they all be distinct (generally $0 \leqslant c_i \leqslant 1$). The collocation polynomial $y(x)$ of degree s is a polynomial satisfying the following properties:

$$y(x_0) = u_0, \quad y'(x_0 + c_i h) = f\left(x_0 + c_i h,\ u\left(x_0 + c_i h\right)\right),$$

for $i = 1, 2, \ldots, s$, and the numerical solution of the collocation method is given as $u_1 = y\left(x_0 + h\right)$.

The node-points are the zeros of the following equation:

$$\frac{d^{s-2}}{dx^{s-2}} \left(x^{s-1} \left(1 - x\right)^{s-1}\right) = 0$$

and the order of the quadrature is $p = 2s - 2$. The corresponding collocation methods are called the Lobatto IIIa methods for historical reasons [14]. For $s = 2$, one reaches the well-known implicit trapezoidal rule. The interested readers are referred to the material in [14, 16] for further details.

In summary, we employ a combination of the methods above in our computations: an efficient and straightforward hybrid method consisting of an asymptotic approach, the so-called SCEM [27], and a numerical method based on a sixth-order finite-difference scheme with a four-stage Lobatto IIIa formula [23].

4 Test computations

In this part of the study, two numerical experiments are given to illustrate the implementation and show the computational characteristics of the proposed method on homogeneous and non-homogeneous dual-layer problems with simple turning points. All computations are performed in Matlab environment.

4.1 A homogeneous problem

Consider the following singularly perturbed homogeneous turning point problem [26, 32]:

$$\varepsilon y''(x) - 2(2x - 1)y'(x) - 4y(x) = 0,\ x \in \Omega = (0, 1) \tag{4.1}$$

with Dirichlet boundary conditions

$$y(0) = 1, \;\; y(1) = 1. \tag{4.2}$$

The exact solution to the problem is available and given as follows:

$$y(x) = \exp\left(-2x(1-x)/\varepsilon\right).$$

The problem has a simple turning point at $x_0 = \frac{1}{2}$, and the solution to the problem exhibits dual boundary layers at both endpoints of the interval $\Omega = (0,1)$. Let us divide the closure of interval Ω as

$$\Omega_{0^-} = \left[0, \frac{1-\delta}{2}\right], \Omega_{0^+} = \left[\frac{1-\delta}{2}, \frac{1}{2}\right], \Omega_{1^-} = \left[\frac{1}{2}, \frac{1+\delta}{2}\right], \Omega_{1^+} = \left[\frac{1+\delta}{2}, 1\right].$$

We start with assuming that the outer approximation is in the following form:

$$y_{out}(x,\varepsilon) = y_0(x,\varepsilon) + \varepsilon y_1(x,\varepsilon) \tag{4.3}$$

for the interval $\Omega_{0^+} = \left[\frac{1-\delta}{2}, \frac{1}{2}\right]$. Substituting(4.3) into (4.1), one obtains

$$\varepsilon\left(y_0''(x,\varepsilon) + \varepsilon y_1''(x,\varepsilon)\right) - 2\left(2x-1\right)\left(y_0'(x,\varepsilon) + \varepsilon y_1'(x,\varepsilon)\right) \\ - 4\left(y_0(x,\varepsilon) + \varepsilon y_1(x,\varepsilon)\right) = 0.$$

Balancing the terms of orders $O(1)$ and $O(\varepsilon)$, we obtain the following initial-value problems:

$$-2\left(2x-1\right)y_0'(x,\varepsilon) - 4y_0(x,\varepsilon) = 0, \;\; y_0(\varepsilon/2) = 0,$$
$$y_0''(x,\varepsilon) - 2\left(2x-1\right)y_1'(x,\varepsilon) - 4y_1(x,\varepsilon) = 0, \;\; y_1(\varepsilon/2) = 0,$$

and the solutions are easily obtained as

$$y_0(x,\varepsilon) \equiv y_1(x,\varepsilon) \equiv 0.$$

Then, the first SCEM approximation is sought as

$$y_0^{sceml}(x,\overline{x},\varepsilon) = 0 + \Psi_0(\overline{x},\varepsilon) = \Psi_0(\overline{x},\varepsilon),$$

where the boundary conditions are given as follows:

$$y_0^{sceml}(0,0,\varepsilon) = 1, \;\; y_0^{sceml}\left(\frac{1}{2}, \frac{1}{2\varepsilon}, \varepsilon\right) = 0.$$

Here, the stretching variable, $\overline{x}$, is defined as $\overline{x} = \frac{x}{\varepsilon}$ for the left boundary layer. The term y_0^{sceml} denotes the first SCEM approximation found for the left part of interval $[0,1]$. One can reach the boundary conditions for the complementary functions as follows:

$$\Psi(0,\varepsilon) = 1, \;\; \Psi\left(\frac{1}{2\varepsilon}, \varepsilon\right) = 0.$$

In order to obtain the complementary functions, let us propose the following expansion for interval $\Omega_{0^-} = \left[0, \frac{1-\delta}{2}\right]$:

$$\Psi(\overline{x},\varepsilon) = \Psi_0(\overline{x},\varepsilon) + \varepsilon\Psi_1(\overline{x},\varepsilon). \tag{4.4}$$

Substituting Equation (4.4) into (4.1) and using the chain rule, i.e.,

$$\frac{d}{dx} = \frac{d\overline{x}}{dx}\frac{d}{d\overline{x}} = \frac{1}{\varepsilon}\frac{d}{d\overline{x}}, \tag{4.5}$$

$$\frac{d^2}{dx^2} = \frac{d}{dx}\frac{d}{dx} = \frac{1}{\varepsilon^2}\frac{d^2}{d\overline{x}^2}, \tag{4.6}$$

and balancing the terms of orders $O(1)$ and $O(\varepsilon)$, we obtain the following boundary-value problems:

$$\Psi_0''(\overline{x},\varepsilon) - 2(2\overline{x}-1)\Psi_0'(\overline{x},\varepsilon) = 0,$$
$$\Psi_0(0,\varepsilon) = 1 \text{ and } \Psi_0(\varepsilon/2,\varepsilon) = 0,$$

and

$$\Psi_1''(\overline{x},\varepsilon) - 2(2\overline{x}-1)\Psi_1'(\overline{x},\varepsilon) - 4\Psi_0(\overline{x},\varepsilon) = 0,$$
$$\Psi_1(0,\varepsilon) = 0 \text{ and } \Psi_1(\varepsilon/2,\varepsilon) = 0.$$

Consequently, since $y_0(x,\varepsilon) \equiv y_1(x,\varepsilon) \equiv 0$, we find the first two iterations of the uniformly valid SCEM approximation using the following expression:

$$y_1^{sceml}(x,\overline{x},\varepsilon) = \Psi_0(\overline{x},\varepsilon) + \varepsilon\Psi_1(\overline{x},\varepsilon).$$

By adopting a stretching variable of the form $\widetilde{x} = (x-1)/\delta(\varepsilon)$, we can follow a similar to that used for the left symmetric interval to obtain y^{scemr}, where $\delta(\varepsilon)$ is an order function, typically, $\delta(\varepsilon) = \varepsilon$.

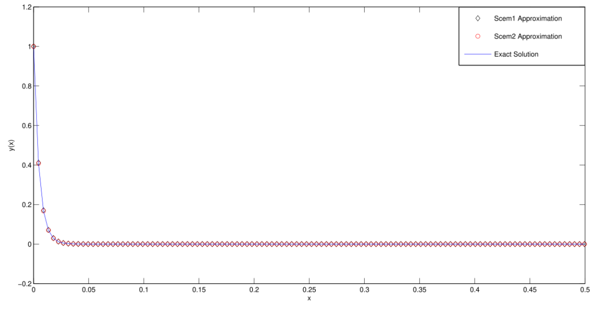

Figure 1. Comparison of the analytical solution and SCEM approximations in solving left symmetric problem for Equations (4.1)–(4.2); $\varepsilon = 0.01$. The exact solution and SCEM approximations are almost identical.

In Figure 1, the exact solution and the SCEM approximations are shown. The difference in the results is almost indistinguishable.

Figure 2 shows the absolute errors in the SCEM approximations for $\varepsilon = 10^{-2}$. The two-term SCEM approximation is clearly better than the one with one term.

In Tables 1 and 2, the results are compared for $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$, respectively. Notice that only the half interval $\left[0, \frac{1}{2}\right]$ is considered since the

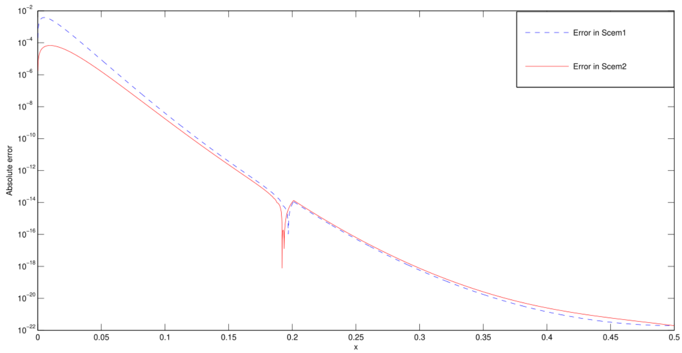

Figure 2. Absolute errors in SCEM approximations y_0^{sceml} and y_1^{sceml} in solving left homogeneous problem (4.1)–(4.2); $\varepsilon = 0.01$. The second SCEM approximations is slightly better, as expected.

Table 1. Comparison of the analytical solutions and SCEM approximations for solving Equations (4.1)–(4.2); $\varepsilon = 0.001$.

x	Exact	y_0^{scem}	y_1^{scem}	Error in y_1^{scem}	Error [32]
0.0000	1.00000000	1.00000000	1.00000000	0.00000000	0.00000000
0.0001	0.81874713	0.81891126	0.81874700	$1.23544e-7$	$5.46502e-7$
0.0003	0.54891043	0.54924068	0.54891013	$2.96916e-7$	$1.09895e-6$
0.0005	0.36806343	0.36843265	0.36806306	$3.66251e-7$	$1.22790e-6$
0.0007	0.24683875	0.24718555	0.24683839	$3.57688e-7$	$1.15264e-6$
0.0009	0.16556689	0.16586609	0.16556659	$3.04402e-7$	$9.93833e-7$
0.0010	0.13560622	0.13587857	0.13560595	$2.69712e-7$	$9.04345e-7$
0.0030	0.00252377	0.00253904	0.00252381	$3.57799e-8$	$5.03922e-8$
0.0050	0.00004773	0.00004821	0.00004773	$5.26713e-9$	$1.58514e-9$
0.0070	0.00000092	0.00000093	0.00000092	$3.18869e-10$	$4.25595e-11$
0.0090	0.00000002	0.00000002	0.00000002	$1.39981e-11$	$1.06632e-12$
0.0100	0.00000000	0.00000000	0.00000000	$2.73735e-12$	$1.66385e-13$
0.5000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000

errors are the same (symmetric) for the other half interval $\left[\frac{1}{2}, 1\right]$. Besides that, it is seen in these tables that the boundary conditions are exactly satisfied.

It is observed from the tables that the absolute errors in the two-term SCEM approximations are smaller than those reported in [32], particularly for the points where boundary layers occur. Around the turning point, even if the current results show similar trends to those obtained in [32], the results of [32] are slightly better in terms of absolute errors.

4.2 A non-homogeneous problem

Consider the following singularly perturbed non-homogeneous turning point problem [32]:

$$\varepsilon y''(x) - 2(2x-1)y'(x) - 4y(x) = 4\,(4x-1)\,, \quad x \in \Omega = (0,1) \tag{4.7}$$

Table 2. Comparison of the analytical solutions and SCEM approximations for solving Equations (4.1)–(4.2); $\varepsilon = 0.0001$.

x	Exact	y_0^{scem}	y_1^{scem}	Error in y_1^{scem}	Error [32]
0.00000	1.00000000	1.00000000	1.00000000	0.00000000	0.00000000
0.00001	0.81873239	0.81874879	0.81873242	$2.45926e-8$	$5.45561e-8$
0.00003	0.54882151	0.54885450	0.54882157	$5.02838e-8$	$1.09710e-7$
0.00005	0.36789784	0.36793469	0.36789789	$5.73031e-8$	$1.22569e-7$
0.00007	0.24662113	0.24665572	0.24662119	$5.50188e-8$	$1.15028e-7$
0.00009	0.16532567	0.16535548	0.16532572	$4.86479e-8$	$9.91402e-8$
0.00010	0.13536235	0.13538947	0.13536240	$4.48692e-8$	$9.01905e-8$
0.00030	0.00248322	0.00248471	0.00248322	$3.56050e-9$	$4.96264e-9$
0.00050	0.00004563	0.00004567	0.00004563	$1.66720e-10$	$1.51945e-10$
0.00070	0.00000084	0.00000084	0.00000084	$6.40360e-12$	$3.91413e-12$
0.00090	0.00000002	0.00000002	0.00000002	$2.15862e-13$	$9.27455e-14$
0.00100	0.00000000	0.00000000	0.00000000	$3.76910e-14$	$1.39978e-14$
0.5000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000

with Dirichlet boundary conditions

$$y(0) = 1, \;\; y(1) = 1. \tag{4.8}$$

The exact solution to the problem is available and given as follows:

$$y(x) = -2x + \left[\frac{\operatorname{erf}\left((2x-1)/\sqrt{2\varepsilon}\right)}{\operatorname{erf}\left(1/\sqrt{2\varepsilon}\right)} + 2 \right] \exp\left(-\frac{2x(1-x)}{\varepsilon} \right),$$

where $\operatorname{erf}(x)$ denotes the error function.
The problem has a simple turning point at $x_0 = \frac{1}{2}$, and the solution to the problem exhibits dual boundary layers at both endpoints of the interval $\Omega = (0,1)$. Let us divide the closure of interval Ω as

$$\Omega_{0^-} = \left[0, \frac{1-\delta}{2}\right], \Omega_{0^+} = \left[\frac{1-\delta}{2}, \frac{1}{2}\right], \Omega_{1^-} = \left[\frac{1}{2}, \frac{1+\delta}{2}\right], \Omega_{1^+} = \left[\frac{1+\delta}{2}, 1\right].$$

Substituting (4.3) into Equation (4.7) for Ω_{0^+}, one obtains

$$\varepsilon\left(y_0''(x,\varepsilon) + \varepsilon y_1''(x,\varepsilon)\right) - 2(2x-1)\left(y_0'(x,\varepsilon) + \varepsilon y_1'(x,\varepsilon)\right) \\ - 4\left(y_0(x,\varepsilon) + \varepsilon y_1(x,\varepsilon)\right) = 4(4x-1).$$

Balancing the terms of orders $O(1)$ and $O(\varepsilon)$, we obtain the following initial-value problems:

$$-2(2x-1)\,y_0'(x,\varepsilon) - 4y_0(x,\varepsilon) = 4(4x-1), \;\; y_0(0.5,\varepsilon) = -1,$$
$$y_0''(x,\varepsilon) - 2(2x-1)\,y_1'(x,\varepsilon) - 4y_1(x,\varepsilon) = 0, \;\; y_1(0.5,\varepsilon) = 0,$$

and the solutions are easily obtained as

$$y_0(x,\varepsilon) = -2x, \;\; y_1(x,\varepsilon) \equiv 0.$$

The first SCEM approximation is sought as

$$y_0^{sceml}(x,\overline{x},\varepsilon) = -2x + \Psi_0(\overline{x},\varepsilon),$$

where the boundary conditions are prescribed as

$$y_0^{sceml}(0,0,\varepsilon) = 1,\ \ y_0^{sceml}(1/2,1/2\varepsilon,\varepsilon) = -1.$$

Here, the stretching variable, $\overline{x}$, is defined as $\overline{x} = \frac{x}{\varepsilon}$ for the left boundary layer. The term y_0^{sceml} denotes the first SCEM approximation found for the left part of interval $[0.0, 1.0]$. One can reach the boundary conditions for the complementary functions as follows: $\Psi(\overline{x}=0,\varepsilon)=1,\ \Psi(\overline{x}=1/2\varepsilon,\varepsilon)=0$.

In order to obtain the complementary functions, let us propose the asymptotic expansion given by (4.4) for interval $\Omega_{0^-} = \left[0, \frac{1-\delta}{2}\right]$. Substituting (4.4) into Equation (4.7) and employing the chain rule (see Equations (4.5)–(4.6)), one can obtain

$$\begin{aligned}(\Psi_0''(\overline{x},\varepsilon) + \varepsilon\Psi_1''(\overline{x},\varepsilon)) - 2(2\overline{x}-1)(\Psi_0'(\overline{x},\varepsilon) + \varepsilon\Psi_1'(\overline{x},\varepsilon))\\ - 4\varepsilon(\Psi_0(\overline{x},\varepsilon) + \varepsilon\Psi_1(\overline{x},\varepsilon)) = 4\varepsilon(4\overline{x}-1).\end{aligned} \tag{4.9}$$

Balancing the terms of orders $O(1)$ and $O(\varepsilon)$, we obtain the following boundary-value problems:

$$\begin{aligned}&\Psi_0''(\overline{x},\varepsilon) - 2(2\overline{x}-1)\Psi_0'(\overline{x},\varepsilon) = 0,\\ &\Psi_0(0,\varepsilon) = 1 \text{ and } \Psi_0(1/2\varepsilon,\varepsilon) = 0,\end{aligned}$$

and

$$\begin{aligned}&\Psi_1''(\overline{x},\varepsilon) - 2(2\overline{x}-1)\Psi_1'(\overline{x},\varepsilon) - 4\Psi_0(\overline{x},\varepsilon) = 4(4\overline{x}-1),\\ &\Psi_1(0,\varepsilon) = 0 \text{ and } \Psi_1(1/2\varepsilon,\varepsilon) = 0.\end{aligned}$$

Consequently, since $y_0(x,\varepsilon) = -2x$ and $y_1(x,\varepsilon) \equiv 0$, we compute the first two iterations of the uniformly valid SCEM approximation as follows:

$$y_1^{sceml}(x,\overline{x},\varepsilon) = -2x + \Psi_0(\overline{x},\varepsilon) + \varepsilon\Psi_1(\overline{x},\varepsilon).$$

In Figure 3, the exact solution and the SCEM approximations are compared for $\varepsilon = 10^{-2}$. The difference in the numerical solutions is almost indistinguishable. Figure 4 shows the absolute errors in the SCEM approximations for $\varepsilon = 10^{-2}$. The two-term SCEM approximation is clearly better than the one with one term.

In Table 3, the results are compared for $\varepsilon = 10^{-3}$. Note that only the half interval $\left[0, \frac{1}{2}\right]$ is considered because the errors are the same (symmetric) for the other half $\left[\frac{1}{2}, 1\right]$. Furthermore, it is obvious from Table 3 that the boundary conditions hold exactly.

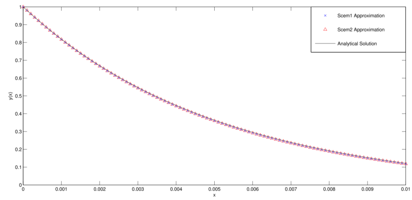

Figure 3. Comparison of the analytical solution and SCEM approximations in solving left symmetric problem for Equations (4.7)–(4.8); $\varepsilon = 0.01$. The analytical solution and SCEM approximations are almost identical.

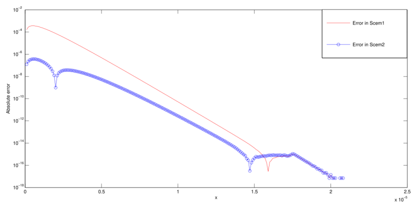

Figure 4. Absolute errors in SCEM approximations y_0^{sceml} and y_1^{sceml} around the boundary layer in solving non-homogeneous problem (4.7)–(4.8); $\varepsilon = 0.01$. The second SCEM approximation is superior over the first one.

Table 3. Comparison of the analytical solutions and SCEM approximations for solving Equations (4.7)–(4.8); $\varepsilon = 0.001$.

x	Exact	y_0^{scem}	y_1^{scem}	Error in y_1^{scem}	Error [32]
0.0000	1.00000000	1.00000000	1.00000000	0.00000000	0.00000000
0.0001	0.81854713	0.81871126	0.81854700	$1.23651e-7$	$1.69213e-5$
0.0003	0.54831043	0.54864068	0.54831013	$2.97120e-7$	$9.98939e-5$
0.0005	0.36706343	0.36743265	0.36706306	$3.66469e-7$	$1.85214e-4$
0.0007	0.24543875	0.24578555	0.24543839	$3.57883e-7$	$2.42936e-4$
0.0009	0.16376689	0.16406609	0.16376658	$3.04561e-7$	$2.68995e-4$
0.0010	0.13360622	0.13387857	0.13360595	$2.69853e-7$	$2.71846e-4$
0.0030	−0.00347623	−0.00346096	−0.00347619	$3.57762e-8$	$4.50719e-5$
0.0050	−0.00995227	−0.00995179	−0.00995227	$5.26714e-9$	$2.32929e-6$
0.0070	−0.01399908	−0.01399907	−0.01399908	$3.18873e-10$	$8.56591e-8$
0.0090	−0.01799998	−0.01799998	−0.01799998	$1.39983e-11$	$2.67941e-9$
0.0100	−0.20000000	−0.20000000	−0.20000000	$2.73738e-12$	$4.56511e-10$
0.5000	−1.00000000	−1.00000000	−1.00000000	0.00000000	0.00000000

5 Conclusions

A hybrid method has been developed to solve singularly perturbed turning point problems with dual boundary layers with a turning point. In this hybrid method, the domain of the original problem is split into sub-intervals first, and the asymptotic approach, SCEM, is employed. Later, a numerical technique is used to approximate the solution to the problem on these sub-intervals. One homogeneous and non-homogeneous problem is provided to test the method's capabilities.

The proposed method performs slightly better near the boundary layers than it does around the turning points for homogeneous problems, according to the results of the test computations. Regarding the non-homogeneous problem, the current method yields good approximations for both regions. Consequently, the present method is numerically efficient and well-suited for singularly perturbed dual boundary layers problems with simple turning points.

The hybrid method studied in this work can be modified to solve singularly perturbed turning point problems containing partial or fractional-order derivatives.

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