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Global Regularity to the Navier-Stokes Equations for a Class of Large Initial Data

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Abstract. In [5], Chemin, Gallagher and Paicu proved the global regularity of solutions to the classical Navier-Stokes equations with a class of large initial data on $\mathbb{T}^2 \times \mathbb{R}$. This data varies slowly in vertical variable and has a norm which blows up as the small parameter (represented by ϵ in the paper) tends to zero. However, to the best of our knowledge, the result is still unclear for the whole spaces $\mathbb{R}^3$. In this paper, we consider the generalized Navier-Stokes equations on $\mathbb{R}^n (n \geq 3)$:

$$\partial_t u + u \cdot \nabla u + D^s u + \nabla P = 0, \quad \text{div } u = 0.$$

For some suitable number s, we prove that the Cauchy problem with initial data of the form $u_0^\epsilon(x) = (v_0^h(x_\epsilon), \epsilon^{-1} v_0^n(x_\epsilon))^T$, $x_\epsilon = (x_h, \epsilon x_n)^T$, is globally well-posed for all small $\epsilon > 0$, provided that the initial velocity profile v_0 is analytic in x_n and certain norm of v_0 is sufficiently small but independent of ϵ. In particular, our result is true for the n-dimensional classical Navier-Stokes equations with $n \geq 4$ and the fractional Navier-Stokes equations with $1 \leq s < 2$ in 3D.

Keywords: large data, global solution, slowly varying.

AMS Subject Classification: 35Q30; 35Q35; 76D03.

1 Introduction

The Cauchy problem of the incompressible Navier-Stokes equations on $\mathbb{R}^n$ is described by the following system

$$\begin{cases} \partial_t u + u\cdot\nabla u - \Delta u + \nabla P = 0, & x\in\mathbb{R}^n,\ t>0,\\ \operatorname{div} u = 0, & x\in\mathbb{R}^n,\ t>0,\\ u(0)=u_0, & x\in\mathbb{R}^n, \end{cases} \tag{1.1}$$

where u represents the velocity field and P is the scalar pressure.

First of all, let us recall some known results on the small-data global regularity of the Navier-Stokes equations on $\mathbb{R}^3$. In the seminal paper [18], Leray proved that the 3D incompressible Navier-Stokes equations are globally well-posed if the initial data u_0 is such that $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2}$ is small enough. This quantity is invariant under the natural scaling of the Navier-Stokes equations. Later on, many authors studied different scaling invariant spaces in which Navier-Stokes equations are well-posed at least for small initial data, which include but are not limited to

$$\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\hookrightarrow L^3(\mathbb{R}^3)\hookrightarrow \dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)\hookrightarrow BMO^{-1}(\mathbb{R}^3),$$

where $3<p<\infty$. The space $BMO^{-1}(\mathbb{R}^3)$ is known to be the largest scaling invariant space so that the Navier-Stokes equations (1.1) are globally well-posed under small initial data. The readers are referred to [3, 9, 14, 15] as references. We also mention that the work of Lei and Lin [16] was the first to quantify the smallness of the initial data to be 1 by introducing a new space $\mathcal{X}^{-1}$.

We remark that the norm in the above scaling invariant spaces are always greater than the norm in the Besov space $\dot{B}^{-1}_{\infty,\infty}$ defined by

$$\|u\|_{\dot{B}^{-1}_{\infty,\infty}} \overset{\text{def}}{=} \sup_{t>0} t^{\frac{1}{2}}\|e^{t\Delta}u_0\|_{L^\infty}.$$

Bourgain and Pavlovic in [2] showed that the Cauchy problem of the 3D Navier-Stokes equations is ill-posed in the sense of norm inflation. Partially because of the result of Bourgain and Pavlovic, data with a large $\dot{B}^{-1}_{\infty,\infty}$ are usually called large data to the Navier-Stokes equations (for instance, see [4, 20]).

Towards this line of research, a well-oiled case is the family of initial data which is slowly varying in vertical variable. The initial velocity field u_0^ϵ is of the form

$$u_0^\epsilon(x) = (v_0^h(x_\epsilon), \epsilon^{-1}v_0^3(x_\epsilon))^T,\quad x_\epsilon = (x_h, \epsilon x_3)^T, \tag{1.2}$$

which allows slowly varying in the vertical variable x_3 when $\epsilon>0$ is a small parameter. This family of initial data is very interesting (as has been pointed out by V.Šverák, see the acknowledgement in [5]) and considered by Chemin, Gallagher and Paicu in [5]. They proved the global regularity of solutions to the Navier-Stokes equations when v_0 is analytic in x_3 and periodic in x_h, and certain norm of v_0 is sufficiently small but independent of $\epsilon>0$. More precisely, they proved the following Theorem:

Theorem 1 [Chemin-Gallagher-Paicu, Ann. Math. 2011]. *Let α be a positive number. There are two positive numbers ϵ_0 and η such that for any divergence free vector field v_0 satisfying*

$$\|e^{\alpha D_3} v_0\|_{B^{\frac{7}{2}}_{2,1}} \leq \eta,$$

then, for any positive ϵ smaller than ϵ_0, the initial data (1.2) *generates a global smooth solution to (1.1) on $\mathbb{T}^2 \times \mathbb{R}$.*

The notation $B^{\frac{7}{2}}_{2,1}$ in the above Theorem denotes the usual inhomogeneous Besov space. The significance of the result lies in that the global regularity of the 3D incompressible Navier-Stokes equations in [5] only requires *very little smallness* imposed on the initial data. It is clear that the $\dot{B}^{-1}_{\infty,\infty}$ norm of u^ϵ_0 can tend to infinity as $\epsilon \to 0$. Let us first focus on the periodic constraint imposed on the initial data in Theorem 1.

As has been pointed out by Chemin, Gallagher and Paicu, the reason why the horizontal variable of the initial data in [5] is restricted to a torus is to be able to deal with very low horizontal frequencies. In the proof of Theorem 1 in [5], functions with zero horizontal average are treated differently to the others, and it is important that no small horizontal frequencies appear other than zero. Later on, many efforts are made towards removing the periodic constraint of v_0 on the horizontal variables. For instance, see [4, 10, 20] and so on. We will review those results a little bit later.

In this paper, we consider the Cauchy problem of the following generalized Navier-Stokes equations on $\mathbb{R}^n$:

$$\begin{cases} \partial_t u + u \cdot \nabla u + D^s u + \nabla P = 0, & x \in \mathbb{R}^n,\ t > 0, \\ \operatorname{div} u = 0, & x \in \mathbb{R}^n,\ t > 0, \\ u(0) = u^\epsilon_0, & x \in \mathbb{R}^n, \end{cases} \tag{1.3}$$

where $D = \sqrt{-\Delta}$. The initial velocity field u^ϵ_0 is of the form

$$u^\epsilon_0(x) = (v^h_0(x_\epsilon), \epsilon^{-1} v^n_0(x_\epsilon))^T, \quad x_\epsilon = (x_h, \epsilon x_n)^T. \tag{1.4}$$

The horizontal variable $x_h = (x_1, x_2, \ldots, x_{n-1})$.

Our main result is the following theorem which generalizes the theorem of Chemin, Gallagher and Paicu to the whole space for the generalized Navier-Stokes equations with some appropriate number s. Definition of notations will be given in Section 2.

Theorem 2. *Let α, ϵ_0, p and s be four positive constants and (p, s) satisfy*

$$1 \leq p < 2(n-1), \ \ 1 \leq s < \min(n-1, 2(n-1)/p).$$

There exists a positive constant η such that for any $0 < \epsilon < \epsilon_0$ and any divergence free vector field v_0 satisfying

$$\|e^{\alpha D_n} v_0\|_{\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1} \cap \dot{B}^{\frac{n-1}{p}+1-s,\frac{1}{2}}_{p,1}} \leq \eta,$$

then the generalized Navier-Stokes equations (1.3) with initial data (1.4) *generate a global smooth solution on $\mathbb{R}^n$.*

Remark 1. 1) When $n \geq 4$, one can find that the classical Navier-Stokes equations on $\mathbb{R}^n$ satisfy the assumption $1 \leq s = 2 < \min(n-1, \frac{2(n-1)}{p})$ for $1 \leq p < 2(n-1)$. Then according to Theorem 2, the n-dimensional incompressible Navier-Stokes equations with initial data (1.4) have a global smooth solution in the whole space case.
2) In the case $n = 3$, we require that $1 \leq s < 2$. The main obstacle when $s = 2$ is that we can not get the product law in $\dot{B}^{\frac{2}{p}-2,\frac{1}{2}}_{p,1}(\mathbb{R}^3)$. This anisotropic Besov space is induced by the *a priori* estimate of anisotropic pressure $(\nabla_h q, \epsilon^2 \partial_3 q)$ (see equation (1.5) and Step 5 for details). From this point of view, the global well-posedness of 3D incompressible Navier-Stokes equations with initial data (1.4) on $\mathbb{R}^3$ is still unclear, even though the higher dimensional cases are settled down.
3) In the present paper, we establish the global solution in the L^p-type Besov space, in which the bilinear estimate of the solution can not be derived by the classical L^2 energy method. Particularly, one can obtain the L^1 estimate in time of the solution by introducing the new quantity

$$\int_0^t \|v^n\|_{\dot{B}^{\frac{n-1}{p},\frac{1}{2}}_{p,1}} \|\partial_n v^h\|_{\dot{B}^{\frac{n-1}{p}+1-s,\frac{1}{2}}_{p,1}} d\tau$$

in the *a priori* estimate. We also mention that in [5, 20, 21], authors did not get the L^1-time estimate of the solution.

Now we mention that many authors make efforts to remove the periodic restriction on horizontal variable. Chemin and Gallagher considered the well-prepared initial data in [4]. They proved the global well-posedness of (1.1) when u_0^ϵ is of the form $u_0^\epsilon = (v_0^h + \epsilon w_0^h, w_0^3)(x_h, \epsilon x_3)$. Later, Gui, Huang, and Zhang in [10] generalized this result to the density dependent Navier-Stokes equations with the same initial velocity. Recently, Paicu and Zhang in [20, 21] considered the global regularity of (1.1) if u_0^ϵ satisfies the form of

$$u_0^\epsilon(x) = \epsilon^\delta (v_0^h(x_h, \epsilon x_3), \epsilon^{-1} v_0^3(x_h, \epsilon x_3))$$

for $\delta = \frac{1}{2}$ and $\delta \in (0, \frac{1}{2})$, then u_0^ϵ generates a global solution of (1.1) on $\mathbb{R}^3$.

Main ideas of the Proof. We will prove our main result by constructing the bilinear estimate (independent of ϵ). Our strategy can be stated as follows.

Step 1. Rescaled system and simplification. As in [5], we define

$$u^\epsilon(t,x) = (v^h(t, x_\epsilon), \epsilon^{-1} v^n(t, x_\epsilon))^T, \quad P^\epsilon(t,x) = q(t, x_\epsilon).$$

Denote

$$\Delta_\epsilon = \Delta_h + \epsilon^2 \partial_n^2, \ \Delta_h = \partial_1^2 + \ldots + \partial_{n-1}^2, \ D_\epsilon = \sqrt{-\Delta_\epsilon}.$$

Using the Navier-Stokes equations (1.1), it is easy to derive the equations governing the rescaled variables v and q (they are still depending on ϵ):

$$\begin{cases} \partial_t v^h + v \cdot \nabla v^h + D_\epsilon^s v^h + \nabla_h q = 0, \\ \partial_t v^n + v \cdot \nabla v^n + D_\epsilon^s v^n + \epsilon^2 \partial_n q = 0, \\ \operatorname{div} v = 0, \quad v(0) = v_0(x), \end{cases} \tag{1.5}$$

where $v^h = (v^1, \dots, v^{n-1})$. The rescaled pressure q can be recovered by the divergence free condition as

$$-\Delta_\epsilon q = \sum_{i,j} \partial_i \partial_j (v^i v^j).$$

The global regularity of solutions to system (1.4) for small initial data v_0 will be presented in Section 3 and 4 for any positive ϵ. But to best illustrate our ideas, let us here focus on the case of $\epsilon = 0$. Formally, by taking $\epsilon = 0$ in system (1.4), we have the following limiting system:

$$\begin{cases} \partial_t v^h + v \cdot \nabla v^h + D_h^s v^h + \nabla_h q = 0, \\ \partial_t v^n + v \cdot \nabla v^n + D_h^s v^n = 0, \\ \operatorname{div} v = 0, \quad v(0) = v_0(x), \end{cases} \tag{1.6}$$

where $D_h = \sqrt{-\Delta_h}$. The pressure q in (1.6) is given by

$$-\Delta_h q = \sum_{i,j} \partial_i \partial_j (v^i v^j).$$

Step 2. Set-up of the *a priori* estimate. Observing that in the rescaled system (1.6), the viscosity is absent in the vertical direction. To make the full use of smoothing effect from operator $\partial_t + D_h^s$, particularly in low frequency parts, we will apply the tool of anisotropic homogeneous Besov spaces. The goal is to derive certain *a priori* estimate of the form:

$$\Psi(t) \lesssim \Psi(0) + \Psi(t)^2.$$

Note that pressure term doesn't explicitly appear in the equation of v^n of the limiting system (1.6), which makes the estimate for v^n easier. So here let us just focus on the equation of v^h. Naturally, we define

$$\Psi(t) = \|v^h\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})} + \|v^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p}+1,\frac{1}{2}})} + \dots.$$

At this step, we assume that the initial data v_0^h belongs to $\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}}$. This ensures that $\Psi(t)$ is a critical quantity with respect to the natural scaling of the generalized Navier-Stokes equations. By Duhamel's principle, we can write the integral equation of v^h by

$$v^h = e^{-tD_h^s} v_0^h - \int_0^t e^{-(t-\tau)D_h^s} (v^h \cdot \nabla_h v^h + v^n \partial_n v^h + \nabla_h q) d\tau.$$

According to the estimates of heat equation, one can formally has

$$\Psi(t) \lesssim \Psi(0) + \|v^h \cdot \nabla_h v^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})} + \|v^n \partial_n v^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})} + \dots.$$

It will be shown that

$$\|v^h \cdot \nabla_h v^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})} \lesssim \|v^h\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})} \|\nabla_h v^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p},\frac{1}{2}})} \lesssim \Psi(t)^2.$$

Step 3. Derivative loss: input the estimate of $\partial_n v^h$. For the quantity $\Psi(t)$, we need to prove the bilinear estimate in the following form:

$$\Psi(t) \lesssim \Psi(0) + \Psi(t)^2 + \|v^n \partial_n v^h\|_{L^1_t(\dot{B}^{\frac{n-1}{p}+1-s,\frac{1}{2}}_{p,1})} + \dots.$$

Certainly, there is ∂_n-derivative loss! By the product law in anisotropic Besov spaces (Lemma 1), the strategy to bound the last term is

$$\|v^n \partial_n v^h\|_{L^1_t(\dot{B}^{\frac{n-1}{p}+1-s,\frac{1}{2}}_{p,1})} \lesssim \int_0^t \|v^n\|_{\dot{B}^{\frac{n-1}{p},\frac{1}{2}}_{p,1}} \|\partial_n v^h\|_{\dot{B}^{\frac{n-1}{p}+1-s,\frac{1}{2}}_{p,1}} d\tau$$

and then we should add the new quantity

$$\int_0^t \|v^n\|_{\dot{B}^{\frac{n-1}{p},\frac{1}{2}}_{p,1}} \|\partial_n v^h\|_{\dot{B}^{\frac{n-1}{p}+1-s,\frac{1}{2}}_{p,1}} d\tau$$

in the definition of $\Psi(t)$. We find that $\|\partial_n v^h\|_{\dot{B}^{\frac{n-1}{p}+1-s,\frac{1}{2}}_{p,1}}$ is the hardest term to estimate. Since we note that by Duhamel's principle,

$$\partial_n v^h = e^{-tD_h^s}\partial_n v_0^h - \int_0^t e^{-(t-\tau)D_h^s}\partial_n(v^n\partial_n v^h)d\tau + \dots.$$

There is still ∂_n-derivative loss in the *a priori* estimates!

Step 4. Recover the derivative loss: analyticity in x_n. Motivated by Chemin-Gallagher-Paicu [5], we add an exponential weight $e^{\Phi(t,D_n)}$ with

$$\Phi(t,|\xi_n|) = (\alpha - \lambda\theta(t))|\xi_n|.$$

Here $\theta(t)$ is defined by

$$\theta(t) = \int_0^t \|v^n_\Phi\|_{\dot{B}^{\frac{n-1}{p},\frac{1}{2}}_{p,1}} d\tau,$$

which will be shown to be small to ensure that $\Phi(t,|\xi_n|)$ satisfies the subadditivity. Denoting $f_\Phi = e^{\Phi(t,D_n)}f$, we then have

$$\partial_n v^h_\Phi = e^{-tD_h^s}e^{\Phi(t,D_n)}\partial_n v_0^h - \int_0^t e^{-(t-\tau)D_h^s}e^{-\lambda\int_\tau^t \dot\theta(t')dt' D_n}\partial_n(v^n\partial_n v^h)_\Phi d\tau + \dots.$$

Hence, we can recover the derivative loss by

$$\begin{aligned}
&\int_0^t \|v^n_\Phi\|_{\dot{B}^{\frac{n-1}{p},\frac{1}{2}}_{p,1}} \|\partial_n v^h_\Phi\|_{\dot{B}^{\frac{n-1}{p}+1-s,\frac{1}{2}}_{p,1}} d\tau \lesssim \sum_{k,j} 2^{k(\frac{n-1}{p}+1-s)}2^{\frac{1}{2}j}\int_0^t \dot\theta(\tau)\\
&\quad\times \int_0^\tau e^{-c\lambda\int_{t'}^\tau \dot\theta(t'')dt''2^j}2^j\|\Delta_{k,j}(v^n\partial_n v^h)_\Phi\|_{L^p_h(L^2_v)}dt'd\tau\\
&\quad\lesssim \sum_{k,j} 2^{k(\frac{n-1}{p}+1-s)}2^{\frac{1}{2}j}\int_0^t\int_{t'}^t e^{-c\lambda\int_{t'}^\tau \dot\theta(t'')dt''2^j}2^j\dot\theta(\tau)d\tau\\
&\times \|\Delta_{k,j}(v^n\partial_n v^h)_\Phi\|_{L^p_h(L^2_v)}dt' \lesssim \frac{1}{\lambda}\int_0^t \|v^n_\Phi\|_{\dot{B}^{\frac{n-1}{p},\frac{1}{2}}_{p,1}} \|\partial_n v^h_\Phi\|_{\dot{B}^{\frac{n-1}{p}+1-s,\frac{1}{2}}_{p,1}} dt' + \dots.
\end{aligned}$$

In this way, the losing derivative term can be absorbed by the left hand side of the above inequality by choosing λ sufficient large.

Step 5. The estimate of the pressure term $\nabla_h q$. To estimate the pressure term, we write

$$\nabla_h q = -2\nabla_h(-\Delta_h)^{-1}(v^n\partial_n \mathrm{div}_h v^h) + 2\nabla_h(-\Delta_h)^{-1}(\mathrm{div}_h v^h \mathrm{div}_h v^h) + \ldots.$$

The $L^1_t(\dot{B}^{\frac{n-1}{p}+1-s,\frac{1}{2}}_{p,1})$ norm of the first term in $\nabla_h q$ can be estimated by

$$\|\nabla_h(-\Delta_h)^{-1}(v^n\partial_n \mathrm{div}_h v^h)\|_{L^1_t(\dot{B}^{\frac{n-1}{p}+1-s,\frac{1}{2}}_{p,1})} \lesssim \|(v^n\partial_n \mathrm{div}_h v^h)\|_{L^1_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})}.$$

Here we should point out that when $s = 2, n = 3$, system (1.3) is nothing but the 3D incompressible Navier-Stokes equations. In this case we have to deal with $\|fg\|_{\dot{B}^{\frac{2}{p}-2,\frac{1}{2}}_{p,1}}$ type estimate. Unfortunately, the product law in $\dot{B}^{\frac{2}{p}-2,\frac{1}{2}}_{p,1}$ is hard to obtain since we can not control the low horizontal frequency part.

Step 6. Estimate of $\theta(t)$. In this step, we want to prove that for any time t, $\theta(t)$ is a small quantity. This ensures that the phase function Φ satisfies the subadditivity property. We will go to derive a stronger estimate for

$$Y(t) = \|v^n_\Phi\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})} + \|v^n_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p},\frac{1}{2}}_{p,1})}.$$

However, when $\epsilon > 0$, we can not get the closed estimate for $Y(t)$. Our strategy is to add an extra term ϵv^h under the same norm which is hidden in the pressure term $\epsilon^2\partial_n q$. See Section 4 for details.

Step 7. Estimate of v^n_Φ. If this is done, we could get a closed *a priori* estimate (see Lemma 4) and finish the proof of Theorem 2. Observing that the nonlinear term $v^n\partial_n v^n$ can be rewritten as $-v^n \mathrm{div}_h v^h$ due to divergence free condition. Hence, in the limiting system (1.6), there is no loss of derivative in vertical direction on v^n. Thus the estimate on v^n is much easier than v^h.

There are also some other type of large initial data so that the Navier-Stokes equations are globally well-posed. For instance, when the domain is thin in the vertical direction, Raugel and Sell [22] were able to establish global solutions for a family of large initial data by using anisotropic Sobolev imbedding theorems (see also the paper [13] by Iftimie, Raugel and Sell). By choosing the initial data to transform the equation into a rotating fluid equations, Mahalov and Nicolaenko [19] obtained global solutions generated by a family of large initial data. A family of axi-symmetric large solutions were established in [11] by Hou, Lei and Li. Recently, Lei, Lin and Zhou in [17] proved the global well-posedness of 3D Navier-Stokes equations for a family of large initial data by making use of the structure of Helicity. The data in [17] are not small in $\dot{B}^{-1}_{\infty,\infty}$ even in the anisotropic sense. We also mention that for the general 3D incompressible Navier-Stokes equations which possess hyper-dissipation in horizontal direction, Fang and Han in [8] obtain the global existence result when the initial data belongs to the anisotropic Besov spaces.

The remaining part of the paper is organized as follows. In Section 2, we present the basic theories of anisotropic Littlewood-Paley decomposition

and anisotropic Besov spaces. Section 3 is devoted to obtaining the *a priori* estimates of solution. The $\theta(t)$ will be studied in Section 4. Finally, the proof of the main result will be given in Section 5.

2 Anisotropic Littlewood-Paley theories and preliminary lemmas

In this section, we first recall the definition of the anisotropic Littlewood-Paley decomposition and some properties about anisotropic Besov spaces. It was introduced by Iftimie in [12] for the study of incompressible Navier-Stokes equations in thin domains. Let us briefly explain how this may be built in $\mathbb{R}^n$. Let (χ, φ) be a couple of C^∞ functions satisfying

$$\begin{aligned}&\mathrm{Supp}\chi \subset \Big\{r \le \frac{4}{3}\Big\}, \ \ \mathrm{Supp}\varphi \subset \Big\{\frac{3}{4} \le r \le \frac{8}{3}\Big\},\\ &\chi(r) + \sum_{k\in\mathbb{N}} \varphi(2^{-k}r) = 1 \ \text{ for } r \in \mathbb{R}, \quad \sum_{j\in\mathbb{Z}} \varphi(2^{-j}r) = 1 \ \text{ for } r \in \mathbb{R}\backslash\{0\}.\end{aligned}$$

For $u \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$, we define the homogeneous dyadic decomposition on the horizontal variables by

$$\Delta_k^h u = \mathcal{F}^{-1}(\varphi(2^{-k}|\xi_h|)\widehat{u}) \ \text{ for } \ k \in \mathbb{Z}.$$

Similarly, on the vertical variable, we define the homogeneous dyadic decomposition by

$$\Delta_j^v u = \mathcal{F}^{-1}(\varphi(2^{-j}|\xi_n|)\widehat{u}) \ \text{ for } \ j \in \mathbb{Z}.$$

The anisotropic Littlewood-Paley decomposition satisfies the property of almost orthogonality:

$$\Delta_k^h\Delta_l^h u \equiv 0 \quad \text{if} \quad |k-l| \ge 2 \quad \text{and} \quad \Delta_k^h(S_{l-1}^h u \Delta_l^h u) \equiv 0 \quad \text{if} \quad |k-l| \ge 5,$$

where S_l^h is defined by $S_l^h u = \sum_{l' \le l-1} \Delta_{l'}^h u$. Similar properties hold for Δ_j^v. In this paper, we shall use the following anisotropic version of Besov spaces [12]. In what follows, we denote for abbreviation $\Delta_{k,j} f \overset{\text{def}}{=} \Delta_k^h \Delta_j^v f$.

Definition 1 [Anisotropic Besov space]. Let $(p, r) \in [1, \infty]^2$, $\sigma, s \in \mathbb{R}$ and $u \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$, we set

$$\|u\|_{\dot{B}^{\sigma,s}_{p,r}} \overset{\text{def}}{=} \|2^{k\sigma}2^{js}\|\Delta_{k,j}u\|_{L^p_h(L^2_v)}\|_{l^r(\mathbb{Z}^2)}.$$

(1) For $\sigma < \frac{n-1}{p}, s < \frac{1}{2}$($\sigma = \frac{n-1}{p}$ or $s = \frac{1}{2}$ if $r = 1$), we define

$$\dot{B}^{\sigma,s}_{p,r}(\mathbb{R}^n) \overset{\text{def}}{=} \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \|u\|_{\dot{B}^{\sigma,s}_{p,r}} < \infty\}.$$

(2) If $k, l \in \mathbb{N}$ and $\frac{n-1}{p} + k < \sigma < \frac{n-1}{p} + k + 1$, $\frac{1}{2} + l < s < \frac{1}{2} + l + 1$ ($\sigma = \frac{n-1}{p} + k + 1$ or $s = \frac{1}{2} + l + 1$ if $r = 1$), then $\dot{B}^{\sigma,s}_{p,r}(\mathbb{R}^n)$ is defined as the subset of $u \in \mathcal{S}'(\mathbb{R}^n)$ such that $\partial_h^\beta \partial_3^\alpha u \in \dot{B}^{\sigma-k,s-l}_{p,r}(\mathbb{R}^n)$ whenever $|\beta| = k, \alpha = l$.

The study of non-stationary equation requires spaces of the type $L^\rho_T(X)$ for appropriate Banach spaces X. In our case, we expect X to be an anisotropic Besov space. So it is natural to localize the equations through anisotropic Littlewood-Paley decomposition. We then get estimates for each dyadic block and perform integration in time. As in [6], we define the so called Chemin-Lerner type spaces:

DEFINITION 2. Let $(p,r)\in[1,\infty]^2$, $\sigma,s\in\mathbb{R}$ and $T\in(0,\infty]$, we set

$$\|u\|_{\widetilde{L}^\rho_T(\dot{B}^{\sigma,s}_{p,r})}\overset{\text{def}}{=}\|2^{k\sigma}2^{js}\|\Delta_{k,j}u\|_{L^\rho_T(L^p_h(L^2_v))}\|_{l^r(\mathbb{Z}^2)}$$

and define the space $\widetilde{L}^\rho_T(\dot{B}^{\sigma,s}_{p,r})(\mathbb{R}^n)$to be the subset of distributions in $u\in\mathcal{S}'(0,T)\times\mathcal{S}'(\mathbb{R}^n)$ with finite $\widetilde{L}^\rho_T(\dot{B}^{\sigma,s}_{p,r})$ norm.

In order to investigate the continuity properties of the products of two temperate distributions f and g in anisotropic Besov spaces, we then recall the isotropic product decomposition which is a simple splitting device going back to the pioneering work by J.-M. Bony [1]. Let $f,g\in\mathcal{S}'(\mathbb{R}^n)$,

$$fg=T(f,g)+\widetilde{T}(f,g)+R(f,g),$$

where the paraproducts $T(f,g)$ and $\widetilde{T}(f,g)$ are defined by

$$T(f,g)=\sum_{k\in\mathbb{Z}}S_{k-1}f\Delta_k g,\quad \widetilde{T}(f,g)=\sum_{k\in\mathbb{Z}}\Delta_k f S_{k-1}g$$

and the remainder

$$R(f,g)=\sum_{k\in\mathbb{Z}}\Delta_k f\widetilde{\Delta}_k g\quad\text{with}\quad\widetilde{\Delta}_k g=\sum_{k'=k-1}^{k+1}\Delta_{k'}g.$$

Similarly, we can define the decompositions for both horizontal variable x_h and vertical variable x_n. Indeed, we have the following split in x_h.

$$fg=T^h(f,g)+\widetilde{T}^h(f,g)+R^h(f,g),$$

with

$$T^h(f,g)=\sum_{k\in\mathbb{Z}}S^h_{k-1}f\Delta^h_k g,\quad \widetilde{T}^h(f,g)=\sum_{k\in\mathbb{Z}}\Delta^h_k f S^h_{k-1}g,$$

$$R^h(f,g)=\sum_{k\in\mathbb{Z}}\Delta^h_k f\widetilde{\Delta}^h_k g,\quad\text{where}\quad\widetilde{\Delta}^h_k g=\sum_{k'=k-1}^{k+1}\Delta^h_{k'}g.$$

The decomposition in vertical variable x_n can be defined by the same line. Thus, we can write fg as

$$\begin{aligned}fg=&(T^h+\widetilde{T}^h+R^h)(T^v+\widetilde{T}^v+R^v)(f,g)=T^hT^v(f,g)+T^h\widetilde{T}^v(f,g)\\&+T^hR^v(f,g)+\widetilde{T}^hT^v(f,g)+\widetilde{T}^h\widetilde{T}^v(f,g)+\widetilde{T}^hR^v(f,g)\\&+R^hT^v(f,g)+R^h\widetilde{T}^v(f,g)+R^hR^v(f,g).\end{aligned}\tag{2.1}$$

Each term of (2.1) has an explicit definition. Here

$$T^hT^v(f,g)=\sum_{(k,j)\in\mathbb{Z}^2} S^h_{k-1}S^v_{j-1}f\Delta_{k,j}g,\ \widetilde{T}^hT^v(f,g)=\sum_{(k,j)\in\mathbb{Z}^2}\Delta^h_kS^v_{j-1}fS^h_{k-1}\Delta^v_jg.$$

Similarly,

$$R^hT^v(f,g)=\sum_{(k,j)\in\mathbb{Z}^2}\Delta^h_kS^v_{j-1}f\widetilde{\Delta}^h_k\Delta^v_jg,\quad R^hR^v(f,g)=\sum_{(k,j)\in\mathbb{Z}^2}\Delta_{k,j}f\widetilde{\Delta}_{k,j}g,$$

and so on.

At this moment, we can state an important product law in anisotropic Besov spaces. The case $p=2,n=3$ was proved in [10]. For completeness, here we prove a similar result in the L^p framework.

Lemma 1. *Let $1\le p<\infty$ and (σ_1,σ_2) be in $\mathbb{R}^2$. If $\sigma_1,\sigma_2\le\frac{n-1}{p}$ and*

$$\sigma_1+\sigma_2>(n-1)\max(0,2/p-1),$$

then we have for any $f\in\dot{B}^{\sigma_1,\frac12}_{p,1}(\mathbb{R}^n)$ and $g\in\dot{B}^{\sigma_2,\frac12}_{p,1}(\mathbb{R}^n)$,

$$\|fg\|_{\dot{B}^{\sigma_1+\sigma_2-\frac{n-1}{p},\frac12}_{p,1}}\lesssim\|f\|_{\dot{B}^{\sigma_1,\frac12}_{p,1}}\|g\|_{\dot{B}^{\sigma_2,\frac12}_{p,1}}.$$

Proof. According to (2.1), we first give the bound of $T^hT^v(f,g)$. Indeed, applying Hölder and Bernstein inequality, we get that

$$\begin{aligned}\|\Delta_{k,j}(T^hT^v(f,g))\|_{L^p_h(L^2_v)}&\lesssim\sum_{\substack{|k-k'|\le4\\|j-j'|\le4}}\|S^h_{k'-1}S^v_{j'-1}f\|_{L^\infty}\|\Delta_{k',j'}g\|_{L^p_h(L^2_v)}\\&\lesssim\sum_{\substack{|k-k'|\le4\\|j-j'|\le4}}\sum_{\substack{k''\le k'-2\\j''\le j'-2}}2^{k''\sigma_1}2^{\frac12j''}\|\Delta_{k'',j''}f\|_{L^p_h(L^2_v)}\cdot2^{k'\sigma_2}2^{\frac12j'}\|\Delta_{k',j'}g\|_{L^p_h(L^2_v)}\\&\quad\times2^{(k''-k')(\frac{n-1}{p}-\sigma_1)}2^{(k-k')(\sigma_1+\sigma_2-\frac{n-1}{p})}2^{\frac12(j-j')}2^{-k(\sigma_1+\sigma_2-\frac{n-1}{p})}2^{-\frac12j}.\end{aligned}$$

Since $\sigma_1\le\frac{n-1}{p}$, we obtain that

$$\|\Delta_{k,j}(T^hT^v(f,g))\|_{L^p_h(L^2_v)}\lesssim c_{k,j}2^{-k(\sigma_1+\sigma_2-\frac{n-1}{p})}2^{-\frac12j}\|f\|_{\dot{B}^{\sigma_1,\frac12}_{p,1}}\|g\|_{\dot{B}^{\sigma_2,\frac12}_{p,1}},$$

where the sequence $\{c_{k,j}\}_{(k,j)\in\mathbb{Z}^2}$ satisfies $\|c_{k,j}\|_{l^1(\mathbb{Z}^2)}=1$. This gives the estimate of $T^hT^v(f,g)$.

Similarly, for $\widetilde{T}^hT^v(f,g)$, we have

$$\begin{aligned}\|\Delta_{k,j}(\widetilde{T}^hT^v(f,g))\|_{L^p_h(L^2_v)}&\lesssim\sum_{\substack{|k-k'|\le4\\|j-j'|\le4}}\|\Delta^h_{k'}S^v_{j'-1}f\|_{L^p_h(L^\infty_v)}\|S^h_{k'-1}\Delta^v_{j'}g\|_{L^\infty_h(L^2_v)}\\&\lesssim\sum_{\substack{|k-k'|\le4\\|j-j'|\le4}}\sum_{\substack{k''\le k'-2\\j''\le j'-2}}2^{k'\sigma_1}2^{\frac12j''}\|\Delta_{k',j''}f\|_{L^p_h(L^2_v)}\cdot2^{k''\sigma_2}2^{\frac12j'}\|\Delta_{k'',j'}g\|_{L^p_h(L^2_v)}\\&\quad\times2^{(k''-k')(\frac{n-1}{p}-\sigma_2)}2^{(k-k')(\sigma_1+\sigma_2-\frac{n-1}{p})}2^{\frac12(j-j')}2^{-k(\sigma_1+\sigma_2-\frac{n-1}{p})}2^{-\frac12j}.\end{aligned}$$

Again, $\sigma_2 \le \frac{n-1}{p}$ implies that

$$\|\Delta_{k,j}(\widetilde{T}^h T^v(f,g))\|_{L_h^p(L_v^2)} \lesssim c_{k,j} 2^{-k(\sigma_1+\sigma_2-\frac{n-1}{p})} 2^{-\frac{1}{2}j} \|f\|_{\dot{B}^{\sigma_1,\frac{1}{2}}_{p,1}} \|g\|_{\dot{B}^{\sigma_2,\frac{1}{2}}_{p,1}}.$$

The estimate on the remainder operator which concerns the horizontal variable $R^h T^v(f,g)$ may be more complicated. When $2 \le p$, the strategy is following:

$$\begin{aligned}
\|\Delta_{k,j}(R^h T^v(f,g))\|_{L_h^p(L_v^2)} &\lesssim \sum_{\substack{k'\ge k-2\\ |j-j'|\le 4}} \|\Delta^h_{k'} S^v_{j'-1} f \widetilde{\Delta}^h_{k'} \Delta^v_{j'} g\|_{L_h^{\frac{p}{2}}(L_v^2)} 2^{\frac{n-1}{p}k} \\
&\lesssim \sum_{\substack{k'\ge k-2\\ |j-j'|\le 4}} \sum_{j''\le j'-2} 2^{k'\sigma_1} 2^{\frac{1}{2}j''} \|\Delta_{k',j''} f\|_{L_h^p(L_v^2)} \cdot 2^{k'\sigma_2} 2^{\frac{1}{2}j'} \|\widetilde{\Delta}^h_{k'} \Delta^v_{j'} g\|_{L_h^p(L_v^2)} \\
&\quad \times 2^{(k-k')(\sigma_1+\sigma_2)} 2^{\frac{1}{2}(j-j')} 2^{-k(\sigma_1+\sigma_2-\frac{n-1}{p})} 2^{-\frac{1}{2}j}.
\end{aligned}$$

As $\sigma_1+\sigma_2>0$ if $2 \le p$, we have

$$\|\Delta_{k,j}(R^h T^v(f,g))\|_{L_h^p(L_v^2)} \lesssim c_{k,j} 2^{-k(\sigma_1+\sigma_2-\frac{n-1}{p})} 2^{-\frac{1}{2}j} \|f\|_{\dot{B}^{\sigma_1,\frac{1}{2}}_{p,1}} \|g\|_{\dot{B}^{\sigma_2,\frac{1}{2}}_{p,1}}.$$

In the case $1 \le p < 2$, we have

$$\begin{aligned}
\|\Delta_{k,j}(R^h T^v(f,g))\|_{L_h^p(L_v^2)} &\lesssim \sum_{\substack{k'\ge k-2\\ |j-j'|\le 4}} \|\Delta^h_{k'} S^v_{j'-1} f \widetilde{\Delta}^h_{k'} \Delta^v_{j'} g\|_{L_h^1(L_v^2)} 2^{(n-1)(1-\frac{1}{p})k} \\
&\lesssim \sum_{\substack{k'\ge k-2\\ |j-j'|\le 4}} \sum_{j''\le j'-2} 2^{\frac{1}{2}j''} \|\Delta_{k',j''} f\|_{L^2} \|\widetilde{\Delta}^h_{k'} \Delta^v_{j'} g\|_{L^2} 2^{(n-1)(1-\frac{1}{p})k} \\
&\lesssim \sum_{\substack{k'\ge k-2\\ |j-j'|\le 4}} \sum_{j''\le j'-2} 2^{k'\sigma_1} 2^{\frac{1}{2}j''} \|\Delta_{k',j''} f\|_{L_h^p(L_v^2)} \cdot 2^{k'\sigma_2} 2^{\frac{1}{2}j'} \|\widetilde{\Delta}^h_{k'} \Delta^v_{j'} g\|_{L_h^p(L_v^2)} \\
&\quad \times 2^{(k-k')(\sigma_1+\sigma_2-(n-1)(\frac{2}{p}-1))} 2^{\frac{1}{2}(j-j')} 2^{-k(\sigma_1+\sigma_2-\frac{n-1}{p})} 2^{-\frac{1}{2}j}.
\end{aligned}$$

As $\sigma_1+\sigma_2 > (n-1)(\frac{2}{p}-1)$ if $1 \le p < 2$, we have

$$\|\Delta_{k,j}(R^h T^v(f,g))\|_{L_h^p(L_v^2)} \lesssim c_{k,j} 2^{-k(\sigma_1+\sigma_2-\frac{n-1}{p})} 2^{-\frac{1}{2}j} \|f\|_{\dot{B}^{\sigma_1,\frac{1}{2}}_{p,1}} \|g\|_{\dot{B}^{\sigma_2,\frac{1}{2}}_{p,1}}.$$

The other terms can be followed exactly in the same way, here we omit the details. These complete the proof of this lemma. □

Throughout this paper, Φ denotes a locally bounded function on $\mathbb{R}^+ \times \mathbb{R}$ which satisfies the following subadditivity (see (3.2) for the explicit expression of Φ)

$$\Phi(t,\xi_n) \le \Phi(t,\xi_n-\eta_n) + \Phi(t,\eta_n).$$

For any function f in $\mathcal{S}'(0,T)\times\mathcal{S}'(\mathbb{R}^n)$, we define

$$f_\Phi(t,x_h,x_n) = \mathcal{F}^{-1}\left(e^{\Phi(t,\xi_n)}\hat{f}(t,x_h,\xi_n)\right).$$

Let us keep the following fact in mind that the map $f \mapsto f^+$ preserves the norm of $L^p_h(L^2_v)$, where $f^+(t, x_h, x_n)$ represents the inverse Fourier transform of $|\hat{f}(t, x_h, \xi_n)|$ on vertical variable, defined as

$$f^+(t, x_h, x_n) \overset{\text{def}}{=} \mathcal{F}^{-1}|\hat{f}(t, x_h, \xi_n)|.$$

On the basis of these facts, we have the following weighted inequality as in Lemma 1.

Lemma 2. *Let $1 \le p < \infty$ and (σ_1, σ_2) be in $\mathbb{R}^2$. If $\sigma_1, \sigma_2 \le (n-1)/p$ and*

$$\sigma_1 + \sigma_2 > (n-1)\max(0, 2/p - 1),$$

then we have for any $f_\Phi \in \dot{B}^{\sigma_1, \frac{1}{2}}_{p,1}(\mathbb{R}^n)$ and $g_\Phi \in \dot{B}^{\sigma_2, \frac{1}{2}}_{p,1}(\mathbb{R}^n)$,

$$\|(fg)_\Phi\|_{\dot{B}^{\sigma_1+\sigma_2-\frac{n-1}{p}, \frac{1}{2}}_{p,1}} \lesssim \|f_\Phi\|_{\dot{B}^{\sigma_1, \frac{1}{2}}_{p,1}} \|g_\Phi\|_{\dot{B}^{\sigma_2, \frac{1}{2}}_{p,1}}.$$

Proof. We only prove the $\dot{B}^{\sigma_1+\sigma_2-\frac{n-1}{p}, \frac{1}{2}}_{p,1}$ norm of $T^h T^v(f, g)_\Phi$. For fixed k, j, we have

$$\begin{aligned}
&\|\Delta_{k,j}(T^h T^v(f, g)_\Phi)\|_{L^p_h(L^2_v)} \\
&\lesssim \sum_{\substack{|k-k'|\le 4 \\ |j-j'|\le 4}} \|e^{\Psi(t,\xi_n)} \mathcal{F}(S^h_{k'-1} S^v_{j'-1} f)(x_h, \cdot) \star \mathcal{F}(\Delta_{k',j'} g)(x_h, \cdot)\|_{L^p_h(L^2_v)} \\
&\lesssim \sum_{\substack{|k-k'|\le 4 \\ |j-j'|\le 4}} \| |\mathcal{F}(S^h_{k'-1} S^v_{j'-1} f_\Phi)(x_h, \cdot)| \star |\mathcal{F}(\Delta_{k',j'} g_\Phi)(x_h, \cdot)| \|_{L^p_h(L^2_v)} \\
&\lesssim \sum_{\substack{|k-k'|\le 4 \\ |j-j'|\le 4}} \sum_{\substack{k''\le k'-2 \\ j''\le j'-2}} \| |\mathcal{F}(\Delta_{k'',j''} f_\Phi)(x_h, \cdot)| \star |\mathcal{F}(\Delta_{k',j'} g_\Phi)(x_h, \cdot)| \|_{L^p_h(L^2_v)} \\
&\lesssim \sum_{\substack{|k-k'|\le 4 \\ |j-j'|\le 4}} \sum_{\substack{k''\le k'-2 \\ j''\le j'-2}} \|(\Delta_{k'',j''} f_\Phi)^+\|_{L^p_h(L^2_v)} \|(\Delta_{k',j'} g_\Phi)^+\|_{L^p_h(L^2_v)} 2^{\frac{n-1}{p} k''} 2^{\frac{1}{2} j''}.
\end{aligned}$$

Using the fact that $f \mapsto f^+$ preserves the norm of $L^p_h(L^2_v)$, we then get by the similar method as in Lemma 1 that

$$\|(T^h T^v(f, g)_\Phi)\|_{\dot{B}^{\sigma_1+\sigma_2-\frac{n-1}{p}, \frac{1}{2}}_{p,1}} \lesssim \|f_\Phi\|_{\dot{B}^{\sigma_1, \frac{1}{2}}_{p,1}} \|g_\Phi\|_{\dot{B}^{\sigma_2, \frac{1}{2}}_{p,1}}.$$

The other terms in (2.1) can be estimated by the same method and finally, we have

$$\|(fg)_\Phi\|_{\dot{B}^{\sigma_1+\sigma_2-\frac{n-1}{p}, \frac{1}{2}}_{p,1}} \lesssim \|f_\Phi\|_{\dot{B}^{\sigma_1, \frac{1}{2}}_{p,1}} \|g_\Phi\|_{\dot{B}^{\sigma_2, \frac{1}{2}}_{p,1}}.$$

□

The following lemma is a direct consequence of Lemma 2.

Lemma 3. *Let $1 \le p < \infty$, $\rho \in [1,\infty]$, $(\rho_1,\rho_2) \in [1,\infty]^2$ and (σ_1,σ_2) be in $\mathbb{R}^2$. Assume that*

$$1/\rho \overset{\text{def}}{=} 1/\rho_1 + 1/\rho_2.$$

If $\sigma_1,\sigma_2 \le \frac{n-1}{p}$ and $\sigma_1+\sigma_2 > (n-1)\max(0,2/p-1)$, then we have for any $f_\Phi \in \widetilde{L}^{\rho_1}_T(\dot{B}^{\sigma_1,\frac{1}{2}}_{p,1}(\mathbb{R}^n))$ and $g_\Phi \in \widetilde{L}^{\rho_2}_T(\dot{B}^{\sigma_2,\frac{1}{2}}_{p,1}(\mathbb{R}^n))$,

$$\|(fg)_\Phi\|_{\widetilde{L}^{\rho}_T(\dot{B}^{\sigma_1+\sigma_2-\frac{n-1}{p},\frac{1}{2}}_{p,1})} \lesssim \|f_\Phi\|_{\widetilde{L}^{\rho_1}_T(\dot{B}^{\sigma_1,\frac{1}{2}}_{p,1})}\|g_\Phi\|_{\widetilde{L}^{\rho_2}_T(\dot{B}^{\sigma_2,\frac{1}{2}}_{p,1})}.$$

3 Estimates for the re-scaled system

This section is devoted to obtaining the *a priori* estimate for the following system

$$\begin{cases} \partial_t v^h + v\cdot\nabla v^h + D^s_\epsilon v^h + \nabla_h q &= 0,\\ \partial_t v^n + v\cdot\nabla v^n + D^s_\epsilon v^n + \epsilon^2\partial_n q &= 0,\\ \operatorname{div} v &= 0,\\ v(0) &= v_0(x). \end{cases}$$

The pressure q can be computed by the formula

$$-\Delta_\epsilon q = \sum_{i,j}\partial_i\partial_j(v^i v^j).$$

Due to the divergence free condition, the pressure can be split into the following three parts

$$\begin{cases} q^1 = (-\Delta_\epsilon)^{-1}\sum\limits_{i,j=1}^{n-1}\partial_i\partial_j(v^i v^j),\\ q^2 = 2(-\Delta_\epsilon)^{-1}\sum\limits_{i=1}^{n-1}\partial_i\partial_n(v^i v^n),\\ q^3 = -2(-\Delta_\epsilon)^{-1}\partial_n(v^n \mathrm{div_h} v^h). \end{cases} \tag{3.1}$$

It is worthwhile to note that there will lose one vertical derivative owing to the term $v^n\partial_n v^h$ and pressure terms q^2, q^3 which appear in the equation on v^h. Thus, we assume that the initial data is analytic in the vertical variable. This method was introduced in [7] to compensate the losing derivative in x_n. Therefore, we introduce two key quantities which we want to control in order to obtain the global bound of v in a certain space. We define the function $\theta(t)$

$$\theta(t) = \int_0^t \|v^n_\Phi(\tau)\|_{\dot{B}^{\frac{n-1}{p},\frac{1}{2}}_{p,1}}d\tau,$$

and denote

$$\Psi(t) = \|v_\Phi\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n-1}{p}+1-s,\frac{1}{2}}_{p,1})} + \|v_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}+1,\frac{1}{2}}_{p,1})} + \int_0^t \|v^n_\Phi\|_{\dot{B}^{\frac{n-1}{p}+1,\frac{1}{2}}_{p,1}} \times \|\partial_n v^h_\Phi\|_{\dot{B}^{\frac{n-1}{p}+1-s,\frac{1}{2}}_{p,1}}d\tau, \quad \Psi(0) = \|e^{\alpha D_n}v_0\|_{\dot{B}^{\frac{n-1}{p}+1-s,\frac{1}{2}}_{p,1}}.$$

The phase function $\Phi(t, D_n)$ is defined by

$$\Phi(t, \xi_n) = (\alpha - \lambda\theta(t))|\xi_n|, \tag{3.2}$$

for some λ that will be chosen later on, α is a positive number. Obviously, we need to ensure that $\theta(t) < \frac{\alpha}{\lambda}$ which implies the subadditivity of Φ.

The following lemma provides the *a priori* estimate of v_Φ in the anisotropic Besov spaces, which is the key bilinear estimate.

Lemma 4. *There exist two constants λ_0 and C_1 such that for any $\lambda > \lambda_0$ and t satisfying $\theta(t) \leq \frac{\alpha}{2\lambda}$, we have*

$$\Psi(t) \leq C_1\Psi(0) + C_1\Psi(t)^2.$$

3.1 Estimates on the horizontal component v^h

According to the definition of v^h_Φ, we find that in each dyadic block, it verifies the following equation

$$\begin{aligned}\Delta_{k,j}v^h_\Phi(t,x) &= e^{-tD^s_\epsilon+\Phi(t,D_n)}\Delta_{k,j}v^h_0 - \int_0^t e^{-(t-\tau)D^s_\epsilon}e^{-\lambda D_n\int_\tau^t\dot\theta(t')dt'}\\ &\times\Delta_{k,j}(v\cdot\nabla v^h)_\Phi(\tau)d\tau - \int_0^t e^{-(t-\tau)D^s_\epsilon}e^{-\lambda D_n\int_\tau^t\dot\theta(t')dt'}\nabla_h\Delta_{k,j}q_\Phi(\tau)d\tau. \quad (3.3)\end{aligned}$$

Taking the $L^p_h(L^2_v)$ norm, we deduce that

$$\begin{aligned}\|\Delta_{k,j}v^h_\Phi\|_{L^p_h(L^2_v)} &\lesssim e^{-c(2^{ks}+\epsilon^s2^{js})t}\|\Delta_{k,j}e^{\alpha D_n}v^h_0\|_{L^p_h(L^2_v)}\\ &+\int_0^t e^{-c(2^{ks}+\epsilon^s2^{js})(t-\tau)}e^{-c\lambda2^j\int_\tau^t\dot\theta(t')dt'}\|\Delta_{k,j}(v\cdot\nabla v^h)_\Phi\|_{L^p_h(L^2_v)}d\tau\\ &+\int_0^t e^{-c(2^{ks}+\epsilon^s2^{js})(t-\tau)}e^{-c\lambda2^j\int_\tau^t\dot\theta(t')dt'}\|\nabla_h\Delta_{k,j}q_\Phi\|_{L^p_h(L^2_v)}d\tau \overset{\text{def}}{=} I_1+I_2+I_3.\end{aligned}$$

We first estimate the linear term I_1. In fact, we have

$$\begin{aligned}\|I_1\|_{L^\infty_t} + 2^{ks}\|I_1\|_{L^1_t} &\lesssim \|\Delta_{k,j}e^{\alpha D_n}v^h_0\|_{L^p_h(L^2_v)}\\ &\lesssim c_{k,j}2^{-k(\frac{n-1}{p}+1-s)}2^{-\frac12 j}\|e^{\alpha D_n}v^h_0\|_{\dot B^{\frac{n-1}{p}+1-s,\frac12}_{p,1}},\end{aligned}$$

where $\{c_{k,j}\}_{(k,j)\in\mathbb{Z}^2}$ is a two dimensional sequence satisfying $\|c_{k,j}\|_{l^1(\mathbb{Z}^2)} = 1$.

The term I_2 can be rewritten as

$$I_2 \lesssim \int_0^t e^{-c2^{ks}(t-\tau)}\|\Delta_{k,j}(v\cdot\nabla v^h)_\Phi\|_{L^p_h(L^2_v)}d\tau.$$

By Young's inequality, we have

$$\begin{aligned}\|I_2\|_{L^\infty_t} + 2^{ks}\|I_2\|_{L^1_t} &\lesssim \|\Delta_{k,j}(v\cdot\nabla v^h)_\Phi\|_{L^1_t(L^p_h(L^2_v))}\\ &\lesssim c_{k,j}2^{-k(\frac{n-1}{p}+1-s)}2^{-\frac12 j}\|(v\cdot\nabla v^h)_\Phi\|_{L^1_t(\dot B^{\frac{n-1}{p}+1-s,\frac12}_{p,1})}.\end{aligned}$$

Thus, we can get by Lemma 2 and 3 that

$$\|I_2\|_{L_t^\infty}+2^{ks}\|I_2\|_{L_t^1}\lesssim c_{k,j}2^{-k(\frac{n-1}{p}+1-s)}2^{-\frac{1}{2}j}\|v_\Phi^h\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})}$$
$$\times\|v_\Phi^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p}+1,\frac{1}{2}})}+c_{k,j}2^{-k(\frac{n-1}{p}+1-s)}2^{-\frac{1}{2}j}\int_0^t\|v_\Phi^n\|_{\dot{B}_{p,1}^{\frac{n-1}{p},\frac{1}{2}}}\|\partial_n v_\Phi^h\|_{\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}}}d\tau.$$

Now we are left with the study of the pressure term I_3. The pressure can be split into $q=q^1+q^2+q^3$ with q^1,q^2,q^3 defined in (3.1). For convenience, we denote that

$$I_{31}=\int_0^t e^{-c(2^{ks}+\epsilon^s2^{js})(t-\tau)}e^{-c\lambda2^j\int_\tau^t\dot\theta(t')dt'}\|\nabla_h\Delta_{k,j}q_\Phi^1\|_{L_h^p(L_v^2)}d\tau,$$

$$I_{32}=\int_0^t e^{-c(2^{ks}+\epsilon^s2^{js})(t-\tau)}e^{-c\lambda2^j\int_\tau^t\dot\theta(t')dt'}\|\nabla_h\Delta_{k,j}q_\Phi^2\|_{L_h^p(L_v^2)}d\tau,$$

$$I_{33}=\int_0^t e^{-c(2^{ks}+\epsilon^s2^{js})(t-\tau)}e^{-c\lambda2^j\int_\tau^t\dot\theta(t')dt'}\|\nabla_h\Delta_{k,j}q_\Phi^3\|_{L_h^p(L_v^2)}d\tau.$$

Hence, using the fact that $(-\Delta_\epsilon)^{-1}\partial_i\partial_j$ is a bounded operator applied for frequency localized functions in $L_h^p(L_v^2)$ when $i,j=1,2,\ldots,n-1$, we get

$$\|\nabla_h\Delta_{k,j}q_\Phi^1\|_{L_h^p(L_v^2)}\lesssim\|\Delta_{k,j}(v^h\cdot\nabla_h v^h)\|_{L_h^p(L_v^2)}.$$

By the same method as in the estimate of I_2, we have

$$\|I_{31}\|_{L_t^\infty}+2^{ks}\|I_{31}\|_{L_t^1}$$
$$\lesssim c_{k,j}2^{-k(\frac{n-1}{p}+1-s)}2^{-\frac{1}{2}j}\|v_\Phi^h\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})}\|v_\Phi^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p}+1,\frac{1}{2}})}.$$

Noting that

$$\nabla_h q^2=2(-\Delta_\epsilon)^{-1}\nabla_h\partial_i(v^n\partial_n v^h-v^h\mathrm{div}_h v^h)$$

and as in the estimate of I_2, it holds that

$$\|I_{32}\|_{L_t^\infty}+2^{ks}\|I_{32}\|_{L_t^1}$$
$$\lesssim c_{k,j}2^{-k(\frac{n-1}{p}+1-s)}2^{-\frac{1}{2}j}\int_0^t\|v_\Phi^n\|_{\dot{B}_{p,1}^{\frac{n-1}{p},\frac{1}{2}}}\|\partial_n v_\Phi^h\|_{\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}}}d\tau$$
$$+c_{k,j}2^{-k(\frac{n-1}{p}+1-s)}2^{-\frac{1}{2}j}\|v_\Phi^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p}+1,\frac{1}{2}})}\|v_\Phi^h\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})}.$$

Using

$$\nabla_h q^3=2(-\Delta_\epsilon)^{-1}\nabla_h(\mathrm{div}_h v^h\mathrm{div}_h v^h-v^n\partial_n\mathrm{div}_h v^h),$$

we write I_{33} as follows

$$I_{33}\lesssim 2^{-k}\int_0^t e^{-c2^{ks}(t-\tau)}\Big(\|\Delta_{k,j}(\mathrm{div}_h v^h\mathrm{div}_h v^h)_\Phi\|_{L_h^p(L_v^2)}$$
$$+\|\Delta_{k,j}(v^n\partial_n\mathrm{div}_h v^h)_\Phi\|_{L_h^p(L_v^2)}\Big)d\tau.$$

By Young's inequality and Lemma 3, as $1 \le s < \min\{n-1, 2\frac{n-1}{p}\}$, we have

$$\begin{aligned}
&\|I_{33}\|_{L_t^\infty} + 2^{ks}\|I_{33}\|_{L_t^1} \\
&\quad\lesssim c_{k,j}2^{-k(\frac{n-1}{p}+1-s)}2^{-\frac{1}{2}j}\|v_\Phi^h\|_{L_t^\infty(\dot{B}^{\frac{n-1}{p}+1-s,\frac{1}{2}}_{p,1})}\|v_\Phi^h\|_{L_t^1(\dot{B}^{\frac{n-1}{p}+1,\frac{1}{2}}_{p,1})} \\
&\quad+ c_{k,j}2^{-k(\frac{n-1}{p}+1-s)}2^{-\frac{1}{2}j}\int_0^t \|v_\Phi^n\|_{\dot{B}^{\frac{n-1}{p},\frac{1}{2}}_{p,1}}\|\partial_n v_\Phi^h\|_{\dot{B}^{\frac{n-1}{p}+1-s,\frac{1}{2}}_{p,1}}d\tau.
\end{aligned}$$

Now we are going to estimate the key quantity

$$\int_0^t \|v_\Phi^n\|_{\dot{B}^{\frac{n-1}{p},\frac{1}{2}}_{p,1}}\|\partial_n v_\Phi^h\|_{\dot{B}^{\frac{n-1}{p}+1-s,\frac{1}{2}}_{p,1}}d\tau.$$

According to (3.3), we find that in each dyadic block $\partial_n v_\Phi^h$ verifies

$$\begin{aligned}
\Delta_{k,j}\partial_n v_\Phi^h(\tau,x) &= e^{-\tau D_\epsilon^s+\Phi(\tau,D_n)}\Delta_{k,j}\partial_n v_0^h \\
&\quad-\int_0^\tau e^{-(\tau-t')D_\epsilon^s}e^{-\lambda D_n\int_{t'}^\tau\dot\theta(t'')dt''}\Delta_{k,j}\partial_n(v\cdot\nabla v^h)_\Phi(t')dt' \\
&\quad-\int_0^\tau e^{-(\tau-t')D_\epsilon^s}e^{-\lambda D_n\int_{t'}^\tau\dot\theta(t'')dt''}\nabla_h\Delta_{k,j}\partial_n q_\Phi(t')dt'.
\end{aligned}\tag{3.4}$$

Taking the $L_h^p(L_v^2)$ norm on both sides of (3.4), we have

$$\begin{aligned}
&\|\Delta_{k,j}\partial_n v_\Phi^h\|_{L_h^p(L_v^2)} \\
&\lesssim e^{-c(2^{ks}+\epsilon^s 2^{js})\tau}e^{-c\lambda 2^j\int_0^\tau\dot\theta(t'')dt''}\|\Delta_{k,j}e^{\alpha D_n}\partial_n v_0^h\|_{L_h^p(L_v^2)} \\
&+\int_0^\tau e^{-c(2^{ks}+\epsilon^s 2^{js})(\tau-t')}e^{-c\lambda 2^j\int_{t'}^\tau\dot\theta(t'')dt''}\|\Delta_{k,j}\partial_n(v\cdot\nabla v^h)_\Phi\|_{L_h^p(L_v^2)}dt' \\
&+\int_0^\tau e^{-c(2^{ks}+\epsilon^s 2^{js})(\tau-t')}e^{-c\lambda 2^j\int_{t'}^\tau\dot\theta(t'')dt''}\|\nabla_h\Delta_{k,j}\partial_n q_\Phi\|_{L_h^p(L_v^2)}dt'.
\end{aligned}\tag{3.5}$$

For fixed k, j, multiplying the (3.5) by $\dot\theta(\tau)$ and integrating over $(0,t)$, one can have

$$\begin{aligned}
&\int_0^t\dot\theta(\tau)\|\Delta_{k,j}\partial_n v_\Phi^h\|_{L_h^p(L_v^2)}d\tau \\
&\quad\lesssim\int_0^t e^{-c\lambda 2^j\int_0^\tau\dot\theta(t'')dt''}\dot\theta(\tau)\|e^{\alpha D_n}\partial_n\Delta_{k,j}v_0^h\|_{L_h^p(L_v^2)}d\tau \\
&\qquad+\int_0^t\int_0^\tau e^{-c\lambda 2^j\int_{t'}^\tau\dot\theta(t'')dt''}2^j\dot\theta(\tau)\|\Delta_{k,j}(v\cdot\nabla v^h)_\Phi\|_{L_h^p(L_v^2)}dt'd\tau \\
&\qquad+\int_0^t\int_0^\tau e^{-c\lambda 2^j\int_{t'}^\tau\dot\theta(t'')dt''}2^j\dot\theta(\tau)\|\nabla_h\Delta_{k,j}q_\Phi\|_{L_h^p(L_v^2)}dt'd\tau \\
&\quad\overset{\text{def}}{=} I_4+I_5+I_6.
\end{aligned}$$

The term I_4 containing initial data can be bounded by

$$\begin{aligned} I_4 &\lesssim \int_0^t e^{-c\lambda 2^j \int_0^\tau \dot\theta(t'')dt''} 2^j \dot\theta(\tau) d\tau \|\Delta_{k,j} e^{\alpha D_n} v_0^h\|_{L_h^p(L_v^2)} \\ &\lesssim \frac{1}{\lambda} c_{k,j} 2^{-k(\frac{n-1}{p}+1-s)} 2^{-\frac{1}{2}j} \|e^{\alpha D_n} v_0^h\|_{\dot B_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}}}. \end{aligned}$$

By Fubini's theorem, the term I_5 can be rewritten as

$$\begin{aligned} I_5 &\lesssim \int_0^t \int_{t'}^t e^{-c\lambda 2^j \int_{t'}^\tau \dot\theta(t'')dt''} 2^j \dot\theta(\tau) d\tau \|\Delta_{k,j}(v\cdot\nabla v^h)_\Phi\|_{L_h^p(L_v^2)} dt' \\ &\lesssim \frac{1}{\lambda} \int_0^t \|\Delta_{k,j}(v\cdot\nabla v^h)_\Phi\|_{L_h^p(L_v^2)} dt' \\ &\lesssim \frac{1}{\lambda} c_{k,j} 2^{-k(\frac{n-1}{p}+1-s)} 2^{-\frac{1}{2}j} \|(v\cdot\nabla v^h)_\Phi\|_{L_t^1(\dot B_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})}. \end{aligned}$$

Thus, we can get by Lemma 2 and 3 that

$$\begin{aligned} I_5 &\lesssim \frac{1}{\lambda} c_{k,j} 2^{-k(\frac{n-1}{p}+1-s)} 2^{-\frac{1}{2}j} \|v_\Phi^h\|_{\widetilde L_t^\infty(\dot B_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})} \|v_\Phi^h\|_{L_t^1(\dot B_{p,1}^{\frac{n-1}{p}+1,\frac{1}{2}})} \\ &\quad + \frac{1}{\lambda} c_{k,j} 2^{-k(\frac{n-1}{p}+1-s)} 2^{-\frac{1}{2}j} \int_0^t \|v_\Phi^n\|_{\dot B_{p,1}^{\frac{n-1}{p},\frac{1}{2}}} \|\partial_n v_\Phi^h\|_{\dot B_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}}} dt'. \end{aligned}$$

As for I_6, for convenience, we denote that

$$\begin{aligned} I_{61} &= \int_0^t \int_0^\tau e^{-c\lambda 2^j \int_{t'}^\tau \dot\theta(t'')dt''} 2^j \dot\theta(\tau) \|\nabla_h \Delta_{k,j} q_\Phi^1\|_{L_h^p(L_v^2)}(t') dt' d\tau, \\ I_{62} &= \int_0^t \int_0^\tau e^{-c\lambda 2^j \int_{t'}^\tau \dot\theta(t'')dt''} 2^j \dot\theta(\tau) \|\nabla_h \Delta_{k,j} q_\Phi^2\|_{L_h^p(L_v^2)}(t') dt' d\tau, \\ I_{63} &= \int_0^t \int_0^\tau e^{-c\lambda 2^j \int_{t'}^\tau \dot\theta(t'')dt''} 2^j \dot\theta(\tau) \|\nabla_h \Delta_{k,j} q_\Phi^3\|_{L_h^p(L_v^2)}(t') dt' d\tau. \end{aligned}$$

By the same method as in the estimate of I_5, we have

$$\begin{aligned} &I_{61} + I_{62} \\ &\lesssim \frac{1}{\lambda} c_{k,j} 2^{-k(\frac{n-1}{p}+1-s)} 2^{-\frac{1}{2}j} \|v_\Phi^h\|_{\widetilde L_t^\infty(\dot B_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})} \|v_\Phi^h\|_{L_t^1(\dot B_{p,1}^{\frac{n-1}{p}+1,\frac{1}{2}})} \\ &\quad + \frac{1}{\lambda} c_{k,j} 2^{-k(\frac{n-1}{p}+1-s)} 2^{-\frac{1}{2}j} \int_0^t \|v_\Phi^n\|_{\dot B_{p,1}^{\frac{n-1}{p},\frac{1}{2}}} \|\partial_n v_\Phi^h\|_{\dot B_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}}} dt'. \end{aligned}$$

Finally, I_{63} can be estimated as follows

$$\begin{aligned} I_{63} &\lesssim \int_0^t \int_0^\tau e^{-c\lambda 2^j \int_{t'}^\tau \dot\theta(t'')dt''} 2^j \dot\theta(\tau) \\ &\quad \times \|\Delta_{k,j}(-\Delta_\epsilon)^{-1} \nabla_h \partial_n (v^n \mathrm{div}_h v^h)_\Phi\|_{L_h^p(L_v^2)}(t') dt' d\tau \\ &\lesssim 2^{-k} \int_0^t \int_{t'}^\tau e^{-c\lambda 2^j \int_{t'}^\tau \dot\theta(t'')dt''} 2^j \dot\theta(\tau) d\tau \|\Delta_{k,j} \partial_n (v^n \mathrm{div}_h v^h)_\Phi\|_{L_h^p(L_v^2)}(t') dt'. \end{aligned}$$

Thus, we can obtain that

$$\begin{aligned}
I_{63} &\lesssim \frac{1}{\lambda}c_{k,j}2^{-k(\frac{n-1}{p}+1-s)}2^{-\frac{1}{2}j}\|\partial_n(v^n\mathrm{div}_h v^h)_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})}\\
&\lesssim \frac{1}{\lambda}c_{k,j}2^{-k(\frac{n-1}{p}+1-s)}2^{-\frac{1}{2}j}\|v^h_\Phi\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n-1}{p}+1-s,\frac{1}{2}}_{p,1})}\|v^h_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}+1,\frac{1}{2}}_{p,1})}\\
&\quad+\frac{1}{\lambda}c_{k,j}2^{-k(\frac{n-1}{p}+1-s)}2^{-\frac{1}{2}j}\int_0^t\|v^n_\Phi\|_{\dot{B}^{\frac{n-1}{p},\frac{1}{2}}_{p,1}}\|\partial_n v^h_\Phi\|_{\dot{B}^{\frac{n-1}{p}+1-s,\frac{1}{2}}_{p,1}}dt'.
\end{aligned}$$

Together with the above estimates on I_1-I_6, we get that

$$\begin{aligned}
&\|v^h_\Phi\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n-1}{p}+1-s,\frac{1}{2}}_{p,1})}+\|v^h_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}+1,\frac{1}{2}}_{p,1})}+\int_0^t\|v^n_\Phi\|_{\dot{B}^{\frac{n-1}{p},\frac{1}{2}}_{p,1}}\\
&\quad\times\|\partial_n v^h_\Phi\|_{\dot{B}^{\frac{n-1}{p}+1-s,\frac{1}{2}}_{p,1}}d\tau\lesssim\Psi(0)+\frac{1}{\lambda}\Psi(t)+\Psi(t)^2.
\end{aligned}\tag{3.6}$$

3.2 Estimates on the vertical component v^n

We begin this part by studying the equation of v^n, which is stated as follows

$$\partial_t v^n+D^s_\epsilon v^n+v\cdot\nabla v^n+\epsilon^2\partial_n q=0.$$

Observing that in the above equation, one can expect that there is no loss of derivative in vertical direction. More precisely, due to divergence free condition, the nonlinear term $v^n\partial_n v^n$ can be rewritten as $-v^n\mathrm{div}_h v^h$. Thus the estimate on v^n is different from v^h.

Applying the anisotropic dyadic decomposition operator $\Delta_{k,j}$ to the equation of v^n, then in each dyadic block, v^n satisfies

$$\partial_t\Delta_{k,j}v^n+D^s_\epsilon\Delta_{k,j}v^n=-\Delta_{k,j}(v^h\cdot\nabla_h v^n)+\Delta_{k,j}(v^n\mathrm{div}_h v^h)-\epsilon^2\Delta_{k,j}\partial_n q.\tag{3.7}$$

Let us define $G\overset{\mathrm{def}}{=}v^h\cdot\nabla_h v^n-v^n\mathrm{div}_h v^h$. We write the solution of (3.7) as follows

$$\begin{aligned}
\Delta_{k,j}v^n_\Phi&=e^{-tD^s_\epsilon+\Phi(t,D_n)}\Delta_{k,j}v^n_0+\int_0^t e^{-(t-\tau)D^s_\epsilon}e^{-\lambda D_n\int_\tau^t\dot\theta(t')dt'}\Delta_{k,j}G_\Phi d\tau\\
&\quad+\epsilon^2\int_0^t e^{-(t-\tau)D^s_\epsilon}e^{-\lambda D_n\int_\tau^t\dot\theta(t')dt'}\Delta_{k,j}\partial_n q_\Phi d\tau.
\end{aligned}$$

Taking the $L^p_h(L^2_v)$ norm, we infer that

$$\begin{aligned}
\|\Delta_{k,j}v^n_\Phi\|_{L^p_h(L^2_v)}&\lesssim e^{-c2^{ks}t}\|\Delta_{k,j}e^{\alpha D_n}v^n_0\|_{L^p_h(L^2_v)}\\
&\quad+\int_0^t e^{-c2^{ks}(t-\tau)}\|\Delta_{k,j}G_\Phi\|_{L^p_h(L^2_v)}d\tau+\epsilon^2\int_0^t e^{-c2^{ks}(t-\tau)}\|\Delta_{k,j}\partial_n q_\Phi\|_{L^p_h(L^2_v)}d\tau.
\end{aligned}$$

By the Young's inequality, we deduce that

$$\begin{aligned}
&\|\Delta_{k,j}v^n_\Phi\|_{L^\infty_t(L^p_h(L^2_v))}+2^{ks}\|\Delta_{k,j}v^n_\Phi\|_{L^1_t(L^p_h(L^2_v))}\lesssim\|\Delta_{k,j}e^{\alpha D_n}v^n_0\|_{L^p_h(L^2_v)}\\
&\quad+\|\Delta_{k,j}G_\Phi\|_{L^1_t(L^p_h(L^2_v))}\quad+\epsilon^2\|\Delta_{k,j}\partial_n q_\Phi\|_{L^1_t(L^p_h(L^2_v))}.
\end{aligned}\tag{3.8}$$

Multiplying both sides of (3.8) by $2^{k(\frac{n-1}{p}+1-s)}2^{j\frac{1}{2}}$ and taking the sum over k, j, we have

$$\begin{aligned}&\|v_{\Phi}^n\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})}+\|v_{\Phi}^n\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p}+1,\frac{1}{2}})}\\&\lesssim\|e^{\alpha D_n}v_0^n\|_{\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}}}+\|G_{\Phi}\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})}+\epsilon^2\|\partial_n q_{\Phi}\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})}.\end{aligned}$$

According to Lemma 3, we can obtain the estimates of nonlinear term by the following:

$$\|(v^h\cdot\nabla_h v^n)_{\Phi}\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})}\lesssim\|v_{\Phi}^h\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})}\|v_{\Phi}^n\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p}+1,\frac{1}{2}})},$$

$$\|(v^n\mathrm{div}_h v^h)_{\Phi}\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})}\lesssim\|v_{\Phi}^n\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})}\|v_{\Phi}^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p}+1,\frac{1}{2}})}.$$

This implies that

$$\|G_{\Phi}\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})}\lesssim\|v_{\Phi}\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})}\|v_{\Phi}\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p}+1,\frac{1}{2}})}.$$

While for the pressure term, we use the decomposition $q=q^1+q^2+q^3$ in (3.1). For q^1, since $\epsilon(-\Delta_\epsilon)^{-1}\partial_i\partial_n$ is a bounded operator applied for frequency localized functions in $L_h^p(L_v^2)$ if $i=1,2,\cdots,n-1$, we have

$$\begin{aligned}\epsilon^2\|\partial_n q_{\Phi}^1\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})}&=\epsilon^2\|(-\Delta_\epsilon)^{-1}\partial_i\partial_j\partial_n(v^iv^j)_{\Phi}\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})}\\&\lesssim\epsilon\|\nabla_h(v^hv^h)_{\Phi}\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})}.\end{aligned}$$

Therefore, we get by using Lemma 3 that

$$\epsilon^2\|\partial_n q_{\Phi}^1\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})}\lesssim\epsilon\|v_{\Phi}^h\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})}\|v_{\Phi}^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p}+1,\frac{1}{2}})}.$$

Similarly, the fact that $\epsilon^2(-\Delta_\epsilon)^{-1}\partial_n^2$ is a bounded operator applied for frequency localized functions in $L_h^p(L_v^2)$ implies

$$\epsilon^2\|\partial_n q_{\Phi}^2\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})}\lesssim\|\nabla_h(v^nv^h)_{\Phi}\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})},$$

$$\epsilon^2\|\partial_n q_{\Phi}^3\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})}\lesssim\|(v^n\mathrm{div}_h v^h)_{\Phi}\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})}.$$

Thus, we have

$$\begin{aligned}&\epsilon^2\|\partial_n q_{\Phi}^2\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})}+\epsilon^2\|\partial_n q_{\Phi}^3\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})}\\&\lesssim\|v_{\Phi}\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})}\|v_{\Phi}\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p}+1,\frac{1}{2}})}.\end{aligned}$$

Then we obtain that

$$\epsilon^2\|\partial_n q_{\Phi}\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})}\lesssim\|v_{\Phi}\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n-1}{p}+1-s,\frac{1}{2}})}\|v_{\Phi}\|_{L_t^1(\dot{B}_{p,1}^{\frac{n-1}{p}+1,\frac{1}{2}})}.$$

Combining the above estimates, we can get the bound of v^n_Φ as follows:

$$\|v^n_\Phi\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n-1}{p}+1-s,\frac{1}{2}}_{p,1})}+\|v^n_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}+1,\frac{1}{2}}_{p,1})}\lesssim \Psi(0)+\Psi(t)^2. \tag{3.9}$$

Together (3.9) with (3.6), we finally get that

$$\Psi(t)\lesssim \Psi(0)+\frac{1}{\lambda}\Psi(t)+\Psi(t)^2.$$

This completes the proof of Lemma 4 by choosing λ large enough.

4 Estimates for $\theta(t)$

In the above section, we have used the fact that $\Phi(t)$ is a subadditivity function. This means we should ensure that $\theta(t)<\frac{\alpha}{\lambda}$. Thus, it is sufficient to prove that for any time t, $\theta(t)$ is a small quantity. By the definition of $\theta(t)$, naturally, we assume that $e^{\alpha D_n}v^n_0$ belongs to $\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1}$. According to the property of the operator $\partial_t+D^s_\epsilon$, then we can get the bound for v^n_Φ in $L^1_t(\dot{B}^{\frac{n-1}{p},\frac{1}{2}}_{p,1})$. However, we can not enclose the estimate for v^n_Φ in $\widetilde{L}^\infty_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})\cap L^1_t(\dot{B}^{\frac{n-1}{p},\frac{1}{2}}_{p,1})$. Our observation is to add an extra term ϵv^h under the same norm which is hidden in the pressure term $\epsilon^2\partial_n q^1$. Hence, we first denote that

$$X_0=\epsilon\|e^{\alpha D_n}v^h_0\|_{\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1}},\quad Y_0=\|e^{\alpha D_n}v^n_0\|_{\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1}},$$

$$\begin{aligned}X(t)&=\epsilon\|v^h_\Phi\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})}+\epsilon\|v^h_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p},\frac{1}{2}}_{p,1})},\\ Y(t)&=\|v^n_\Phi\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})}+\|v^n_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p},\frac{1}{2}}_{p,1})}.\end{aligned}$$

In order to get the desired estimates, it suffices to prove the following lemma.

Lemma 5. *There exists a constant C_2 such that for any $\lambda>0$ and t satisfying $\theta(t)\le\frac{\alpha}{2\lambda}$, we have*

$$X(t)+Y(t)\le C_2(X_0+Y_0)+C_2(X(t)+Y(t))\Psi(t).$$

Proof. We apply the same method as in the above section to prove $Y(t)$. Indeed, multiplying both sides of (3.8) by $2^{(\frac{n-1}{p}-s)k}2^{\frac{1}{2}j}$ and taking the sum over k,j, we can get that

$$\begin{aligned}&\|v^n_\Phi\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})}+\|v^n_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p},\frac{1}{2}}_{p,1})}\\ &\quad\le\|e^{\alpha D_n}v^n_0\|_{\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1}}+\|G_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})}+\epsilon^2\|\partial_n q_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})}.\end{aligned}$$

According to Lemma 3, we can obtain the estimates of nonlinear terms that

$$\begin{aligned}&\|\mathrm{div}_h(v^nv^h)_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})}+\|(v^n\mathrm{div}_hv^h)_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})}\\ &\qquad\lesssim\|v^n_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p},\frac{1}{2}}_{p,1})}\|v^h_\Phi\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n-1}{p}+1-s,\frac{1}{2}}_{p,1})}.\end{aligned}$$

This implies that

$$\|G_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})} \lesssim Y(t)\Psi(t).$$

While for the pressure term, we use the decomposition $q = q^1 + q^2 + q^3$ in (3.1). For q_1, since $\epsilon(-\Delta_\epsilon)^{-1}\partial_i\partial_n$ is a bounded operator applied for frequency localized functions in $L^p_h(L^2_v)$ if $i = 1, 2, \cdots, n-1$, we have

$$\begin{aligned}\epsilon^2\|\partial_n q^1_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})} &= \epsilon^2\|(-\Delta_\epsilon)^{-1}\partial_i\partial_j\partial_n(v^iv^j)_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})}\\ &\lesssim \epsilon\|\nabla_h(v^hv^h)_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})} \lesssim \epsilon\|v^h_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p},\frac{1}{2}}_{p,1})}\|v^h_\Phi\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n-1}{p}+1-s,\frac{1}{2}}_{p,1})}.\end{aligned}$$

Similarly, the fact that $\epsilon^2(-\Delta_\epsilon)^{-1}\partial_n^2$ is a bounded operator applied for frequency localized functions in $L^p_h(L^2_v)$ implies

$$\epsilon^2\|\partial_n q^2_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})} \lesssim \|\nabla_h(v^nv^h)_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})},$$

$$\epsilon^2\|\partial_n q^3_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})} \lesssim \|(v^n\mathrm{div}_h v^h)_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})}.$$

Thus, applying Lemma 3, we have

$$\begin{aligned}&\epsilon^2\|\partial_n q^2_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})} + \epsilon^2\|\partial_n q^3_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})}\\ &\lesssim \|v^n_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p},\frac{1}{2}}_{p,1})}\|v^h_\Phi\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n-1}{p}+1-s,\frac{1}{2}}_{p,1})}.\end{aligned}$$

Then we obtain that

$$\epsilon^2\|\partial_n q_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p},\frac{1}{2}}_{p,1})} \lesssim Y(t)\Psi(t) + X(t)\Psi(t).$$

Combining all the above estimates, we can get the bound of v^n_Φ in $\widetilde{L}^\infty_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1}) \cap L^1_t(\dot{B}^{\frac{n-1}{p},\frac{1}{2}}_{p,1})$ by the following:

$$\|v^n_\Phi\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})} + \|v^n_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p},\frac{1}{2}}_{p,1})} \lesssim Y_0 + (X(t)+Y(t))\Psi(t). \tag{4.1}$$

This completes the proof of $Y(t)$ in Lemma 5.

The following is devoted to getting the estimate of $X(t)$. The horizontal component v^h in each dyadic block satisfies

$$\partial_t\Delta_{k,j}v^h + D^s_\epsilon\Delta_{k,j}v^h = -\Delta_{k,j}\mathrm{div}_h(v^h\otimes v^h) - \Delta_{k,j}\partial_n(v^nv^h) - \nabla_h\Delta_{k,j}q.$$

Denote $F \overset{\mathrm{def}}{=} -\mathrm{div}_h(v^h\otimes v^h) - \partial_n(v^nv^h)$, then we infer that

$$\begin{aligned}\|\Delta_{k,j}v^h_\Phi\|_{L^p_h(L^2_v)} &\lesssim e^{-c(2^{ks}+\epsilon^s2^{js})t}\|\Delta_{k,j}e^{\alpha D_n}v^h_0\|_{L^p_h(L^2_v)} + \int_0^t e^{-c(2^{ks}+\epsilon^s2^{js})(t-\tau)}\\ &\times\|\Delta_{k,j}F_\Phi\|_{L^p_h(L^2_v)}d\tau + \int_0^t e^{-c(2^{ks}+\epsilon^s2^{js})(t-\tau)}\|\Delta_{k,j}\nabla_h q_\Phi\|_{L^p_h(L^2_v)}d\tau.\end{aligned}$$

By Young's inequality, it holds that

$$\begin{aligned}
&\|\int_0^t e^{-c(2^{ks}+\epsilon^s 2^{js})(t-\tau)}\|\Delta_{k,j}\partial_n(v^n v^h)_\Phi\|_{L^p_h(L^2_v)}d\tau\|_{L^\infty_t}\\
&\quad\lesssim \frac{1}{\epsilon}\|\Delta_{k,j}(v^n v^h)_\Phi\|_{L_t^{\frac{s}{s-1}}(L^p_h(L^2_v))},\\
&2^{ks}\|\int_0^t e^{-c(2^{ks}+\epsilon^s 2^{js})(t-\tau)}\|\Delta_{k,j}\partial_n(v^n v^h)_\Phi\|_{L^p_h(L^2_v)}d\tau\|_{L^1_t}\\
&\quad\lesssim \frac{1}{\epsilon}2^k\|\Delta_{k,j}(v^n v^h)_\Phi\|_{L^1_t(L^p_h(L^2_v))}.
\end{aligned}$$

Here and in what follows, if $s=1$, the quantity $\|\Delta_{k,j}(v^n v^h)_\Phi\|_{L_t^{\frac{s}{s-1}}(L^p_h(L^2_v))}$ should be regarded as $\|\Delta_{k,j}(v^n v^h)_\Phi\|_{L^\infty_t(L^p_h(L^2_v))}$. Therefore, taking L^∞ norm and L^1 norm on $[0,t]$, we deduce that

$$\begin{aligned}
&\|\Delta_{k,j}v^h_\Phi\|_{L^\infty_t(L^p_h(L^2_v))}+2^{ks}\|\Delta_{k,j}v^h_\Phi\|_{L^1_t(L^p_h(L^2_v))}\lesssim\|e^{\alpha D_n}\Delta_{k,j}v^h_0\|_{L^p_h(L^2_v)}\\
&\quad+2^k\|\Delta_{k,j}(v^h\otimes v^h)_\Phi\|_{L^1_t(L^p_h(L^2_v))}+\frac{1}{\epsilon}2^k\|\Delta_{k,j}(v^n v^h)_\Phi\|_{L^1_t(L^p_h(L^2_v))}\\
&\quad+\frac{1}{\epsilon}\|\Delta_{k,j}(v^n v^h)_\Phi\|_{L_t^{\frac{s}{s-1}}(L^p_h(L^2_v))}+\|\Delta_{k,j}\nabla_h q_\Phi\|_{L^1_t(L^p_h(L^2_v))}.
\end{aligned}\tag{4.2}$$

Multiplying both sides of (4.2) by $2^{(\frac{n-1}{p}-s)k}2^{\frac{1}{2}j}$ and taking the sum over k,j, we finally get

$$\begin{aligned}
&\epsilon\|v^h_\Phi\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})}+\epsilon\|v^h_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p},\frac{1}{2}}_{p,1})}\\
&\lesssim\epsilon\|e^{\alpha D_n}v^h_0\|_{\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1}}+\epsilon\|(v^h\otimes v^h)_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}+1-s,\frac{1}{2}}_{p,1})}\\
&\quad+\|(v^n v^h)_\Phi\|_{\widetilde{L}_t^{\frac{s}{s-1}}(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})}+\|(v^n v^h)_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}+1-s,\frac{1}{2}}_{p,1})}\\
&\quad+\epsilon\|\nabla_h q_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})}.
\end{aligned}\tag{4.3}$$

For the pressure term $q=q^1+q^2+q^3$, we find that

$$\begin{aligned}
\|\nabla_h q^1_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})}&=\|(-\Delta_\epsilon)^{-1}\nabla_h\partial_i\partial_j(v^i v^j)_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})}\\
&\lesssim\|\nabla_h(v^h v^h)_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})},
\end{aligned}$$

where we have used that $(-\Delta_\epsilon)^{-1}\partial_i\partial_j$ is a bounded operator for frequency localized functions in $L^p_h(L^2_v)$. Similarly,

$$\begin{aligned}
\|\nabla_h q^2_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})}&=2\|(-\Delta_\epsilon)^{-1}\nabla_h\partial_i\partial_n(v^i v^n)_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})}\\
&\lesssim\frac{1}{\epsilon}\|\nabla_h(v^n v^h)_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})},\\
\|\nabla_h q^3_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})}&=2\|(-\Delta_\epsilon)^{-1}\nabla_h\partial_n(v^n\mathrm{div}_h v^h)_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})}\\
&\lesssim\frac{1}{\epsilon}\|(v^n\mathrm{div}_h v^h)_\Phi\|_{L^1_t(\dot{B}^{\frac{n-1}{p}-s,\frac{1}{2}}_{p,1})}.
\end{aligned}$$

According to Lemma 3, the right hand side of (4.3) can be bounded by following:

$$\begin{aligned}
&\epsilon\|(v^h\otimes v^h)_\Phi\|_{L^1_t(\dot B^{\frac{n-1}{p}+1-s,\frac12}_{p,1})}\lesssim \epsilon\|v^h_\Phi\|_{L^1_t(\dot B^{\frac{n-1}{p},\frac12}_{p,1})}\|v^h_\Phi\|_{\widetilde L^\infty_t(\dot B^{\frac{n-1}{p}+1-s,\frac12}_{p,1})},\\
&\|(v^nv^h)_\Phi\|_{L^1_t(\dot B^{\frac{n-1}{p}+1-s,\frac12}_{p,1})}\lesssim \|v^h_\Phi\|_{\widetilde L^\infty_t(\dot B^{\frac{n-1}{p}+1-s,\frac12}_{p,1})}\|v^n_\Phi\|_{L^1_t(\dot B^{\frac{n-1}{p},\frac12}_{p,1})},\\
&\epsilon\|\nabla_h q_\Phi\|_{L^1_t(\dot B^{\frac{n-1}{p}-s,\frac12}_{p,1})}\lesssim \Big(\epsilon\|v^h_\Phi\|_{L^1_t(\dot B^{\frac{n-1}{p},\frac12}_{p,1})}+\|v^n_\Phi\|_{L^1_t(\dot B^{\frac{n-1}{p},\frac12}_{p,1})}\Big)\\
&\qquad\times\|v^h_\Phi\|_{\widetilde L^\infty_t(\dot B^{\frac{n-1}{p}+1-s,\frac12}_{p,1})},\\
&\|(v^nv^h)_\Phi\|_{\widetilde L^{\frac{s}{s-1}}_t(\dot B^{\frac{n-1}{p}-s,\frac12}_{p,1})}\lesssim \|v^n_\Phi\|_{\widetilde L^{\frac{s}{s-1}}_t(\dot B^{\frac{n-1}{p}-1,\frac12}_{p,1})}\|v^h_\Phi\|_{\widetilde L^\infty_t(\dot B^{\frac{n-1}{p}+1-s,\frac12}_{p,1})}\\
&\quad\lesssim\Big(\|v^n_\Phi\|_{\widetilde L^\infty_t(\dot B^{\frac{n-1}{p}-s,\frac12}_{p,1})}+\|v^n_\Phi\|_{L^1_t(\dot B^{\frac{n-1}{p},\frac12}_{p,1})}\Big)\|v^h_\Phi\|_{\widetilde L^\infty_t(\dot B^{\frac{n-1}{p}+1-s,\frac12}_{p,1})}.
\end{aligned}$$

These imply that

$$\epsilon\|v^h_\Phi\|_{\widetilde L^\infty_t(\dot B^{\frac{n-1}{p}-s,\frac12}_{p,1})}+\epsilon\|v^h_\Phi\|_{L^1_t(\dot B^{\frac{n-1}{p},\frac12}_{p,1})}\lesssim X_0+(X(t)+Y(t))\Psi(t). \tag{4.4}$$

Combining (4.4) with (4.1), we finally obtain that there exists a constant C_2 such that

$$X(t)+Y(t)\le C_2(X_0+Y_0)+C_2(X(t)+Y(t))\Psi(t).$$

This completes the proof of Lemma 5. □

5 Proof of the main result

In this section, we will prove the Theorem 2. It relies on a continuation argument. For any $\lambda>\lambda_0$ and η_1, we define τ by

$$\tau=\max\{t\ge0|\ X(t)+Y(t)\le\eta_1,\quad \Psi(t)\le\eta_1\}. \tag{5.1}$$

In what follows, we shall prove that $\tau=\infty$ under the assumption (1.3) for some small number η_1. Assume that this is not true. We choose η_1 small enough such that

$$\theta(\tau)\le CY(\tau)\le C\eta_1\le\frac{\alpha}{2\lambda},\quad (C_1+C_2)\eta_1\le\frac14.$$

For such fixed η_1, we select the following norms of initial data sufficiently small enough such that

$$C_1\Psi(0)+C_2(X(0)+Y(0))\le C\eta\le\frac{\eta_1}{4}.$$

Hence, we obtain from Lemma 4 and 5 that

$$\Psi(\tau)\le C_1\Psi(0)+C_1\eta_1^2,\quad X(\tau)+Y(\tau)\le C_2(X(0)+Y(0))+C_2\eta_1^2.$$

This implies that

$$X(\tau)+Y(\tau)\le\frac{\eta_1}{2},\ \ \Psi(\tau)\le\frac{\eta_1}{2}.$$

However, this contradicts (5.1) and hence completes the proof.

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