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Generalised Two-Component Modified Weakly Dissipative Dullin-Gottwald-Holm System: Invariance Analysis and Conservation Laws

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Abstract. The Dullin-Gottwald-Holm equation models the unidirectional propagation of shallow regime water waves. In this work, the Lie symmetry analysis of the generalised two-component modified weakly dissipative Dullin-Gottwald-Holm system is performed. Using symmetry reduction, the exact solutions are obtained in the form of power series and trigonometric functions. Also using multiplier approach, the conservation laws are obtained. The 3D graphical representations are also shown for obtained solutions.

Keywords: weakly dissipative Dullin-Gottwald-Holm system, Lie symmetries, exact solutions, conservation laws.

AMS Subject Classification: 35B06; 35C07; 35C09; 35C10; 35C11; 35G50.

1 Introduction

Dullin et al. [7] discussed the $1+1$ quadratically nonlinear equation

$$u_t+(c_0+3u)u_x+\beta u_{xxx}-\alpha^2(2u_xu_{xx}+uu_{xxx}+u_{xxt})=0, \quad x\in\mathbb{R},\ t>0, \quad (1.1)$$

known as the Dullin-Gottwald-Holm (DGH) equation. Here, u represents the fluid velocit in x direction; the constants $\frac{\beta}{c_0}$ and α^2 are squares of length scales, and $c_0=\sqrt{gh}>0$ is the linear wave speed for undisturbed water at rest at spatial infinity where g is the gravitational constant and h is the mean fluid depth. DGH equation models the unidirectional propagation of shallow water waves over a flat bottom. For $\alpha = 0$, it becomes the Korteweg-de Vries equation; while for $\beta = 0, \alpha = 1$, (1.1) recovers the well known Camassa-Holm (CH) equation.

The CH equation arises while studying some non-Newtonian fluids. It models small amplitude, finite length radial deformation waves in hyperelastic rods. This completely integrable equation has bi-Hamiltonian structure and so, it possesses infinitely many conservation laws [2]. It's property of presence of breaking waves has attracted lot of attention [6]. For $\beta = 0, \alpha = 1, c_0 = 2k$ (k an arbitrary constant), (1.1) recovers the Fuchssteiner-Fokas-Camassa-Holm equation, which has "peakon" solitary wave solution [5].

Lie group method [1,19] is a powerful method to find the invariant solutions of system of nonlinear differential equations. This method is used for finding the symmetries, for symmetry reduction [15] and for finding the invariant solutions of system of nonlinear partial differential equations (PDEs) [11,12,13,14]. By using this method, the invariant solutions of the DGH equation have been found by Gupta and Anupma [9] as well as, the invariant solutions of the dissipative DGH equation have been found by Wei and Wang [24].

Guo et al. [8] followed the Ivanov's approach in the presence of a linear shear flow and non-zero vorticity to derive the following two-component DGH system [3,4] as

$$\begin{aligned} &u_t + (c_0 + 3u)u_x + \beta u_{xxx} - \alpha^2(2u_x u_{xx} + u u_{xxx} + u_{xxt}) + \rho\rho_x = 0, \\ &\rho_t + (\rho u)_x = 0. \end{aligned} \tag{1.2}$$

To allow the dependence on the average density $\bar{\rho}$ as well as the pointwise density ρ, this system (1.2) has been modified to modified two-component DGH system [25] as

$$\begin{aligned} &u_t + (c_0 + 3u)u_x + \beta u_{xxx} - \alpha^2(2u_x u_{xx} + u u_{xxx} + u_{xxt}) - \gamma\rho\bar{\rho}_x = 0, \\ &\rho_t + (\rho u)_x = 0. \end{aligned}$$

Here, $\rho = \left(1 - \partial_x^2\right)(\bar{\rho} - \bar{\rho_0})$, with $\bar{\rho_0}$ to be constant and γ is the downward acceleration of gravity in applications to shallow water waves.

It is important to include energy dissipation mechanisms in experiments for real waves. Therefore, Tian [21] considered weakly dissipative modified two-component DGH system (mDGH2) as

$$\begin{aligned} &u_t + (c_0 + 3u)u_x + \beta u_{xxx} - \alpha^2(2u_x u_{xx} + u u_{xxx} + u_{xxt}) \\ &\quad + \lambda(u - \alpha^2 u_{xx}) - \gamma\rho\bar{\rho}_x = 0, \\ &\rho_t + (\rho u)_x + \lambda\rho = 0, \end{aligned}$$

with weakly dissipative terms $\lambda(u - \alpha^2 u_{xx})$ and $\lambda\rho$, where λ is a dissipative parameter. Tian derived asymptotic behaviour [21] and infinite propagation speed [22] of weakly dissipative mDGH2 system. The blow up phenomena of weakly dissipative mDGH2 system has also been derived by Tian et al. [23].

In this paper, weakly dissipative mDGH2 system is considered in a generalised form as

$$\begin{aligned}&u_t - \alpha^2 u_{xxt} + (c_0 + 3u)u_x + \beta u_{xxx} - \sigma\alpha^2(2u_x u_{xx} + u u_{xxx})\\&\qquad + \lambda(u - \alpha^2 u_{xx}) - \gamma\rho\bar{\rho}_x = 0,\\&\rho_t + (\rho u)_x + \lambda\rho = 0.\end{aligned} \tag{1.3}$$

This system includes several classical shallow water wave models. For example, if $\sigma = 1$, this system becomes the weakly dissipative mDGH2 system.

Let $w = \bar{\rho} - \bar{\rho}_0$ in the above system (1.3), then it can be transformed as

$$\begin{aligned}&u_t - \alpha^2 u_{xxt} + (c_0 + 3u)u_x + \beta u_{xxx} - \sigma\alpha^2(2u_x u_{xx} + u u_{xxx})\\&\quad + \lambda(u - \alpha^2 u_{xx}) - \gamma w_x(w - w_{xx}) = 0,\\&w_t - w_{xxt} + u(w_x - w_{xxx}) + u_x(w - w_{xx}) + \lambda(w - w_{xx}) = 0.\end{aligned} \tag{1.4}$$

The symmetries of generalised weakly dissipative mDGH2 system (1.4) are obtained by using Lie symmetry method. The exact solutions of system (1.4) are found in the form of power series [16] and other exact solutions in the form of hyperbolic and trigonometric functions. The separated and combined 3D plots are shown for the obtained solutions. The conservation laws [10] are also derived by using the multiplier approach [17, 18].

2 Symmetry reduction

The Lie classical method [1, 12, 19] is used in this section for symmetry analysis in order to solve the above system of nonlinear PDEs (1.4).

Proposition 1. *Equation* (1.4) *admits the following Lie symmetry operators:*

$$V_1 = \partial_x, \quad V_2 = \partial_t.$$

Proof. Consider the Lie group of point transformations

$$\begin{aligned}x^* &= x + \epsilon\xi(x,t,u,w) + O(\epsilon^2), \quad t^* = t + \epsilon\tau(x,t,u,w) + O(\epsilon^2),\\u^* &= u + \epsilon\eta(x,t,u,w) + O(\epsilon^2), \quad w^* = w + \epsilon\zeta(x,t,u,w) + O(\epsilon^2),\end{aligned} \tag{2.1}$$

such that if u, w satisfy (1.4), then u^*, w^* also satisfy (1.4). By using the above group of transformations (2.1) in system of PDEs (1.4), the invariance conditions are

$$\begin{aligned}&\eta^t - \alpha^2\eta^{xxt} + \eta^x(c_0 + 3u - 2\sigma\alpha^2 u_{xx}) - \alpha^2\eta^{xx}(\lambda + 2\sigma u_x)\\&\quad + \eta^{xxx}(\beta - \sigma\alpha^2 u) + \eta(3u_x + \lambda - \alpha^2\sigma u_{xxx}) = 0,\\&\zeta w_x - \zeta^{xx} w_x + \zeta^x(w - w_{xx}) = 0.\end{aligned}$$

By substituting the values of extended infinitesimals, and equating the coefficients of differentials of u and w to 0, obtain a set of determining equations.

The infinitesimals $\xi,\ \tau,\ \eta,\ \zeta$ obtained by solving the determining equations are found as

$$\xi = C_1, \quad \tau = C_2, \quad \eta = 0, \quad \zeta = 0,$$

where $C_1,\ C_2$ are arbitrary constants. The corresponding vector fields are $V_1 = \partial_x$, $V_2 = \partial_t$. $\square$

On considering the vector field $V_2+\epsilon V_1$, the similarity variables are obtained as

$$r = x - \epsilon t, \quad u = g(r), \quad w = h(r),$$

where r is new independent variable and $g,\ h$ are new dependent functions. Back substituting these variables in equation (1.4), the reduced system of ODEs is obtained as

$$\begin{aligned}&(\alpha^2\epsilon+\beta-\sigma\alpha^2 g)g'''-\alpha^2(2\sigma g'+\lambda)g''+(3g+c_0-\epsilon)g'+\lambda g+\gamma(h''-h)h' = 0,\\ &(\epsilon - g)h''' - (\lambda + g')h'' + (g-\epsilon)h' + (g'+\lambda)h = 0, \end{aligned} \tag{2.2}$$

where $'$ denotes the differentiation with respect to r.

3 Exact solutions

In this part, the exact solutions of system of ODEs (2.2) are obtained in the form of power series and other exact solutions in the form of hyperbolic and trigonometric functions.

3.1 Power series solutions

Let power series solution [16] be of the form

$$g(r) = \sum_{m=0}^{\infty} a_m r^m, \quad h(r) = \sum_{m=0}^{\infty} b_m r^m, \tag{3.1}$$

for system of ODEs (2.2). Computing third order derivatives, product terms and substituting these computed values in (2.2), it can be expressed generally as

$$\begin{aligned}&-\sigma\alpha^2 \sum_{m=1}^{\infty}\left(\sum_{k=0}^{m}(m-k+1)(m-k+2)(m-k+3)a_k a_{m-k+3}\right)r^m + 3a_0a_1\\ &+3\sum_{m=1}^{\infty}\left(\sum_{k=0}^{m}(m-k+1)a_k a_{m-k+1}\right)r^m - \alpha^2\lambda\sum_{m=1}^{\infty}(m+1)(m+2)a_{m+2}r^m\\ &+\lambda a_0 - 2\sigma\alpha^2\sum_{m=1}^{\infty}\left(\sum_{k=0}^{m}(k+1)(m-k+1)(m-k+2)a_{k+1}a_{m-k+2}\right)r^m\\ &+(c_0-\epsilon)\sum_{m=1}^{\infty}(m+1)a_{m+1}r^m + (\alpha^2\epsilon+\beta)\sum_{m=1}^{\infty}(m+1)(m+2)(m+3)a_{m+3}r^m\end{aligned}$$

$$
\begin{aligned}
&+6a_3(\alpha^2\epsilon+\beta)+\gamma\sum_{m=1}^{\infty}\left(\sum_{k=0}^{m}(k+1)(m-k+1)(m-k+2)b_{k+1}b_{m-k+2}\right)r^m \\
&+2b_1b_2\gamma-a_0a_1\gamma-\gamma\sum_{m=1}^{\infty}\left(\sum_{k=0}^{m}(m-k+1)b_kb_{m-k+1}\right)r^m+(c_0-\epsilon)a_1 \\
&-6\sigma\alpha^2a_0a_3+\lambda\sum_{m=1}^{\infty}a_mr^m-4\sigma\alpha^2a_1a_2-2a_2\alpha^2\lambda=0,
\end{aligned}
\tag{3.2a}
$$

$$
\begin{aligned}
&-6a_0b_3-\sum_{m=1}^{\infty}\left(\sum_{k=0}^{m}(m-k+1)(m-k+2)(m-k+3)a_kb_{m-k+3}\right)r^m+a_0b_1 \\
&+\sum_{m=1}^{\infty}\left(\sum_{k=0}^{m}(m-k+1)a_kb_{m-k+1}\right)r^m-2\lambda b_2-\lambda\sum_{m=1}^{\infty}(m+1)(m+2)b_{m+2}r^m \\
&-\sum_{m=1}^{\infty}\left(\sum_{k=0}^{m}(k+1)(m-k+1)(m-k+2)a_{k+1}b_{m-k+2}\right)r^m \\
&+\epsilon\sum_{m=1}^{\infty}(m+1)(m+2)(m+3)b_{m+3}r^m+\sum_{m=1}^{\infty}\left(\sum_{k=0}^{m}(m-k+1)b_ka_{m-k+1}\right)r^m \\
&+6\epsilon b_3+a_1b_0-\epsilon b_1+\lambda b_0+\lambda\sum_{m=1}^{\infty}b_mr^m-2a_1b_2-\epsilon\sum_{m=1}^{\infty}(m+1)b_{m+1}r^m=0.
\end{aligned}
\tag{3.2b}
$$

From above Equations (3.2a) and (3.2b) collecting coefficients of r^m, for $m=0$, it can be observed that

$$
\begin{aligned}
a_3&=\frac{\big(\lambda a_0+(c_0-\epsilon+3a_0-a_0\gamma)a_1+2b_1b_2\gamma-2\alpha^2a_2(\lambda+2a_1\sigma)\big)}{6\alpha^2(a_0\sigma-\epsilon)-6\beta}, \\
b_3&=\frac{1}{6(a_0-\epsilon)}\left((a_0-\epsilon)b_1-2(\lambda+a_1)b_2+(\lambda+a_1)b_0\right).
\end{aligned}
\tag{3.3}
$$

In general, for $m\geq 1$, it can be seen that

$$
\begin{aligned}
a_{m+3}=&\frac{1}{(\alpha^2(a_0\sigma-\epsilon)-\beta)(m+1)(m+2)(m+3)}\Big(3\sum_{k=0}^{m}(m-k+1)a_ka_{m-k+1} \\
&-\sigma\alpha^2\sum_{k=1}^{m}(m-k+1)(m-k+2)(m-k+3)a_ka_{m-k+3}+\lambda a_m \\
&-2\sigma\alpha^2\sum_{k=0}^{m}(k+1)(m-k+1)(m-k+2)a_{k+1}a_{m-k+2}
\end{aligned}
$$

$$+ (c_0-\epsilon)(m+1)a_{m+1} + \gamma\sum_{k=0}^{m}(k+1)(m-k+1)(m-k+2)b_{k+1} \times b_{m-k+2} - \gamma\sum_{k=0}^{m}(m-k+1)b_k b_{m-k+1} - \alpha^2\lambda(m+1)(m+2)a_{m+2}\Big), \tag{3.4a}$$

$$b_{m+3} = \frac{1}{(a_0-\epsilon)(m+1)(m+2)(m+3)}\Big(-\sum_{k=0}^{m}(k+1)(m-k+1)(m-k+2) \times a_{k+1}b_{m-k+2} - \sum_{k=1}^{m}(m-k+1)(m-k+2)(m-k+3)a_k b_{m-k+3} + \sum_{k=0}^{m}(m-k+1)a_k b_{m-k+1} + \lambda b_m - \lambda(m+1)(m+2)b_{m+2} - \epsilon(m+1)b_{m+1} + \sum_{k=0}^{m}(m-k+1)b_k a_{m-k+1}\Big). \tag{3.4b}$$

Thus, all the coefficients a_m, b_m, for $m \geq 4$ of power series (3.1) can be obtained from (3.3) and (3.4) as follows

$$a_4 = \frac{1}{24\alpha^2(a_0\sigma-\epsilon)-24\beta}\Big(\lambda a_1 - 6\alpha^2(3\sigma a_1 a_3 + 2\sigma a_2^2 + \lambda a_3) + 2(c_0-\epsilon+3a_0)a_2 + 3a_1^2 + (6b_1b_3 + 4b_2^2 - 2b_0b_2 - b_1^2)\gamma\Big),$$

$$b_4 = \frac{1}{24(a_0-\epsilon)}\Big(2(a_0-\epsilon)b_2 - 12a_1b_3 + 2a_1b_1 + \lambda(b_1-6b_3) + 2(b_0-2b_2)a_2\Big),$$

$$a_5 = \frac{1}{60\alpha^2(a_0\sigma-\epsilon)-60\beta}\Big(\lambda a_2 - 6\alpha^2(8\sigma a_1a_4 + 7\sigma a_2a_3 + 2\lambda a_4) + 9a_1a_2 + 3(c_0-\epsilon+3a_0)a_3 + 3(6b_2b_3 - b_1b_2 - b_0b_3 + 4b_1b_4)\gamma\Big),$$

$$b_5 = \frac{1}{60(a_0-\epsilon)}\Big(3(a_0-\epsilon)b_3 + 3a_1(b_2-12b_4) + 3a_2(b_1-6b_3) + \lambda(b_2-12b_4) + 3(b_0-2b_2)a_3\Big),$$

and so on. Thus, for arbitrary chosen constants a_0, a_1, a_2 and b_0, b_1, b_2, the other terms of the sequences $\{a_m\}_{m=0}^{\infty}$ and $\{b_m\}_{m=0}^{\infty}$ can be determined successively from (3.3) and (3.4) in a unique manner. Therefore,

$$g(r) = a_0 + a_1r + a_2r^2 + \frac{1}{6\alpha^2(a_0\sigma-\epsilon)-6\beta} \times \big(\lambda a_0 + (c_0-\epsilon+3a_0-a_0\gamma)a_1 + 2b_1b_2\gamma - 2\alpha^2a_2(\lambda+2a_1\sigma)\big)r^3 + \ldots,$$

$$h(r) = b_0 + b_1r + b_2r^2 + \frac{1}{6(a_0-\epsilon)}\left((a_0-\epsilon)b_1 - 2(\lambda+a_1)b_2 + (\lambda+a_1)b_0\right)r^3 + \ldots.$$

Convergence of power series

Now we will check the convergence of the power series solution (3.1) of system of ODEs (2.2). From (3.4a), it can be observed that

$$\begin{aligned}|a_{m+3}| \leq & A\Bigg(\sum_{k=0}^{m}|a_k||a_{m-k+1}| + |a_m| + \sum_{k=1}^{m}|a_k||a_{m-k+3}| + |a_{m+2}| \\ & + \sum_{k=0}^{m}|a_{k+1}||a_{m-k+2}| + |a_{m+1}| + \sum_{k=0}^{m}|b_{k+1}||b_{m-k+2}| \\ & + \sum_{k=0}^{m}|b_k||b_{m-k+1}|\Bigg), \quad m=1,2,3,\ldots,\end{aligned}$$

where

$$\begin{aligned}A = \max\Bigg\{ & \frac{|\lambda|}{|\alpha^2(a_0\sigma-\epsilon)-\beta|}, \frac{|2\alpha^2\sigma|}{|\alpha^2(a_0\sigma-\epsilon)-\beta|}, \frac{|c_0-\epsilon|}{|\alpha^2(a_0\sigma-\epsilon)-\beta|}, \\ & \frac{|\alpha^2\lambda|}{|\alpha^2(a_0\sigma-\epsilon)-\beta|}, \frac{|\gamma|}{|\alpha^2(a_0\sigma-\epsilon)-\beta|}, \frac{|3|}{|\alpha^2(a_0\sigma-\epsilon)-\beta|}\Bigg\}.\end{aligned}$$

Similarly, from (3.4b), it can be observed that

$$\begin{aligned}|b_{m+3}| \leq & B\Bigg(\sum_{k=0}^{m}|a_{k+1}||b_{m-k+2}| + \sum_{k=1}^{m}|a_k||b_{m-k+3}| + \sum_{k=0}^{m}|a_k||b_{m-k+1}| \\ & + |b_{m+2}| + |b_{m+1}| + \sum_{k=0}^{m}|b_k||a_{m-k+1}|\Bigg), \quad m=1,2,3,\ldots,\end{aligned}$$

where $B = \max\{|\lambda|/|a_0-\epsilon|, |\epsilon|/|a_0-\epsilon|, |1|/|a_0-\epsilon|\}$. Defining other power series as

$$\mu = P(r) = \sum_{m=0}^{\infty} p_m r^m, \quad \nu = Q(r) = \sum_{m=0}^{\infty} q_m r^m, \tag{3.5}$$

such that $p_i = |a_i|$ and $q_i = |b_i|$ for $i = 0,1,2,3$ and

$$\begin{aligned}p_{m+3} = & A\Bigg(\sum_{k=0}^{m} p_k p_{m-k+1} + p_m + \sum_{k=1}^{m} p_k p_{m-k+3} + p_{m+2} + \sum_{k=0}^{m} p_{k+1} p_{m-k+2} \\ & + p_{m+1} + \sum_{k=0}^{m} q_{k+1} q_{m-k+2} + \sum_{k=0}^{m} q_k q_{m-k+1}\Bigg), \\ q_{m+3} = & B\Bigg(\sum_{k=0}^{m} p_{k+1} q_{m-k+2} + \sum_{k=1}^{m} p_k q_{m-k+3} + \sum_{k=0}^{m} p_k q_{m-k+1} + q_m \\ & + q_{m+2} + q_{m+1} + \sum_{k=0}^{m} q_k p_{m-k+1}\Bigg),\end{aligned}$$

for $m = 1,2,3,\ldots$. Then, it can be easily seen that $|a_m| \leq |p_m|$, $|b_m| \leq |q_m|$, for $m = 0,1,2,\ldots$ Therefore, the series $\mu = P(r) = \sum_{m=0}^{\infty} p_m r^m$ and $\nu = Q(r) = \sum_{m=0}^{\infty} q_m r^m$ are the majorant series of (3.1).

Now, it remains to show that these series (3.5) have a positive radius of convergence. It can be calculated that

$$\begin{aligned}
P(r) &= p_0 + p_1 r + p_2 r^2 + p_3 r^3 + \sum_{m=1}^{\infty} p_{m+3} r^{m+3} \\
&= p_0 + p_1 r + p_2 r^2 + p_3 r^3 + A\Big(\sum_{m=1}^{\infty}\sum_{k=0}^{m} p_k p_{m-k+1} r^{m+3} + \sum_{m=1}^{\infty} p_m r^{m+3} \\
&\quad + \sum_{m=1}^{\infty}\sum_{k=1}^{m} p_k p_{m-k+3} r^{m+3} + \sum_{m=1}^{\infty} p_{m+2} r^{m+3} + \sum_{m=1}^{\infty}\sum_{k=0}^{m} p_{k+1} p_{m-k+2} r^{m+3} \\
&\quad + \sum_{m=1}^{\infty} p_{m+1} r^{m+3} + \sum_{m=1}^{\infty}\sum_{k=0}^{m} q_{k+1} q_{m-k+2} r^{m+3} + \sum_{m=1}^{\infty}\sum_{k=0}^{m} q_k q_{m-k+1} r^{m+3}\Big) \\
&= p_0 + p_1 r + p_2 r^2 + p_3 r^3 + A\Big(2P^2 - 2p_0 P + Q^2 - q_0 Q \\
&\quad + (P - p_0 - 2p_0 p_1 - q_0 q_1) r + (P^2 + P - p_0^2 - p_0 - p_1 + Q^2 - q_0^2) r^2 \\
&\quad + (-q_1^2 - q_0 q_2 - 2p_0 p_2 - 2p_1^2 - 4p_0 p_3 - 4p_1 p_2) r^2 \\
&\quad + (P - p_0 + 2p_0 p_3 - 2p_0 p_1 - p_1 - p_2 - 2q_0 q_3 - 2q_1 q_2 - q_0 q_3 - 2q_0 q_1) r^3\Big),
\end{aligned}$$

and

$$\begin{aligned}
Q(r) &= q_0 + q_1 r + q_2 r^2 + q_3 r^3 + \sum_{m=1}^{\infty} q_{m+3} r^{m+3} \\
&= q_0 + q_1 r + q_2 r^2 + q_3 r^3 + B\Big(\sum_{m=1}^{\infty}\sum_{k=0}^{m} p_{k+1} q_{m-k+2} r^{m+3} \\
&\quad + \sum_{m=1}^{\infty}\sum_{k=0}^{m} p_k q_{m-k+1} r^{m+3} + \sum_{m=1}^{\infty} q_{m+2} r^{m+3} + \sum_{m=1}^{\infty} q_{m+1} r^{m+3} \\
&\quad + \sum_{m=1}^{\infty}\sum_{k=0}^{m} q_k p_{m-k+1} r^{m+3} + \sum_{m=1}^{\infty}\sum_{k=1}^{m} p_k q_{m-k+3} r^{m+3} + \sum_{m=1}^{\infty} q_m r^{m+3}\Big) \\
&= q_0 + q_1 r + q_2 r^2 + q_3 r^3 + B\Big(2PQ - 2p_0 Q + (Q - q_0 - 2p_1 q_0) r \\
&\quad + (2PQ + Q - q_0 - q_1 - 2q_0 p_2 - 2p_1 q_1 - 2p_0 q_0) r^2 \\
&\quad + (Q - q_0 - q_1 - q_2 - 2p_0 q_1 - 2q_1 p_2 - 2q_0 p_3 - 2q_2 p_1 - 2p_1 q_0) r^3\Big).
\end{aligned}$$

Now, considering the implicit functional equations in the form of independent variable r as

$$\begin{aligned}
F(r,\mu,\nu) &= \mu - p_0 - p_1 r - p_2 r^2 - p_3 r^3 - A\Big(2\mu^2 - 2p_0\mu + \nu^2 - q_0\nu \\
&\quad + (\mu - p_0 - 2p_0 p_1 - q_0 q_1) r + \Big(\mu^2 + \mu - p_0^2 - p_0 - p_1 + \nu^2 - q_0^2 \\
&\quad - q_1^2 - q_0 q_2 - 2p_0 p_2 - 2p_1^2 - 4p_0 p_3 - 4p_1 p_2\Big) r^2 + (-p_0 + 2p_0 p_3) r^3
\end{aligned}$$

$$+\left(\mu-2p_0p_1-p_1-p_2-2q_0q_3-2q_1q_2-q_0q_3-2q_0q_1)r^3\right),$$

$$G(r,\mu,\nu)=\nu-q_0-q_1r-q_2r^2-q_3r^3-B\Big(2PQ-2p_0Q+(Q-q_0-2p_1q_0)r + (2PQ+Q-q_0-q_1-2q_0p_2-2p_1q_1-2p_0q_0)r^2 + (Q-q_0-q_1-q_2-2p_0q_1-2q_1p_2-2q_0p_3-2q_2p_1-2p_1q_0)r^3\Big).$$

Since $F(r,\mu,\nu)$ and $G(r,\mu,\nu)$ are analytic in the neighborhood of $(0,p_0,q_0)$ and $F(0,p_0,q_0)=0,\ G(0,p_0,q_0)=0$. Furthermore, the Jacobian determinant

$$J=\frac{\partial(F,G)}{\partial(\mu,\nu)}=1-2Ap_0-2ABq_0^2\neq 0,$$

if the parameters $p_0=|a_0|$ and $q_0=|b_0|$ are chosen suitably. By the implicit function theorem [20], it can be seen that $\mu=P(r)$ and $\nu=Q(r)$ are analytic in a neighborhood of point $(0,p_0,q_0)$. This shows that both the power series (3.1) converge in a neighborhood of point $(0,p_0,q_0)$. This completes the proof. Thus, the exact analytic power series solution of generalised weakly dissipative mDGH2 system (1.4) can be written as

$$\begin{aligned} u(x,t)&=a_0+a_1(x-\epsilon t)+a_2(x-\epsilon t)^2+\frac{1}{6\alpha^2(a_0\sigma-\epsilon)-6\beta}\\ &\times\big(\lambda a_0+(c_0-\epsilon+3a_0-a_0\gamma)a_1+2b_1b_2\gamma-2\alpha^2a_2(\lambda+2a_1\sigma)\big)\\ &\times(x-\epsilon t)^3+\sum_{m=1}^{\infty}a_{m+3}(x-\epsilon t)^{m+3},\\ w(x,t)&=b_0+b_1(x-\epsilon t)+b_2(x-\epsilon t)^2+((a_0-\epsilon)b_1-2(\lambda+a_1)b_2+(\lambda+a_1)b_0)\\ &\times\frac{(x-\epsilon t)^3}{6(a_0-\epsilon)}+\sum_{m=1}^{\infty}b_{m+3}(x-\epsilon t)^{m+3}, \end{aligned}\tag{3.6}$$

where a_{m+3} and b_{m+3} are given by recurrence relation (3.4).

3.2 Travelling wave solutions

In order to obtain travelling wave solutions [9] of the weakly dissipative mDGH2 system (1.4), the different particular values of parameters of ODE system (2.2) are considered.

3.2.1 $\alpha=\pm\frac{1}{3},\ c_0=-9\beta,\ \sigma=1$

For $\alpha=\pm\frac{1}{3},\ c_0=-9\beta,\ \sigma=1$ and keeping all other parameters same, the ODE system (2.2) reduces to

$$\begin{aligned} &(\epsilon+9\beta-g)g'''-(2g'+\lambda)g''+9(3g-9\beta-\epsilon)g'+9\lambda g+9\gamma(h''-h)h'=0,\\ &(\epsilon-g)h'''-(\lambda+g')h''+(g-\epsilon)h'+(g'+\lambda)h=0. \end{aligned}\tag{3.7}$$

Equation (3.7) possesses travelling wave solutions as

(i) $\quad g(r)=C_3\iota\sinh(r)(4\cosh^2(r)-1),\quad h(r)=C_4\cosh(r),$

(ii) $\quad g(r)=C_7e^{\pm 3r+3C_6},\quad h(r)=C_5e^{\pm r+C_6},$

where ι represents an imaginary number, C_3, C_4, C_5, C_6 and C_7 are arbitrary constants. Thus, the travelling wave solutions of the mDGH2 system (1.4) are

$$\begin{aligned} &(i) \qquad u(x,t) = C_3 \iota \sinh(x-\epsilon t)(4\cosh^2(x-\epsilon t)-1), \\ &\qquad\quad w(x,t) = C_4 \cosh(x-\epsilon t), \\ &(ii) \qquad u(x,t) = C_7 e^{\pm 3(x-\epsilon t)+3C_6}, \quad w(x,t) = C_5 e^{\pm(x-\epsilon t)+C_6}. \end{aligned} \tag{3.8}$$

3.2.2 $\alpha = \pm 1,\ \beta = -c_0,\ \sigma = 1$

For $\alpha = \pm 1,\ \beta = -c_0,\ \sigma = 1$ and keeping all other parameters same, the ODE system (2.2) reduces to

$$\begin{aligned} &(\epsilon - c_0 - g)g''' - (2g' + \lambda)g'' + (3g + c_0 - \epsilon)g' + \lambda g + \gamma(h'' - h)h' = 0, \\ &(\epsilon - g)h''' - (\lambda + g')h'' + (g - \epsilon)h' + (g' + \lambda)h = 0. \end{aligned} \tag{3.9}$$

Equation (3.9) possesses travelling wave solutions as

$$\begin{aligned} &(i) \qquad g(r) = C_3 \cosh^3\left(\frac{r}{3}\right) - \frac{3}{4} C_3 \cosh\left(\frac{r}{3}\right), \\ &\qquad\quad h(r) = C_4 \iota \sinh\left(\pm\frac{r}{3}\right)\left(4\cosh^2\left(\frac{r}{3}\right) - 1\right), \\ &(ii) \qquad g(r) = C_6 \sinh^3\left(\pm\frac{1}{3}r + C_7\right) + \frac{3}{4} C_6 \sinh\left(\pm\frac{1}{3}r + C_7\right), \\ &\qquad\quad h(r) = C_5 \sinh^3\left(\pm\frac{1}{3}r + C_7\right) + \frac{3}{4} C_5 \sinh\left(\pm\frac{1}{3}r + C_7\right), \end{aligned}$$

where C_3, C_4, C_5, C_6 and C_7 are arbitrary constants. Thus, the travelling wave solutions of the mDGH2 system (1.4) are

$$\begin{aligned} &(i) \quad u(x,t) = C_3 \cosh^3\left(\frac{x-\epsilon t}{3}\right) - \frac{3}{4} C_3 \cosh\left(\frac{x-\epsilon t}{3}\right), \\ &\qquad w(x,t) = C_4 \iota \sinh\left(\pm\frac{(x-\epsilon t)}{3}\right)\left(4\cosh^2\left(\frac{x-\epsilon t}{3}\right) - 1\right), \\ &(ii) \quad u(x,t) = C_6 \sinh^3\left(\pm\frac{(x-\epsilon t)}{3} + C_7\right) + \frac{3}{4} C_6 \sinh\left(\pm\frac{1}{3}(x-\epsilon t) + C_7\right), \\ &\qquad w(x,t) = C_5 \sinh^3\left(\pm\frac{(x-\epsilon t)}{3} + C_7\right) + \frac{3}{4} C_5 \sinh\left(\pm\frac{1}{3}(x-\epsilon t) + C_7\right). \end{aligned} \tag{3.10}$$

3.2.3 $\alpha = \pm\frac{3}{2},\ \beta = -\frac{9}{4}c_0,\ \sigma = 1$

For $\alpha = \pm\frac{3}{2},\ \beta = -\frac{9}{4}c_0,\ \sigma = 1$ and keeping all other parameters same, the ODE system (2.2) reduces to

$$\begin{aligned} &\frac{9}{4}(\epsilon - c_0 - g)g''' - \frac{9}{4}(2g' + \lambda)g'' + (3g + c_0 - \epsilon)g' + \lambda g + \gamma(h'' - h)h' = 0, \\ &(\epsilon - g)h''' - (\lambda + g')h'' + (g - \epsilon)h' + (g' + \lambda)h = 0. \end{aligned} \tag{3.11}$$

Equation (3.11) possesses travelling wave solutions as

$$g(r) = C_3 \sinh^2\left(\pm\frac{1}{3}r + C_2\right) + \frac{1}{2}C_3,$$
$$h(r) = C_4 \sinh^3\left(\pm\frac{1}{3}r + C_2\right) + \frac{3}{4}C_4 \sinh\left(\pm\frac{1}{3}r + C_2\right),$$

where C_2, C_3 and C_4 are arbitrary constants. Thus, the travelling wave solutions of the mDGH2 system (1.4) are

$$\begin{aligned} u(x,t) &= C_3 \sinh^2\left(\pm\frac{1}{3}(x-\epsilon t) + C_2\right) + \frac{1}{2}C_3, \\ w(x,t) &= C_4 \sinh^3\left(\pm\frac{(x-\epsilon t)}{3} + C_2\right) + \frac{3}{4}C_4 \sinh\left(\pm\frac{(x-\epsilon t)}{3} + C_2\right). \end{aligned} \tag{3.12}$$

3.2.4 $\alpha = \pm 3,\ \beta = -9c_0,\ \sigma = 1$

For $\alpha = \pm 3,\ \beta = -9c_0,\ \sigma = 1$ and keeping all other parameters same, the ODE system (2.2) reduces to

$$\begin{aligned} &9(\epsilon - c_0 - g)g''' - 9(2g' + \lambda)g'' + (3g + c_0 - \epsilon)g' + \lambda g + \gamma(h'' - h)h' = 0, \\ &(\epsilon - g)h''' - (\lambda + g')h'' + (g - \epsilon)h' + (g' + \lambda)h = 0. \end{aligned} \tag{3.13}$$

Equation (3.13) possesses travelling wave solutions as

$$g(r) = C_3 \cosh\left(\pm\frac{1}{3}r + C_2\right),$$
$$h(r) = C_4 \cosh^3\left(\pm\frac{1}{3}r + C_2\right) - \frac{3}{4}C_4 \cosh\left(\pm\frac{1}{3}r + C_2\right),$$

where C_2, C_3 and C_4 are arbitrary constants. Thus, the travelling wave solutions of the mDGH2 system (1.4) are

$$\begin{aligned} u(x,t) &= C_3 \cosh\left(\pm\frac{1}{3}(x-\epsilon t) + C_2\right), \\ w(x,t) &= C_4 \cosh^3\left(\pm\frac{(x-\epsilon t)}{3} + C_2\right) - \frac{3}{4}C_4 \cosh\left(\pm\frac{(x-\epsilon t)}{3} + C_2\right). \end{aligned} \tag{3.14}$$

4 Conservation laws

In this part, the local conservation laws [10] of generalised weakly dissipative mDGH2 system (1.4) are obtained by using the multiplier approach [17, 18]. Consider the multipliers of the form

$$\Lambda(x,t,u,w,u_x,w_x,u_{xx},w_{xx}), quad\psi(x,t,u,w,u_x,w_x,u_{xx},w_{xx}).$$

The simplified determining equations to be solved are

$$\Lambda_t = -\frac{1}{\gamma}\left((u\psi_w - \gamma\Lambda)\lambda\right), \quad \Lambda_u = -\frac{\psi_w}{\gamma}, \quad \psi_t = \psi_w\lambda w + \lambda\psi, \;\; \psi_{ww} = 0,$$
$$\Lambda_{u_{xx}} = 0, \;\; \Lambda_x = 0, \;\; \Lambda_{u_x} = 0, \;\; \Lambda_{w_{xx}} = 0, \;\; \Lambda_w = 0, \;\; \Lambda_{w_x} = 0,$$
$$\psi_{u_{xx}} = 0, \;\; \psi_x = 0, \;\; \psi_{u_x} = 0, \;\; \psi_{w_{xx}} = 0, \;\; \psi_u = 0, \;\; \psi_{w_x} = 0. \qquad (4.1)$$

The solution of above determining equations (4.1) yields three multipliers as

$$\Lambda^{(1)} = -\frac{u}{\gamma}e^{2\lambda t}, \;\; \psi^{(1)} = we^{2\lambda t}, \quad \Lambda^{(2)} = 0, \;\; \psi^{(2)} = e^{\lambda t},$$
$$\Lambda^{(3)} = e^{\lambda t}, \quad \psi^{(3)} = 0. \qquad (4.2)$$

Thus, the conserved fluxes in accordance to each multiplier (4.2) are

$$T_t^{(1)} = \frac{e^{2\lambda t}}{2\gamma}\left(-u^2 + \alpha^2 u u_{xx} + \gamma w^2 - \gamma w w_{xx}\right),$$
$$T_x^{(1)} = \frac{e^{2\lambda t}}{2\gamma}\Big(\alpha^2\sigma(2u^2u_{xx} + uu_{xt} - u_t u_x) + \beta(u_x^2 - 2uu_{xx} - u^2(c_0 + 2u))$$
$$+ \gamma(2uw^2 - 2uww_{xx} + w_x w_t - ww_{xt})\Big),$$
$$T_t^{(2)} = e^{\lambda t}(w - w_{xx}), \quad T_x^{(2)} = ue^{\lambda t}(w - w_{xx}), \quad T_t^{(3)} = e^{\lambda t}(u - \alpha^2 u_{xx}),$$
$$T_x^{(3)} = -\frac{e^{\lambda t}}{2}\left(\alpha^2\sigma(2uu_{xx} + u_x^2) + \gamma(w^2 - w_x^2) - 2\beta u_{xx} - u(2c_0 + 3u)\right).$$

5 Discussion

The solution (3.12) contains arbitrary constants C_2, C_3, C_4, ϵ which lead to many possible solutions. The graphical representations are shown by considering a possibility of these constants.

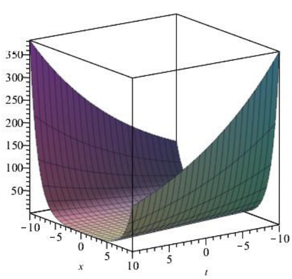

Figure 1. 3D plot of $u(x,t)$ of solution (3.12) by considering $C_2 = 0,\; C_3 = 1,\; C_4 = -\frac{1}{5},\; \epsilon = \frac{1}{10}$ and positive sign for angle.

Figures 1 and 2 depict the 3D graphs of solution (3.12) of (1.4) by considering $C_2 = 0,\; C_3 = 1,\; C_4 = -\frac{1}{5},\; \epsilon = \frac{1}{10}$ and positive sign for angle. The solution

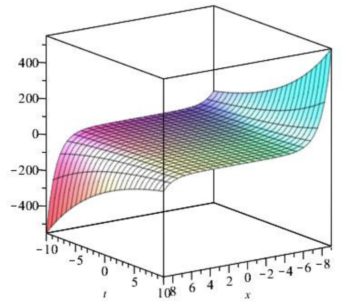

Figure 2. 3D plot of $w(x,t)$ of solution (3.12) by considering $C_2 = 0,\ C_3 = 1,\ C_4 = -\frac{1}{5},\ \epsilon = \frac{1}{10}$ and positive sign for angle.

(3.14) also contains arbitrary constants C_2, C_3, C_4, ϵ leading to infinite possible solutions. The graphical representations are shown by considering a possibility of these constants. Figures 3 and 4 depict the 3D graphs of solution (3.14) of (1.4) by considering $C_2 = 0,\ C_3 = -13,\ C_4 = 1,\ \epsilon = \frac{1}{100}$ and positive sign for angle. The relation of solutions $u(x,t)$ and $w(x,t)$ can also be observed from

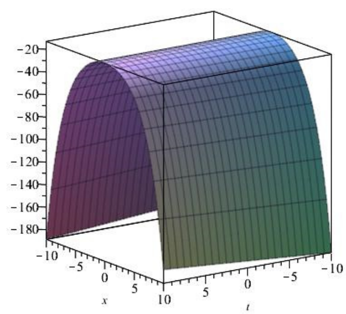

Figure 3. 3D plot of $u(x,t)$ of solution (3.14) by considering $C_2 = 0,\ C_3 = -13,\ C_4 = 1,\ \epsilon = \frac{1}{100}$ and positive sign for angle.

their combined 3D plot as in Figure 5. The overlapping and non-overlapping parts, the smoothness, non-singural regions of $u(x,t)$ and $w(x,t)$ can be examined easily, due to their different shading. The lining shade is used for $u(x,t)$ while plain shade is used for $w(x,t)$. Figure 5 represents combined solution (3.14) by considering $C_2 = 0,\ C_3 = 10,\ C_4 = -\frac{1}{12},\ \epsilon = \frac{1}{50}$ and positive sign for angle. For $w(x,t) = 0,\ \lambda = 0,\ \sigma = 1$, the solutions (3.6), (3.8), (3.10), (3.12), (3.14), are the new solutions of the DGH equation considered by Gupta and Anupma [9].

For $w(x,t) = 0,\ \alpha = \sigma = 1,\ \beta = -c_0$, the solutions (3.6), (3.10), are the new solutions of the dissipative DGH equation considered by Wei and Wang [24].

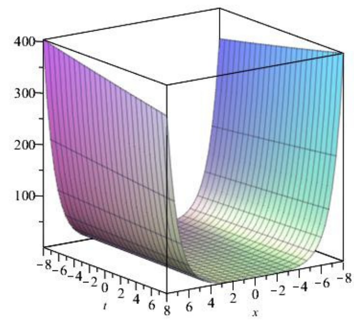

Figure 4. 3D plot of $w(x,t)$ of solution (3.14) by considering $C_2 = 0$, $C_3 = -13$, $C_4 = 1$, $\epsilon = \frac{1}{100}$ and positive sign for angle.

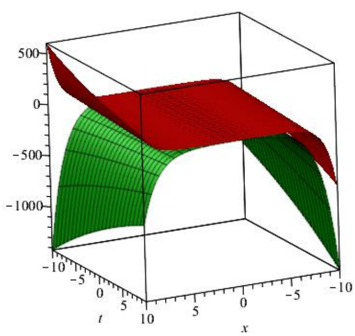

Figure 5. 3D plot of combined solution (3.14) by considering $C_2 = 0$, $C_3 = 10$, $C_4 = -\frac{1}{12}$, $\epsilon = \frac{1}{50}$, positive sign for angle. Here, lining shade is used for $u(x,t)$ (upper) while plain shade is used for $w(x,t)$ (lower).

6 Conclusions

In this work, the Lie symmetry analysis of the generalised weakly dissipative modified two-component Dullin-Gottwald-Holm system has been performed. This system has been reduced to a system of ODEs by using the classical symmetries. Using symmetry reduction, the exact solutions of this system have been obtained in the form of power series (3.6) and other exact solutions in the form of hyperbolic and trigonometric functions (3.8),(3.10),(3.12),(3.14). The periodic solutions of this system has been found for the first time, to the best of author's knowledge. Also using multiplier approach, the conservation laws have been obtained. The 3D graphical representations have also been shown for obtained solutions.

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