Mathematical Modelling and Analysis
Volume 23, Issue 2, 240–261, 2018
https://doi.org/10.3846/mma.2018.016

http://mma.vgtu.lt
ISSN: 1392-6292
eISSN: 1648-3510

Mixed Jacobi-Fourier Spectral Method for Fisher Equation

Yujian Jiao[a], Tianjun Wang[b], Xiandong Shi[a] and Wenjie Liu[c]

[a] *Department of Mathematics, Shanghai Normal University and Scientific Computing Key Laboratory of Shanghai Universities*
200234 Shanghai, China
[b] *Henan University of Science and Technology*
471003 Luoyang, China
[c] *Department of Mathematics, Harbin Institute of Technology*
150001 Harbin, China
E-mail(*corresp.*): wangtianjun64@163.com
E-mail: yj-jiao@shnu.edu.cn

Received July 10, 2017; revised February 3, 2018; accepted February 4, 2018

Abstract. In this paper, we propose a mixed Jacobi-Fourier spectral method for solving the Fisher equation in a disc. Some mixed Jacobi-Fourier approximation results are established, which play important roles in numerical simulation of various problems defined in a disc. We use the generalized Jacobi approximation to simulate the singularity of solution at the regional center. This also simplifies the theoretical analysis and provides a sparse system. The stability and convergence of the proposed scheme are proved. Numerical results demonstrate the efficiency of this new algorithm and coincide well with the theoretical analysis.

Keywords: Fisher equation in a disc, mixed Jacobi-Fourier approximation, spectral method, problem with end-point weak singularity, nonlinear problem.

AMS Subject Classification: 41A30; 76M22; 41A63; 33C45; 33C55.

1 Introduction

Fisher equation serves as a foundation in various mathematical investigations in ecology and biology, see, e.g., [6, 7, 16, 17, 20, 23]. There are so many works concerned with analytic solutions of the Fisher equation, see, e.g., [12, 25, 28] and the references therein. Recently, Olmos and Shizgal [21] provided the numerical solution with the Chebyshev collocation method. Mittal and Jiwari [19] numerically discussed the Fisher equation by using the differential quadrature method. Jiwari [15] used Haar wavelet quasilinearization approach to solve Burges' equation. Wang [26] studied the Fisher equation on a semi-infinite domain using the generalized Laguerre functions. Wang and Jiao [27] considered the Fisher equation on unbounded domain using the generalized Hermite functions.

In this paper, we consider the Fisher equation in a disc. Let $\tilde{\Omega} = \{(\rho, \theta) \mid 0 \le \rho < 2,\ 0 \le \theta < 2\pi\}$ with the boundary $\partial\tilde{\Omega}$ and $T > 0$. $W_0(\rho, \theta)$ and $\nu > 0$ are the initial state and the kinetic viscosity, respectively. For simplicity, let $\partial_z W = \dfrac{\partial W}{\partial z}$, etc. The Fisher problem in a disc is of the form

$$\begin{cases} \partial_t W(\rho,\theta,t) - \nu \Delta W(\rho,\theta,t) - W(\rho,\theta,t)(1 - W(\rho,\theta,t)) = g(\rho,\theta,t), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in } \tilde{\Omega}, t \in [0,T], \\ W(2,\theta,t) = 0, \quad t \in [0,T], \\ W(\rho,\theta,0) = W_0(\rho,\theta), \quad \text{in } \bar{\tilde{\Omega}}. \end{cases} \tag{1.1}$$

For a scalar function $w(\rho, \theta)$,

$$\Delta w(\rho,\theta) = \frac{1}{\rho}\partial_\rho(\rho \partial_\rho w(\rho,\theta)) + \frac{1}{\rho^2}\partial_\theta^2 w(\rho,\theta), \tag{1.2}$$

$$\nabla w(\rho,\theta) = (\partial_\rho w(\rho,\theta), \frac{1}{\rho}\partial_\theta w(\rho,\theta))^T. \tag{1.3}$$

Observing from (1.1)–(1.3) that it is a nonlinear evolution equation. Particularly, the problem has singularity at the center of the domain. This often destroys the merit of high accuracy of spectral method. To overcome the disadvantage, some techniques have been proposed. Boyd [3,4] used some orthogonal polynomials to approximate solutions with end-pointed weak singularities, and in [5] compared seven methods for solving problems defined on a disk. Stenger [24] used Sinc base functions to fit the singular solutions. Matsushima and Marcus [18] studied spectral method for problems defined in polar coordinates using a set of orthogonal polynomials. Bernardi and Maday [2] considered ultraspherical approximations in some weighted Sobolev spaces. Guo [11] studied Jacobi approximations in certain Hilbert spaces and with applications to singular differential equations. On the other hand, some problems in unbounded domain can be reformulated as singular ones in bounded domain by variable transformations. Guo [8] used some Jacobi approximations to numerically simulate these resulting problems.

In this paper, we employ the generalized Jacobi-Fourier spectral method to simulate (1.1) numerically. This approach has several merits:

(i) We use the generalized Jacobi approximation in the radial direction similar to the method used by Yu and Wang in [29,30]. It avoids effectively the singularity at the regional center.

(ii) This reduces the difficulty of the theoretical analysis and provides a sparse system of the unknown coefficients of numerical solution.

(iii) Moreover, the numerical solution possesses spectral accuracy in space with the smooth solutions.

The rest of the paper is organized as follows. In the next section, we recall some basic results of the generalized Jacobi and Fourier orthogonal approximations. In Section 3, we introduce the mixed generalized Jacobi-Fourier orthogonal approximation. In Section 4, a mixed spectral scheme for the Fisher equation in a disc is constructed, and it's stability and convergence are proved. In Section 5, we present some numerical results to demonstrate the efficiency of this new approach. The final section is for some concluding remarks.

2 Preliminaries

Let $\Lambda = \{x|\ |x| < 1\}$ and $\chi(x)$ be a certain weight function. Denote by $\mathbb{N}$ the set of all non-negative integers. For any $r \in \mathbb{N}$, we define the weighted Sobolev space $H^r_\chi(\Lambda)$ in the usual way, with the inner product, semi-norm and norm $(u,v)_{r,\chi,\Lambda}$, $|u|_{r,\chi,\Lambda}$, and $\|u\|_{r,\chi,\Lambda}$, respectively. In particular, $L^2_\chi(\Lambda) = H^0_\chi(\Lambda)$, $(u,v)_{\chi,\Lambda} = (u,v)_{0,\chi,\Lambda}$ and $\|u\|_{\chi,\Lambda} = \|u\|_{0,\chi,\Lambda}$. For any $r > 0$, we define the space $H^r_\chi(\Lambda)$ by space interpolation as in [1]. We omit the subscript χ whenever $\chi(x) \equiv 1$. Let $\alpha, \beta > -1$ and $\chi^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$. The Jacobi polynomial of degree l in the interval Λ is given by

$$(1-x)^\alpha(1+x)^\beta J_l^{(\alpha,\beta)}(x) = \frac{(-1)^l}{2^l l!}\partial_x^l((1-x)^{\alpha+l}(1+x)^{\beta+l}).$$

The set of $\{J_l^{(\alpha,\beta)}(x)\}$ is a complete $L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$-orthogonal system, namely

$$(J_l^{(\alpha,\beta)}(x), J_{l'}^{(\alpha,\beta)}(x))_{\chi^{(\alpha,\beta)}} = \gamma_l^{(\alpha,\beta)}\delta_{l,l'}, \tag{2.1}$$

where $\delta_{l,l'}$ is the Kronecker delta symbol, and

$$\gamma_l^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}\Gamma(l+\alpha+1)\Gamma(l+\beta+1)}{(2l+\alpha+\beta+1)\Gamma(l+1)\Gamma(l+\alpha+\beta+1)}.$$

Let $N \in \mathbb{N}$, and denote by $\mathcal{P}_N(\Lambda)$ the set of all algebraic polynomials of degree at most N. For $\gamma, \delta > -1$, we introduce the space $H^\mu_{\alpha,\beta,\gamma,\delta}(\Lambda)$, $0 \le \mu \le 1$. For $\mu = 0$, $H^0_{\alpha,\beta,\gamma,\delta}(\Lambda) = L^2_{\chi^{(\gamma,\delta)}}(\Lambda)$. For $\mu = 1$,

$$H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) = \{u \mid u \text{ is measurable and } \|u\|_{1,\alpha,\beta,\gamma,\delta,\Lambda} < \infty\},$$

equipped with the norm

$$\|u\|_{1,\alpha,\beta,\gamma,\delta,\Lambda} = (|u|^2_{1,\chi^{(\alpha,\beta)},\Lambda} + \|u\|^2_{\chi^{(\gamma,\delta)},\Lambda})^{\frac{1}{2}}.$$

The space $H^\mu_{\alpha,\beta,\gamma,\delta}(\Lambda)$ with $0 < \mu < 1$ is defined by space interpolation as in [1].

For describing approximation results, we define the following two spaces,

$$H^r_{\chi^{(\alpha,\beta)},A}(\Lambda) = \{u \mid u \text{ is measurable and } \|u\|_{r,\chi^{(\alpha,\beta)},A} < \infty\}, \quad r \in \mathbb{N},$$

where

$$\|u\|_{r,\chi^{(\alpha,\beta)},A} = \big(\sum_{k=0}^{r} \|\partial_x^k u\|^2_{\chi^{(\alpha+k,\beta+k)}}\big)^{\frac{1}{2}} \quad \textit{and} \quad |u|_{r,\chi^{(\alpha,\beta)},A} = \|\partial_x^r u\|_{\chi^{(\alpha+r,\beta+r)}}$$

and

$$H^r_{\chi^{(\alpha,\beta)},*}(\Lambda) = \{u \mid u \text{ is measurable and } \|u\|_{r,\chi^{(\alpha,\beta)},*} < \infty\}, \quad r \geq 1, r \in \mathbb{N},$$

where

$$\|u\|_{r,\chi^{(\alpha,\beta)},*} = \big(\sum_{k=0}^{r-1} |u|^2_{k+1,\chi^{(\alpha,\beta)},*}\big)^{\frac{1}{2}} \quad \textit{and} \quad |u|_{r,\chi^{(\alpha,\beta)},*} = \|\partial_x^r u\|_{\chi^{(\alpha+r-1,\beta+r-1)}}.$$

Clearly, we have $|u|_{r-1,\chi^{(\alpha,\beta)},A} = |u|_{r,\chi^{(\alpha,\beta)},*}$.

Lemma 1. *(cf. Lemma 3.5 of [13]) If $\beta < 1$ then for any $u \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)$, u is continuous on any subinterval $\Lambda^* = [-1, a]$ with $a < 1$, and $\max_{x\in\Lambda^*} |u(x)| \leq c\|U\|_{1,\chi^{(\alpha,\beta)}}$. If, in addition, $\alpha < 1$, then these results can be extended to $\bar{\Lambda}$.*

Lemma 2. *(cf. Lemma 2.3 of [30]) If one of the following conditions holds,*

$$\alpha \leq \gamma + 2, \quad \alpha < 1, \quad \beta \leq 0, \quad \delta \geq 0, \tag{2.2}$$

$$\alpha \leq 0, \quad \beta \leq \delta + 2, \quad \gamma \geq 0, \tag{2.3}$$

$$\alpha \leq \gamma + 2, \quad \beta \leq \delta + 1, \quad \alpha < 1, \quad 0 < \beta < 1, \tag{2.4}$$

then for any $u \in H^1_{\chi^{(\alpha,\beta)}}(\Lambda)$ with u(1)=0,

$$\|u\|_{\chi^{(\gamma,\delta)}} \leq c|u|_{1,\chi^{(\alpha,\beta)}}.$$

Lemma 3. *(Lemma 2.2 of [13]) There exists a mapping $\bar{P}^1_{N,\alpha,\beta} : H^1_{\chi^{(\alpha,\beta)},A}(\Lambda) \to \mathcal{P}_N(\Lambda)$ such that $\bar{P}^1_{N,\alpha,\beta}u(-1) = u(-1)$, and for any $u \in H^1_{\chi^{(\alpha,\beta)},A}(\Lambda)$,*

$$(\partial_x(\bar{P}^1_{N,\alpha,\beta}u - u),\ \partial_x\phi)_{\chi^{(\alpha+1,\beta+1)}} = 0, \quad \forall\phi \in \mathcal{P}_N(\Lambda).$$

Moreover, for any $u \in H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$, $\mu, r \in N$, $1 \leq r \leq N+1$ and $0 \leq \mu \leq r$,

$$\|\bar{P}^1_{N,\alpha,\beta}u - u\|_{\mu,\chi^{(\alpha,\beta)},A} \leq cN^{\mu-r}|u|_{r,\chi^{(\alpha,\beta)},A}.$$

Next, let $\alpha < 1$ and

$${}^0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) = \{u \mid u \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda), u(1) = 0\},$$
$${}^0\mathcal{P}_N(\Lambda) = \mathcal{P}_N(\Lambda) \cap {}^0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda).$$

Due to Lemma 1, the set ${}^0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)$ is meaningful. Next, we consider the orthogonal projection ${}^0P^1_{N,\alpha,\beta,\gamma,\delta}: {}^0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \to {}^0\mathcal{P}_N(\Lambda)$ is defined by

$$\tilde{a}_{\alpha,\beta,\gamma,\delta}({}^0P^1_{N,\alpha,\beta,\gamma,\delta}u - u, \phi) = 0, \qquad \forall \phi \in {}^0\mathcal{P}_N(\Lambda),$$

where $\tilde{a}_{\alpha,\beta,\gamma,\delta}(u,v) = (\partial_x u, \partial_x v)_{\chi^{(\alpha,\beta)}} + (u,v)_{\chi^{(\gamma,\delta)}}$.

Using similar manner to the proof of Lemma 3.4 of [30], we can get the following result.

Lemma 4. *If one of the conditions of* (2.2)–(2.4) *holds, then for any* $u \in {}^0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \cap H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ *with* $r \in N$ *and* $r \geq 1$,

$$\|{}^0P^1_{N,\alpha,\beta,\gamma,\delta}u - u\|_{1,\alpha,\beta,\gamma,\delta} \leq cN^{1-r}|u|_{r,\chi^{(\alpha,\beta)},*}. \tag{2.5}$$

In addition, if $\alpha \leq \gamma+1, \beta \leq \delta+1$, *then for* $0 \leq \mu \leq 1$,

$$\|{}^0P^1_{N,\alpha,\beta,\gamma,\delta}u - u\|_{\mu,\alpha,\beta,\gamma,\delta} \leq cN^{\mu-r}|u|_{r,\chi^{(\alpha,\beta)},*}.$$

Lemma 5. *If one of the conditions of* (2.2)–(2.4) *holds, then there exists a quasi-orthogonal projection* ${}^0\hat{P}^1_{\alpha,\beta,\gamma,\delta}: {}^0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \to {}^0\mathcal{P}_N(\Lambda)$, *such that*

$${}^0\hat{P}^1_{\alpha,\beta,\gamma,\delta}u(-1) = u(-1),$$

then for any $u \in {}^0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \cap H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ *with* $r \in N$ *and* $r \geq 1$,

$$\|{}^0\hat{P}^1_{\alpha,\beta,\gamma,\delta}u - u\|_{1,\alpha,\beta,\gamma,\delta} \leq cN^{1-r}|u|_{r,\chi^{(\alpha,\beta)},*}. \tag{2.6}$$

In particular, if (2.2) *or* (2.4) *holds, then we have*

$$\|{}^0\hat{P}^1_{\alpha,\beta,\gamma,\delta}u - u\|_{(\gamma,-1)} \leq cN^{1-r}|u|_{r,\chi^{(\alpha,\beta)},*}. \tag{2.7}$$

Proof. For any $u \in {}^0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)$, we define a quasi-orthogonal projection:

$$\begin{aligned}{}^0\hat{P}^1_{N,\alpha,\beta,\gamma,\delta}u(x) = &\int_1^x {}^0P^1_{N-1,\alpha,\beta,\gamma,\delta}\partial_\xi u(\xi)d\xi \\ &+ \frac{1-x}{2}\Big(u(-1) + \int_{-1}^1 {}^0P^1_{N-1,\alpha,\beta,\gamma,\delta}\partial_x u(x)dx\Big).\end{aligned}$$

We can check that ${}^0\hat{P}^1_{N,\alpha,\beta,\gamma,\delta}u(-1) = u(-1)$. Set

$$D = u(-1) + \int_{-1}^1 {}^0P^1_{N-1,\alpha,\beta,\gamma,\delta}\partial_x u(x)dx.$$

By (2.5), we derive that

$$\begin{aligned}
&\|{}^0\hat{P}^1_{N,\alpha,\beta,\gamma,\delta}u(x)-u(x)\|_{1,\alpha,\beta,\gamma,\delta} \qquad (2.8)\\
&=\|\int_1^x {}^0P^1_{N-1,\alpha,\beta,\gamma,\delta}\partial_\xi u(\xi)d\xi+\frac{1-x}{2}D-u(x)\|_{1,\alpha,\beta,\gamma,\delta}\\
&\le \|{}^0P^1_{N-1,\alpha,\beta,\gamma,\delta}\partial_x u(x)-\partial_x u(x)\|_{\chi^{(\alpha,\beta)}}+\frac{1}{2}(\gamma_0^{(\alpha,\beta)})^{\frac{1}{2}}|D|\\
&\quad+\|\int_1^x {}^0P^1_{N-1,\alpha,\beta,\gamma,\delta}\partial_\xi u(\xi)d\xi+\frac{1-x}{2}D-u(x)\|_{\chi^{(\gamma,\delta)}}\\
&\le c\|{}^0P^1_{N-1,\alpha,\beta,\gamma,\delta}\partial_x u(x)-\partial_x u(x)\|_{\chi^{(\gamma,\delta)}}+\frac{1}{2}(\gamma_0^{(\alpha,\beta)})^{\frac{1}{2}}|D|\\
&+\|\int_1^x {}^0P^1_{N-1,\alpha,\beta,\gamma,\delta}\partial_\xi u(\xi)d\xi+\frac{1-x}{2}D-\int_1^x\partial_\xi u(\xi)d\xi\|_{\chi^{(\gamma,\delta)}}\\
&\le c\|{}^0P^1_{N-1,\alpha,\beta,\gamma,\delta}\partial_x u(x)-\partial_x u(x)\|_{\chi^{(\gamma,\delta)}}+\frac{1}{2}(\gamma_0^{(\alpha,\beta)})^{\frac{1}{2}}|D|\\
&\quad+\|\int_{-1}^1\big({}^0P^1_{N-1,\alpha,\beta,\gamma,\delta}\partial_x u(x)-\partial_x u(x)\big)dx\|_{\chi^{(\gamma,\delta)}}+\frac{1}{2}(\gamma_0^{(\gamma+2,\delta)})^{\frac{1}{2}}|D|.
\end{aligned}$$

Thanks to $\gamma,\delta<1$,

$$\begin{aligned}
|D|&=\Big|\int_{-1}^1\big({}^0P^1_{N-1,\alpha,\beta,\gamma,\delta}\partial_x u(x)-\partial_x u(x)\big)dx\Big|\\
&\le\frac{1}{2}(\gamma_0^{(-\gamma,-\delta)})^{\frac{1}{2}}\|{}^0P^1_{N-1,\alpha,\beta,\gamma,\delta}\partial_x u(x)-\partial_x u(x)\|_{\chi^{(\gamma,\delta)}}. \qquad (2.9)
\end{aligned}$$

Substituting (2.9) into (2.8) and using Lemma 4, we obtain the desired result (2.6). We next prove (2.7). Let $\bar{\delta}\in(-1,1)$. By virtue of (2.15) of [14], we have that

$$\begin{aligned}
&\|{}^0\hat{P}^1_{\alpha,\beta,\gamma,\delta}u-u\|_{(\gamma,-1)}\le 2^{1-\bar{\delta}}2\zeta_{-\gamma,-\bar{\delta}}|{}^0\hat{P}^1_{\alpha,\beta,\gamma,\delta}u-u|_{1,\chi^{(\gamma,\bar{\delta})}}\\
&\le 2^{1-\bar{\delta}}2\zeta_{-\gamma,-\bar{\delta}}\big(\|{}^0P^1_{N-1,\alpha,\beta,\gamma,\delta}\partial_x u(x)-\partial_x u(x)\|_{\chi^{(\gamma,\bar{\delta})}}+\frac{1}{2}(\gamma_0^{(\gamma,\bar{\delta})})^{\frac{1}{2}}|D|\big)\\
&\le c\|{}^0P^1_{N-1,\alpha,\beta,\gamma,\delta}\partial_x u(x)-\partial_x u(x)\|_{\chi^{(\gamma,\bar{\delta})}}\le cN^{1-r}|u|_{r,\chi^{(\alpha,\beta)},*}, \qquad (2.10)
\end{aligned}$$

where

$$\begin{aligned}
&\xi_\gamma=\max(2^\gamma,1),\zeta_{\gamma,\bar{\delta}}=\max(\xi_\gamma^{\frac{1}{2}}(\gamma+1)^{-\frac{1}{2}},\xi_{\bar{\delta}}^{\frac{1}{2}}(\bar{\delta}+1)^{-\frac{1}{2}}),\\
&c=2^{1-\bar{\delta}}2\zeta_{-\gamma,-\bar{\delta}}(1+\frac{1}{2}(\gamma_0^{(\gamma,\bar{\delta})})^{\frac{1}{2}})(\gamma_0^{(-\gamma,-\bar{\delta})})^{\frac{1}{2}}).
\end{aligned}$$

□

Moreover, we recall some results of Fourier approximation, which will be used in the forthcoming discussion. Let $I=[0,2\pi)$, s be any positive integer, and $H^s(I)$ be Sobolev space with the norm $\|\cdot\|_{s,I}$ and the semi-norm $|\cdot|_{s,I}$ as usual. We denote by $H^s_p(I)$ the subspace of $H^s(I)$, consisting of all functions whose derivatives of order up to $s-1$ have the period 2π. For any $r>0$, the space $H^r_p(I)$ is defined by space interpolation as in [1].

Let M be any positive integer, and $\tilde{V}_M(I) = \text{span}\{e^{il\theta} \mid |l| \leq M\}$. We define $V_M(I)$ as the subset of $\tilde{V}_M(I)$ consisting of all real-valued functions. The orthogonal projection $P_{M,I} : L^2(I) \to V_M(I)$ is defined by

$$\int_I (P_{M,I}v(\theta) - v(\theta))\phi(\theta)d\theta = 0, \qquad \forall \phi \in V_M(I).$$

It was shown in [9] that for any $v \in H_p^s(I)$, and $0 \leq \mu \leq s$,

$$\|P_{M,I}v - v\|_{\mu,I} \leq cM^{\mu-s}|v|_{s,I}. \tag{2.11}$$

3 Mixed Fourier-Jacobi approximation

In this section, we consider the mixed Jacobi-Fourier orthogonal approximation. Let $\Omega = \{(x,\theta)| -1 \leq x < 1,\ 0 \leq \theta < 2\pi\}$ and $L^2_{\chi^{(\alpha,\beta)}}(\Omega) = L^2_{\chi^{(\alpha,\beta)}}(\Lambda, L^2(I))$ equipped with the inner product

$$(u,v)_{\chi^{(\alpha,\beta)},\Omega} = \int_\Omega (1-x)^\alpha (1+x)^\beta u(x,\theta)v(x,\theta)dxd\theta$$

and the norm

$$\|u\|_{L^2_{\chi^{(\alpha,\beta)}}(\Omega)} = (u,u)^{\frac{1}{2}}_{\chi^{(\alpha,\beta)}}.$$

In order to describe the approximation results, we introduce the space

$$H^1_{\alpha,\beta,\gamma,\delta,\eta,\xi}(\Omega) = \{u \mid u \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda; H^1_p(I)),\ \|u\|_{1,\alpha,\beta,\gamma,\delta,\eta,\xi,\Omega} < \infty\},$$

where

$$\begin{aligned}\|u\|_{1,\alpha,\beta,\gamma,\delta,\eta,\xi,\Omega} &= (\|\partial_x u\|^2_{L^2_{\chi^{(\alpha,\beta)}}(\Lambda;L^2(I))} + \|\partial_\theta u\|^2_{L^2_{\chi^{(\eta,\xi)}}(\Lambda;L^2(I))} \\ &\quad + \|u\|^2_{L^2_{\chi^{(\gamma,\delta)}}(\Lambda;L^2(I))})^{\frac{1}{2}}.\end{aligned}$$

Let

$$\begin{aligned}{}^0\mathcal{F}(\Omega) &= \{u \mid u \in H^1_{\alpha,\beta,\gamma,\delta,\eta,\xi}(\Omega) \text{ and there exists a finite trace of} \\ &\qquad u(x,\theta) \text{ at } x = 1\}, \\ {}^0H^1_{\alpha,\beta,\gamma,\delta,\eta,\xi}(\Omega) &= \{u \mid u \in {}^0\mathcal{F}(\Omega) \text{ and } u(1,\theta) = 0\}.\end{aligned}$$

According to Lemma 1, the above space is meaningful. For simplicity, let

$$\begin{aligned}&|u|_{H^1_{0,1,0,-1}(\Omega)} = (\|\partial_x u\|^2_{L^2_{\chi^{(0,1)}}(\Lambda;L^2(I))} + \|\partial_\theta u\|^2_{L^2_{\chi^{(0,-1)}}(\Lambda;L^2(I))})^{\frac{1}{2}}, \\ &\|u\|_{L^2(\Omega)} = \|u\|_{L^2_{\chi^{(0,0)}}(\Lambda;L^2(I))}, \\ &\|u\|_{1,0,1,0,-1,0,0,\Omega} = \|u\|_{H^1_{0,1,0,-1,0,0}(\Omega)} = (|u|^2_{H^1_{0,1,0,-1}(\Omega)} + \|u\|^2_{L^2(\Omega)})^{\frac{1}{2}}.\end{aligned}$$

In order to deal with the polar condition $\partial_\theta u(x,\theta)|_{x=-1} = 0$, we define the space

$${}^0\mathcal{V}(\Omega) = \{u \mid u \in {}^0H^1_{\alpha,\beta,\gamma,\delta,\eta,\xi}(\Omega) \text{ and } \partial_\theta u(-1,\theta) = 0\}.$$

We introduce the bilinear form

$$a_{\alpha,\beta,\eta,\xi}(u,v)=(\partial_x u,\partial_x v)_{\chi^{(\alpha,\beta)},\Omega}+(\partial_\theta u,\partial_\theta v)_{\chi^{(\eta,\xi)},\Omega}$$

and the notation

$${}^0\mathcal{P}_{N,M}(\Omega)=({}^0\mathcal{P}_N(\Lambda)\otimes V_{M,I}(I))\cap {}^0\mathcal{V}(\Omega).$$

The orthogonal projection ${}^0P^1_{N,M,\Omega}:{}^0\mathcal{V}(\Omega)\to {}^0\mathcal{P}_{N,M}(\Omega)$ is defined by

$$A({}^0P^1_{N,M,\Omega}v-v,\phi)=0,\quad \forall\phi\in {}^0\mathcal{P}_{N,M}(\Omega),$$

where $A(u,v)=a_{0,1,0,-1}(u,v)+(u,v)_{\chi^{(0,1)},\Omega}$.

Lemma 6. *For any $v(\cdot,\theta)\in\ L^2(I)$ and $\partial_\theta v(-1,\theta)=0$, then*

$$\partial_\theta P_{M,I}v(-1,\theta)=0.$$

Proof. For any $\delta>0$, owing to $\partial_\theta v(-1,\theta)=0$, we can rewrite $v(x,\theta)$ as $v(x,\theta)=(1+x)^\delta u(x,\theta)+w(x)$. Thanks to $v(\cdot,\theta)\in L^2(I)$, we can deduce that $u(\cdot,\theta)\in L^2(I)$. So the $P_{M,I}u(x,\theta)$ is meaningful. Furthermore, $P_{M,I}v(x,\theta)=(1+x)^\delta P_{M,I}u(x,\theta)+w(x)$, hence, $\partial_\theta P_{M,I}v(-1,\theta)=0$. □

Theorem 1. *If integer $2\le r\le N+1,\ s\ge 1$, then for any*

$$v\in {}^0\mathcal{V}(\Omega)\bigcap H^r_{\chi^{(0,1)},*}(\Lambda,H^1_p(I))\bigcap H^1_{0,1,0,1}(\Lambda,H^s_p(I))\bigcap L^2_{\chi^{(0,-1)}}(\Lambda,H^s_p(I))$$

we have

$$\begin{aligned}\|{}^0P^1_{N,M,\Omega}v-v\|_{1,0,1,0,-1,0,1,\Omega}\quad &\le c\big(N^{1-r}+M^{1-s}\big)\big(|v|_{H^r_{\chi^{(0,1)},*}(\Lambda,L^2_p(I))}\\ &+|v|_{H^r_{\chi^{(0,1)},*}(\Lambda,H^1_p(I))}+|\partial_x v|_{L^2_{\chi^{(0,1)}}(\Lambda,H^s_p(I))}+|v|_{L^2_{\chi^{(0,-1)}}(\Lambda,H^s_p(I))}\big).\end{aligned}\tag{3.1}$$

Proof. Setting $\phi={}^0\hat{P}^1_{N,0,1,0,1,\Lambda}\cdot P_{M,I}v$. Then

$$\phi(-1,\theta)={}^0\hat{P}^1_{N,0,1,0,1,\Lambda}\cdot P_{M,I}v(-1,\theta)=P_{M,I}v(-1,\theta).$$

Since $\partial_\theta v(-1,\theta)=0$, by Lemma 6, we check that $\partial_\theta P_{M,I}v(-1,\theta)=0$. So, $\partial_\theta\phi(-1,\theta)=0$. On the other hand, $\phi(1,\theta)=0$. Then, $\phi\in {}^0\mathcal{P}_{N,M}(\Omega)$. By the projection theorem we have

$$\|{}^0P^1_{N,M,\Omega}v-v\|_{1,0,1,0,-1,0,1,\Omega}\le\|\phi-v\|_{1,0,1,0,-1,0,1,\Omega},\quad\forall\phi\in {}^0\mathcal{P}_{N,M}(\Omega).$$

Taking $\phi={}^0\hat{P}^1_{N,0,1,0,1,\Lambda}\cdot P_{M,I}v$ in the above inequality, we get that

$$\begin{aligned}\|{}^0P^1_{N,M,\Omega}v-v\|_{1,0,1,0,-1,0,1,\Omega}&\le\|{}^0\hat{P}^1_{N,0,1,0,1,\Lambda}\cdot P_{M,I}v-v\|_{H^1_{0,1,0,1}(\Lambda,L^2(I))}\\ &+\|\partial_\theta({}^0\hat{P}^1_{N,0,1,0,1,\Lambda}\cdot P_{M,I}v-v)\|_{L^2_{\chi^{(0,-1)}}(\Lambda,L^2(I))}.\end{aligned}$$

It remains to estimate the terms $\|{}^0\hat{P}^1_{N,0,1,0,1,\Lambda}\cdot P_{M,I}v-v\|_{H^1_{0,1,0,1}(\Lambda,L^2(I))}$ and $\|\partial_\theta({}^0\hat{P}^1_{N,0,1,0,1,\Lambda}\cdot P_{M,I}v-v)\|_{L^2_{\chi^{(0,-1)}}(\Lambda,L^2(I))}$.

Thanks to (2.6) and (2.11) we deduce that for integer $1\le r\le N+1$ and $s\ge 0$

$$\begin{aligned}
&\|{}^0\hat{P}^1_{N,0,1,0,1,\Lambda}\cdot P_{M,I}v-v\|_{H^1_{0,1,0,1}(\Lambda,L^2(I))}\le \|{}^0\hat{P}^1_{N,0,1,0,1,\Lambda}\cdot P_{M,I}v\\
&\quad -P_{M,I}v\|_{H^1_{0,1,0,1}(\Lambda,L^2(I))}+\|P_{M,I}v-v\|_{H^1_{0,1,0,1}(\Lambda,L^2(I))}\\
&\le cN^{1-r}|P_{M,I}v|_{H^r_{\chi^{(0,1)},*}(\Lambda,L^2(I))}+cM^{-s}|\partial_x v|_{L^2_{\chi^{(0,1)}}(\Lambda,H^s_p(I))}\\
&\quad +cM^{-s}|v|_{L^2_{\chi^{(0,1)}}(\Lambda,H^s_p(I))}\le cN^{1-r}|v|_{H^r_{\chi^{(0,1)},*}(\Lambda,L^2(I))}\\
&\quad +cM^{-s}|\partial_x v|_{L^2_{\chi^{(0,1)}}(\Lambda,H^s_p(I))}+cM^{-s}|v|_{L^2_{\chi^{(0,1)}}(\Lambda,H^s_p(I))}. \qquad (3.2)
\end{aligned}$$

Using (2.7) and (2.11) again, we obtain that

$$\begin{aligned}
&\|\partial_\theta({}^0\hat{P}^1_{N,0,1,0,1,\Lambda}\cdot P_{M,I}v-v)\|_{L^2_{\chi^{(0,-1)}}(\Lambda,L^2(I))}\\
&\le \|{}^0\hat{P}^1_{N,0,1,0,1,\Lambda}\cdot \partial_\theta P_{M,I}v-\partial_\theta P_{M,I}v\|_{L^2_{\chi^{(0,-1)}}(\Lambda,L^2(I))}\\
&\quad +\|\partial_\theta(P_{M,I}v-v)\|_{L^2_{\chi^{(0,-1)}}(\Lambda,L^2(I))} \qquad (3.3)\\
&\le cN^{1-r}|\partial_\theta P_{M,I}v|_{H^r_{\chi^{(0,1)},*}(\Lambda,L^2(I))}+cM^{1-s}|v|_{L^2_{\chi^{(0,-1)}}(\Lambda,H^s_p(I))}\\
&\le cN^{1-r}|v|_{H^r_{\chi^{(0,1)},*}(\Lambda,H^1(I))}+cM^{1-s}|v|_{L^2_{\chi^{(0,-1)}}(\Lambda,H^s_p(I))}.
\end{aligned}$$

Therefore, a combination of (3.2), (3.3) and Poincaré inequality leads to (3.1). □

Lemma 7. *For any* $u\in {}^0H^1(\Omega)$,

$$\|u\|_{L^p,\chi^{(0,1)}(\Omega)}\le c|u|_{H^1_{0,1,0,-1}(\Omega)},\quad 2\le p\le\infty.$$

Proof. By embedding theory and Lemma 2 we can easily obtain the desired result. □

For nonlinear problem we need the following lemma (cf. Lemma 3.1 of [10]).

Lemma 8. *Assume that*
(1) the constants $b_1>0, b_2\ge 0, b_3\ge 0$ *and* $d\ge 0$,
(2) $Z(t)$ *and* $A(t)$ *are non-negative functions of* t,
(3) $d\le \dfrac{b_1^2}{b_2^2}\exp(-b_3 t)$ *for certain* $t_1>0$, *and for all* $t\le t_1$,

$$Z(t)+\int_0^t\big(b_1-b_2Z^{\frac12}(\eta)\big)A(\eta)d\eta\le d+b_3\int_0^t Z(\eta)d\eta.$$

Then for all $t\le t_1$, *we have* $Z(t)\le d\exp(b_3t)$.

4 Mixed spectral method for Fisher equation

In this section, we propose the mixed spectral method for the Fisher equation (1.1). We define the space $L^2(\tilde{\Omega})$ as usual with the following inner product and norm

$$(w,v)_{L^2(\tilde{\Omega})} = \int_{\tilde{\Omega}} \rho w(\rho,\theta)v(\rho,\theta)d\rho d\theta, \quad \|w\|_{L^2(\tilde{\Omega})} = (w,w)^{\frac{1}{2}}_{L^2(\tilde{\Omega})}.$$

We also define the space $H^1(\tilde{\Omega})$ equipped with the following inner product, semi-norm and norm,

$$(w,v)_{H^1(\tilde{\Omega})} = \int_{\tilde{\Omega}} \big(\rho\partial_\rho w(\rho,\theta)\partial_\rho v(\rho,\theta) + \frac{1}{\rho}\partial_\theta w(\rho,\theta)\partial_\theta v(\rho,\theta)\big)d\rho d\theta,$$
$$|w|_{H^1(\tilde{\Omega})} = (w,v)^{\frac{1}{2}}_{H^1(\tilde{\Omega})}, \quad \|w\|_{H^1(\tilde{\Omega})} = (|w|_{H^1(\tilde{\Omega})} + \|w\|_{L^2(\tilde{\Omega})})^{\frac{1}{2}}.$$

In order to use the generalized Jacobi approximation, we make the variable transformation $\rho = 1 + x$. Then we can check that

$$\begin{aligned}(w,v)_{L^2(\tilde{\Omega})} &= \int_{\tilde{\Omega}} \rho w(\rho,\theta)v(\rho,\theta)d\rho d\theta \\ &= \int_{\Omega} (1+x)w(x,\theta)v(x,\theta)dxd\theta = (w,v)_{\chi^{(0,1)},\Omega},\end{aligned}$$

$$\begin{aligned}(w,v)_{H^1(\tilde{\Omega})} &= \int_{\tilde{\Omega}} \big(\rho\partial_\rho w(\rho,\theta)\partial_\rho v(\rho,\theta) + \frac{1}{\rho}\partial_\theta w(\rho,\theta)\partial_\theta v(\rho,\theta)\big)d\rho d\theta \\ &= \int_{\Omega} \Big((1+x)\partial_x w(x,\theta)\partial_x v(x,\theta) + \frac{1}{(1+x)}\partial_\theta w(x,\theta)\partial_\theta v(x,\theta)\Big)dxd\theta \\ &= (\partial_x w, \partial_x v)_{\chi^{(0,1)},\Omega} + (\partial_\theta w, \partial_\theta v)_{\chi^{(0,-1)},\Omega} = a_{0,1,0,-1}(w,v),\end{aligned}$$
$$\|w\|_{L^2(\tilde{\Omega})} = \|w\|_{L^2_{\chi^{(0,1)}}(\Omega)}, \quad |w|^2_{1,H^1(\tilde{\Omega})} = \|\partial_x w\|^2_{L^2_{\chi^{(0,1)}}(\Omega)} + \|\partial_\theta w\|^2_{L^2_{\chi^{(0,-1)}}(\Omega)}.$$

Moreover, because the problem is defined in a disc, so we need the polar condition $\partial_\theta u(-1,\theta) = 0$ for $0 \le \theta < 2\pi$. Taking $U(x,\theta,t) = W(1+x,\theta,t)$, $f(x,\theta,t) = g(1+x,\theta,t)$. Then (1.1) is reformulated to

$$\begin{cases}\partial_t U(x,\theta,t) = \nu\Big(\frac{1}{1+x}\partial_x((1+x)\partial_x U(x,\theta,t)) + \frac{1}{(1+x)^2}\partial_\theta^2 U(x,\theta,t)\Big) \\ \quad +U(x,\theta,t)\big(1-U(x,\theta,t)\big) + f(x,\theta,t), \quad \text{in } \Omega, \quad t \in [0,T], \\ \partial_\theta U(-1,\theta,t) = 0, \quad U(1,\theta,t) = 0, \quad t \in [0,T], \\ U(x,\theta,0) = U_0(x,\theta), \quad \text{on } \bar{\Omega}.\end{cases} \tag{4.1}$$

Therefore, a weak formulation of (4.1) is to find $U(x,\theta,t) \in {}^0\mathcal{V}(\Omega)\otimes[0,T]$ such that for all $v(x,\theta) \in^0 \mathcal{V}(\Omega)$ satisfy

$$\begin{cases}(\partial_t U, v)_{L^2_{\chi^{(0,1)}}(\Omega)} + \nu a_{0,1,0,-1}(U,v) \\ \quad = (U(1-U),v)_{L^2_{\chi^{(0,1)}}(\Omega)} + (f,v)_{L^2_{\chi^{(0,1)}}(\Omega)}, \quad \forall v \in {}^0\mathcal{V}(\Omega),\ 0 \le t \le T, \\ U(x,\theta,0) = U_0(x,\theta), \quad \text{in } \Omega.\end{cases}$$

The mixed spectral scheme for the above equation is to find $u_{N,M} \in {}^0\mathcal{P}_{N,M}(\Omega)$ such that

$$\begin{cases} (\partial_t u_{N,M}(x,\theta,t),\phi)_{L^2_{\chi^{(0,1)}}(\Omega)} + \nu a_{0,1,0,-1}\big(u_{N,M}(x,\theta,t),\phi\big) \\ \qquad = \big(u_{N,M}(x,\theta,t)(1-u_{N,M}(x,\theta,t)),\phi\big)_{L^2_{\chi^{(0,1)}}(\Omega)} \\ \qquad\quad +(f(x,\theta,t),\phi)_{L^2_{\chi^{(0,1)}}(\Omega)}, \quad \forall\phi\in {}^0\mathcal{P}_{N,M}(\Omega),\ t\in[0,T], \\ u_{N,M}(x,\theta,0) = u_{0,N,M}(x,\theta) = {}^0P^1_{N,M,\Omega}U_0, \quad \text{in} \quad \bar{\Omega}. \end{cases} \tag{4.2}$$

We now consider the stability of scheme (4.2). Since (4.2) is a nonlinear problem, it does not possess the usual stability. But it might be of the generalized stability as described in [9]. Suppose that $u_{0,N,M}$, the right-hand side of the equation of (4.2) have the errors $\tilde{u}_0$ and $\tilde{f}$, respectively. They induce the error of $u_{N,M}$ denoted by $\tilde{u}_{N,M}$. Then, we obtain from (4.2) that

$$\begin{aligned} &(\partial_t\tilde{u}_{N,M}(t),\phi)_{L^2_{\chi^{(0,1)}}(\Omega)} + \nu a_{0,1,0,-1}(\tilde{u}_{N,M}(t),\phi) \\ &\qquad - (\tilde{u}_{N,M}(t) - 2u_{N,M}(t)\tilde{u}_{N,M}(t) - \tilde{u}^2_{N,M}(t),\phi)_{L^2_{\chi^{(0,1)}}(\Omega)} \\ &\quad = (\tilde{f}(t),\phi)_{L^2_{\chi^{(0,1)}}(\Omega)}, \quad \forall\phi\in {}^0\mathcal{P}_{M,N}(\Omega), \quad t\in[0,T], \\ &\tilde{u}_{N,M}(x,\theta,0) = \tilde{u}_{0,N,M}(x,\theta), \quad \text{in} \quad \bar{\Omega}. \end{aligned} \tag{4.3}$$

Now, let

$$\begin{aligned} C_1(t) &= 3 + 4\sup_{0\le t\le T}\|v(t)\|_{\infty,\Omega}, \\ \rho_1(u,w,t) &= \|u(t)\|^2_{L^2_{\chi^{(0,1)}}(\Omega)} + \int_0^t \|w(\xi)\|^2_{L^2_{\chi^{(0,1)}}(\Omega)}d\xi, \\ E(u,t) &= \|u(t)\|^2_{L^2_{\chi^{(0,1)}}(\Omega)} + \int_0^t |u(\xi)|^2_{1,H^1(\Omega)}d\xi. \end{aligned}$$

Taking $\phi = 2\tilde{u}_{N,M}(t)$ in (4.3), we derive that

$$\begin{aligned} \partial_t\|\tilde{u}_{N,M}(t)\|^2_{L^2_{\chi^{(0,1)}}(\Omega)} &+ 2\nu|\tilde{u}_{N,M}(t)|^2_{H^1_{0,1,0,-1}(\Omega)} = 2\big(\tilde{f}(t),\tilde{u}_{N,M}(t)\big)_{L^2_{\chi^{(0,1)}}(\Omega)} \\ &+ 2\big(\tilde{u}_{N,M}(t) - 2\tilde{u}_{N,M}(t)u_{N,M}(t) - \tilde{u}^2_{N,M}(t),\tilde{u}_{N,M}(t)\big)_{L^2_{\chi^{(0,1)}}(\Omega)}. \end{aligned} \tag{4.4}$$

Thanks to Cauchy inequality and Lemma 7, we yield that

$$\begin{aligned} &|2\big(\tilde{u}_{N,M}(t) - 2\tilde{u}_{N,M}(t)u_{N,M}(t) - \tilde{u}^2_{N,M}(t),\tilde{u}_{N,M}(t)\big)_{L^2_{\chi^{(0,1)}}(\Omega)}| \\ &\le |2\big(\tilde{u}_{N,M}(t),\tilde{u}_{N,M}(t)\big)_{L^2_{\chi^{(0,1)}}(\Omega)}| + |4\big(u_{N,M}(t)\tilde{u}_{N,M}(t),\tilde{u}_{N,M}(t)\big)_{L^2_{\chi^{(0,1)}}(\Omega)}| \\ &\quad + |2\big(\tilde{u}^2_{N,M}(t),\tilde{u}_{N,M}(t)\big)_{L^2_{\chi^{(0,1)}}(\Omega)}| \le 2\|\tilde{u}_{N,M}(t)\|^2_{L^2_{\chi^{(0,1)}}(\Omega)} \\ &\quad + 4\|u_{N,M}(t)\|_\infty\|\tilde{u}_{N,M}(t)\|^2_{L^2_{\chi^{(0,1)}}(\Omega)} + 2\|\tilde{u}_{N,M}(t)\|_{L^2_{\chi^{(0,1)}}(\Omega)} \\ &\quad \times\|\tilde{u}_{N,M}(t)\|^2_{L^4_{\chi^{(0,1)}}(\Omega)} \le \big(2+4\|u_{N,M}(t)\|_\infty\big)\|\tilde{u}_{N,M}(t)\|^2_{L^2_{\chi^{(0,1)}}(\Omega)} \\ &\quad + 2c^2\|\tilde{u}_{N,M}(t)\|_{L^2_{\chi^{(0,1)}}(\Omega)}|\tilde{u}_{N,M}(t)|^2_{H^1_{0,1,0,-1}(\Omega)}. \end{aligned}$$

Substituting the above inequality into (4.4), we deduce that

$$\partial_t\|\tilde{u}_{N,M}(t)\|^2_{L^2_{\chi^{(0,1)}}(\Omega)} + (2\nu - 2c^2\|\tilde{u}_{N,M}(t)\|_{L^2_{\chi^{(0,1)}}(\Omega)})|\tilde{u}_{N,M}(t)|^2_{H^1_{0,1,0,-1}(\Omega)} \\ \le (3+4\|u_{N,M}(t)\|_\infty)\|\tilde{u}_{N,M}(t)\|^2_{L^2_{\chi^{(0,1)}}(\Omega)} + \|\tilde{f}(t)\|^2_{L^2_{\chi^{(0,1)}}(\Omega)}. \qquad (4.5)$$

Integrating the inequality (4.5) from 0 to t with respect to t, we obtain that

$$\|\tilde{u}_{N,M}(t)\|^2_{L^2_{\chi^{(0,1)}}(\Omega)} + \int_0^t \big(2\nu - 2c^2\|\tilde{u}_{N,M}(\xi)\|_{L^2_{\chi^{(0,1)}}(\Omega)}\big)|\tilde{u}_{N,M}(\xi)|^2_{H^1_{0,1,0,-1}(\Omega)}d\xi \\ \le \rho_1(\tilde{u}_{0,N,M}, \tilde{f}(t), t) + C_1(t)\int_0^t \|\tilde{u}_{N,M}(\xi)\|^2_{L^2_{\chi^{(0,1)}}(\Omega)}d\xi.$$

Finally, applying Lemma 8, we get the following result of stability.

Theorem 2. *Suppose that* $\rho(\tilde{u}_{0,N,M}, \tilde{f}, T) \le \frac{\nu}{c^2}\exp(-C_1(u(t)))T$, *then for all* $0 \le t \le T$,

$$E(\tilde{u}_{N,M}, t) \le \rho(\tilde{u}_{0,N,M}, \tilde{f}(t), t)\exp(c_1 t),$$

where $C_1(u_{N,M}(t))$ *depends on* $\|u_{N,M}(t)\|_\infty$.

We next deal with the convergence of (4.2). To do this, let $U^*_{N,M} = {}^0P^1_{N,M,\Omega}U(t)$. By the definition of ${}^0P^1_{N,M,\Omega}$, we derive that

$$a_{0,1,0,-1}(U^*_{N,M}(t), \phi) + (U^*_{N,M}(t), \phi)_{\chi^{(0,1)}} \\ = a_{0,1,0,-1}(U(t), \phi) + (U(t), \phi)_{\chi^{(0,1)}}, \quad \forall \phi \in {}^0\mathcal{P}_{M,N}(\Omega).$$

Then, we have that

$$(\partial_t U^*_{N,M}(t), \phi)_{L^2_{\chi^{(0,1)}}(\Omega)} + \nu a_{0,1,0,-1}(U^*_{N,M}(t), \phi) \\ - \big(U^*_{N,M}(t) - U^{*2}_{N,M}(t), \phi\big)_{L^2_{\chi^{(0,1)}}(\Omega)} \\ = (f(t), \phi)_{L^2_{\chi^{(0,1)}}(\Omega)} + \big(G_1(t) + G_2(t) + G_3(t), \phi\big)_{L^2_{\chi^{(0,1)}}(\Omega)} \\ + \nu(G_3(t), \phi)_{L^2_{\chi^{(0,1)}}(\Omega)}, \quad \forall \phi \in^0 \mathcal{P}_{M,N}(\Omega), \quad t \in [0,T],$$

where

$$G_1(t) = \partial_t(U^*_{N,M}(t) - U(t)),\ G_2(t) = U^{*2}_{N,M}(t) - U^2(t),\ G_3(t) = U(t) - U^*_{N,M}(t).$$

Next, taking $\tilde{U}(t) = U^*_{N,M}(t) - u_{N,M}(t)$, we check that

$$(\partial_t \tilde{U}(t), \phi)_{L^2_{\chi^{(0,1)}}(\Omega)} + \nu a_{0,1,0,-1}(\tilde{U}(t), \phi) \\ - \big(\tilde{U}(t) - 2\tilde{U}(t)U^*_{N,M}(t) + \tilde{U}^2(t), \phi\big)_{L^2_{\chi^{(0,1)}}(\Omega)} \\ = \big(G_1(t) + G_2(t) + (1+\nu)G_3(t), \phi\big)_{L^2_{\chi^{(0,1)}}(\Omega)}, \ \forall \phi \in {}^0\mathcal{P}_{M,N}(\Omega), \quad t \in [0,T].$$

Using the similar manner of the derivation of (4.5), we can deduce that

$$\begin{aligned}&\partial_t\|\tilde{U}(t)\|^2_{L^2_{\chi^{(0,1)}}(\Omega)}+(2\nu-2c^2\|\tilde{U}(t)\|^2_{L^2_{\chi^{(0,1)}}(\Omega)})|\tilde{U}(t)|^2_{H^1_{0,1,0,-1}(\Omega)}\\&\quad\le(3+4\|U^*_{N,M}(t)\|_{\infty,\Omega})\|\tilde{U}(t)\|^2_{L^2_{\chi^{(0,1)}}(\Omega)}\\&\qquad+\|G_1(t)+G_2(t)+(1+\nu)G_3(t)\|^2_{L^2_{\chi^{(0,1)}}(\Omega)}.\end{aligned}\tag{4.6}$$

Integrating the inequality (4.6) from 0 to t with respect to t, we obtain that

$$\begin{aligned}&E(\tilde{U}(t),t)+\int_0^t(2\nu-2c^2\|\tilde{U}(\xi)\|^2_{L^2_{\chi^{(0,1)}}(\Omega)})|\tilde{U}(\xi)|^2_{H^1_{0,1,0,-1}(\Omega)}d\xi\\&\quad\le\rho_2(t)+C_2(t)\int_0^t\|\tilde{U}(\xi)\|^2_{L^2_{\chi^{(0,1)}}(\Omega)}d\xi,\end{aligned}\tag{4.7}$$

where

$$\rho_2(t)=\int_0^t\Big(\|G_1(\xi)\|^2_{L^2_{\chi^{(0,1)}}(\Omega)}+\|G_2(\xi)\|^2_{L^2_{\chi^{(0,1)}}(\Omega)}+(1+\nu)\|G_3(\xi)\|^2_{L^2_{\chi^{(0,1)}}(\Omega)}\Big)d\xi.$$

Similarly, applying Lemma 8 to (4.7), we have the following conclusion. Suppose that $\rho_2(t)\le\frac{\nu}{c^2}e^{-C_2T}$. Then, for all $0\le t\le T$

$$E(\tilde{U}(t),t)\le\rho_2(t)e^{-C_2T},\tag{4.8}$$

where C_2 depend on $\|U(t)\|_\infty$. Now, we only need to estimate $\rho_2(t)$. According to Theorem 1, we have that

$$\|G_1(t)\|^2_{L^2_{\chi^{(0,1)}}(\Omega)}\le c\|G_1(t)\|^2_{L^2_{\chi^{(0,0)}}(\Omega)}\le c(N^{2-2r}+M^{2-2s})B(\partial_tU(t)).$$

Similarly, we deduce that

$$\begin{aligned}\|G_2(t)\|^2_{L^2_{\chi^{(0,1)}}(\Omega)}&\le c\|G_2(t)\|^2_{L^2_{\chi^{(0,0)}}(\Omega)}\\&\le c(\|U^*_{N,M}(t)\|^2_\infty+\|U(t)\|^2_\infty)(\|U^*_{N,M}(t)-U(t)\|^2_{L^2(\Omega)})\\&\le c(N^{2-2r}+M^{2-2s})(\|U^*_{N,M}(t)\|^2_\infty+\|U(t)\|^2_\infty)B(U(t))\\&\le c(N^{2-2r}+M^{2-2s})\|U(t)\|^2_\infty B(U(t)),\\\|G_3(t)\|^2_{L^2_{\chi^{(0,1)}}(\Omega)}&\le c\|G_3(t)\|^2_{L^2_{\chi^{(0,0)}}(\Omega)}\le c(N^{2-2r}+M^{2-2s})B(U(t)),\end{aligned}$$

where

$$\begin{aligned}B(U(t))\;&=\big(|U|_{H^r_{\chi^{(0,1)},*}(\Lambda,L^2_p(I))}+|U|_{H^r_{\chi^{(0,1)},*}(\Lambda,H^1_p(I))}\\&\qquad+|\partial_xU|_{L^2_{\chi^{(0,1)}}(\Lambda,H^s_p(I))}+|U|_{L^2_{\chi^{(0,-1)}}(\Lambda,H^s_p(I))}\big)^2.\end{aligned}$$

The above inequality implies $\rho_2(t)=\mathcal{O}(N^{2-2r}+M^{2-2s})$, when N and M are large enough. We can obtain from (4.8) that

$$E(\tilde{U}(t),t)\le C(N^{2-2r}+M^{2-2s}),\quad 0\le t\le T,$$

where C depends on $\|U(t)\|_\infty,\nu$ and $\int_0^t B(U(\xi))d\xi$. Then, we have the following result of convergence.

Theorem 3. *Let $U(x,\theta,t)$ be the solution of problem (4.1) and $u_{N,M}(x,\theta,t)$ be the solution of scheme (4.2), if $U(1,\theta,t)$=0 and $\partial_\theta u(-1,\theta,t)$=0, $\alpha,\beta,\gamma>-1$, $\delta\geq -1$, then for any $U\in L^\infty(0,T: H^r_{\chi^{(\alpha,\beta)},*}(\Lambda, L^2_p(I))\cap L^2_{\chi^{(\gamma,\delta)}}(\Lambda, H^s_p(I))\cap H^r_{\chi^{(\alpha,\beta)},*}(\Lambda, H^1_p(I)))$, $\partial_x U\in L^2_{\chi^{(\alpha,\beta)}}(\Lambda; H^s_p(I))$, $1\leq r\leq N+1, s\geq 1$, all $0\leq t\leq T$, we have*

$$E(u_{N,M}-U,t)\leq c(N^{2-2r}+M^{2-2s}),$$

where c depends on ν and the above norms of U.

5 Numerical results

In this section, we describe the numerical implementations and present some numerical results. Let

$$\varphi_l(x)=J_l^{(-1,0)}(x),\quad 1\leq l\leq N,\ \ \psi_l(x)=J_l^{(-1,-1)}(x),\quad 2\leq l\leq N.$$

Clearly, $\varphi_l(1)=0,\ 1\leq l\leq N$ and $\psi_l(\pm1)=0,\ \ 2\leq l\leq N$. We set

$$\begin{cases} \phi^{(1)}_{l,m}(x,\theta)=\dfrac{1}{\sqrt{2\pi}}\psi_l(x)\sin(m\theta), & 2\leq l\leq N,\ 1\leq m\leq M,\\ \phi^{(2)}_{l,m}(x,\theta)=\dfrac{1}{\sqrt{2\pi}}\psi_l(x)\cos(m\theta), & 2\leq l\leq N,\ 1\leq m\leq M,\\ \phi^{(3)}_{l}(x,\theta)=\dfrac{1}{\sqrt{2\pi}}\varphi_l(x), & 1\leq l\leq N. \end{cases}$$

It is easy to check that

$$\begin{aligned} &\phi^{(q)}_{l,m}(\pm1,\theta)=0,\quad \partial_\theta\phi^{(q)}_{l,m}(-1,\theta)=0,\quad q=1,2\\ &\phi^{(3)}_{l}(1)=0,\quad \partial_\theta\phi^{(3)}_{l}(-1)=0. \end{aligned}$$

Clearly, the set $\phi^{(q)}_{l,m}(x,\theta),\ 2\leq l\leq N,\ 1\leq m\leq M,\ q=1,2$ and $\phi^{(3)}_l(x),\ 1\leq l\leq N$ conform the basis of the space ${}^0\mathcal{P}_{M,N}(\Omega)$.

Remark 1. Observing that the basis consists of three groups of functions $\phi^{(1)}_{l,m}(x,\theta)$, $\phi^{(2)}_{l,m}(x,\theta)$, $2\leq l\leq N$, $1\leq m\leq M$ and $\phi^{(3)}_{l,m}(x,\theta)$, $1\leq l\leq N$. The first two groups of base functions contain the generalized Jacobi polynomial $J_l^{(-1,-1)}(x)$, and the third group contains $J_l^{(-1,0)}(x)$. The generalized Jacobi polynomials with indices $\alpha=\beta=-1$ and $\alpha=-1,\beta=0$ fit the behaviours of the true solution $U(x,\theta,t)$ and the polar condition $\partial_\theta U(-1,\theta,t)=0$ well.

The numerical solution $u_{N,M}(x,\theta,t)$ can be expanded as

$$\begin{aligned} u_{N,M}(x,\theta,t)\ &=\sum_{l=2}^{N}\sum_{m=1}^{M}\tilde{u}^{(1)}_{l,m}(t)\phi^{(1)}_{l,m}(x,\theta)+\sum_{l=2}^{N}\sum_{m=1}^{M}\tilde{u}^{(2)}_{l,m}(t)\phi^{(2)}_{l,m}(x,\theta)\\ &\quad+\sum_{l=1}^{N}\tilde{u}^{(3)}_{l}(t)\phi^{(3)}_{l}(x). \end{aligned}$$

Taking $\phi=\phi^{(q)}_{l',m'}(x,\theta)$, $q=1,2$ and $\phi=\phi^{(3)}_{l'}(x)$ in (4.2). Then, by the orthogonality of trigonometric functions, we deduce that

$$\begin{aligned}
&\sum_{l=2}^{N}\partial_t\tilde{u}^{(q)}_{l,m}(t)\int_{\Lambda}(1+x)\psi_l(x)\psi_{l'}(x)dx\\
&\quad+\nu\sum_{l=2}^{N}\tilde{u}^{(q)}_{l,m}(t)\int_{\Lambda}((1+x)\partial_x\psi_l(x)\partial_x\psi_{l'}(x)+\frac{1}{1+x}m'^2\psi_l(x)\psi_{l'}(x))dx\\
&\quad=2f^{(q)}_{l',m'},\qquad q=1,2,\ l'=2,3,\cdots,N,\ m'=1,2,\cdots,M,\\
&\sum_{l=1}^{N}\partial_t\tilde{u}^{(3)}_{l}(t)\int_{\Lambda}(1+x)\varphi_l(x)\varphi_{l'}(x)dx+\nu\sum_{l=1}^{N}\tilde{u}^{(3)}_{l,m}(t)\\
&\quad\times\int_{\Lambda}(1+x)\partial_x\varphi_l(x)\partial_x\varphi_{l'}(x)dx=f^{(3)}_{l'},\quad l'=1,2\cdots,N,\quad m'=1,2,\cdots,M,
\end{aligned}$$

where

$$\begin{aligned}
f^{(q)}_{l',m'}&=\int_{\Omega}(1+x)(f(t)+u_{N,M}(t)(1-u_{N,M}(t)))\phi^{(q)}_{l',m'}dxd\theta,\quad q=1,2,\\
f^{(3)}_{l'}&=\int_{\Omega}(1+x)(f(t)+u_{N,M}(t)(1-u_{N,M}(t)))\phi^{(3)}_{l'}dxd\theta.
\end{aligned}$$

We introduce the matrices $\mathbb{A}=(a_{l,l'})$, $\mathbb{B}=(b_{l,l'})$, $\mathbb{C}=(c_{l,l'})$, $\mathbb{G}=(g_{l,l'})$ and $\mathbb{H}=(h_{l,l'})$ with the following entries

$$\begin{aligned}
a_{l,l'}&=\int_{-1}^{1}(1+x)\psi_{l+1}(x)\psi_{l'+1}(x)dx,\quad 1\le l,l'\le N-1,\\
b_{l,l'}&=\int_{-1}^{1}(1+x)\partial_x\psi_{l+1}(x)\partial_x\psi_{l'+1}(x),\quad 1\le l,l'\le N-1,\\
c_{l,l'}&=\int_{-1}^{1}\frac{1}{1+x}\psi_{l+1}(x)\psi_{l'+1}(x)dx,\quad 1\le l,l'\le N-1,\\
g_{l,l'}&=\int_{-1}^{1}(1+x)\varphi_{l+1}(x)\varphi_{l'+1}(x)dx,\quad 0\le l,l'\le N-1,\\
h_{l,l'}&=\int_{-1}^{1}(1+x)\partial_x\varphi_{l+1}(x)\partial_x\varphi_{l'+1}(x)dx,\quad 0\le l,l'\le N-1.
\end{aligned}$$

We next calculate the non-zero elements of the matrices $\mathbb{A},\mathbb{B},\mathbb{C},\mathbb{G}$ and $\mathbb{H}$. We denote by $L_l(x)$ the Legendre polynomial of degree l. We have

$$\int_{-1}^{1}L_l(x)L_{l'}(x)dx=\frac{2}{2l+1}\delta_{ll'},\tag{5.1}$$

where $\delta_{ll'}$ is the Kronecker symbol. We also have the recurrence relation

$$xL_l(x)=\frac{l}{2l+1}L_{l-1}(x)+\frac{l+1}{2l+1}L_{l+1}(x),\quad l\ge 1.\tag{5.2}$$

Thanks to (6.8) of [22] and (5.2), we have that

$$\begin{aligned}(1+x)J_{l+1}^{(-1,-1)}(x) &= (1+x)\frac{2l}{2l+1}(L_{l-1}(x)-L_{l+1}(x))\\ &= \frac{2l}{2l+1}\Big(\frac{l-1}{2l-1}L_{l-2}(x)+L_{l-1}(x)+\frac{2l+1}{(2l-1)(2l+3)}L_l(x)\\ &\quad - L_{l+1}(x)-\frac{l+1}{2l+3}L_{l+2}(x)\Big).\end{aligned}$$

Then, we can obtain that

$$a_{l,l'} = \begin{cases} -\frac{2l}{2l+1}\frac{2l+6}{2l+7}\frac{l+2}{2l+3}\frac{2}{2l+5}, & l=l'-3,\\ -\frac{2l}{2l+1}\frac{2l+4}{2l+5}\frac{2}{2l+3}, & l=l'-2,\\ \frac{2l}{2l+1}\frac{2l+2}{2l+3}\Big(\frac{2}{(2l-1)(2l+3)}+\frac{l+2}{2l+3}\frac{2}{2l+5}\Big), & l=l'-1,\\ \frac{2l}{2l+1}\frac{2l}{2l+1}(\frac{2}{2l-1}+\frac{2}{2l+3}), & l=l',\\ \frac{2l}{2l+1}\frac{2l-2}{2l-1}\Big(\frac{l-1}{2l-1}\frac{2}{2l-3}-\frac{2}{(2l-1)(2l+3)}\Big), & l=l'+1,\\ -\frac{2l}{2l+1}\frac{2l-4}{2l-3}\frac{2}{2l-1}, & l=l'+2,\\ -\frac{2l}{2l+1}\frac{2l-6}{2l-5}\frac{l-1}{2l-1}\frac{2}{2l-3}, & l=l'+3,\\ 0, & \text{otherwise.}\end{cases}$$

According to (6.12) of [22], we deduce that

$$\begin{aligned}b_{l,l'} &= \int_{-1}^{1}(1+x)\partial_x J_{l+1}^{(-1,-1)}(x)\partial_x J_{l'+1}^{(-1,-1)}(x)dx\\ &= 4ll'\int_{-1}^{1}(1+x)L_l(x)L_{l'}(x)dx.\end{aligned}$$

With the aid of (5.2) and (5.1), we get that

$$b_{l,l'} = \begin{cases} \frac{8l(l+1)^2}{(2l+1)(2l+3)}, & l=l'-1,\\ \frac{8l^2}{(2l+1)}, & l=l',\\ \frac{8(l-1)l^2}{(2l-1)(2l+1)}, & l=l'+1,\\ 0, & \text{otherwise.}\end{cases}$$

Using (6.1) of [22], we yield that

$$\begin{aligned}c_{l,l'} &= \int_{-1}^{1}\frac{1}{(1+x)}J_{l+1}^{(-1,-1)}(x)J_{l'+1}^{(-1,-1)}(x)dx\\ &= \int_{-1}^{1}\frac{1}{(1+x)}\Big((1+x)(1-x)J_{l-1}^{(1,1)}(x))((1+x)(1-x)J_{l'-1}^{(1,1)}(x)\Big)dx \quad (5.3)\\ &= \int_{-1}^{1}(1-x)J_{l-1}^{(1,1)}(x)J_{l'-1}^{(1,1)}(x)(1+x)(1-x)dx.\end{aligned}$$

Recurrence relation (3.110) of [22] leads to

$$(1-x)J_{l-1}^{(1,1)}(x) = -\frac{l}{2l+1}J_{l-2}^{(1,1)}(x) + J_{l-1}^{(1,1)}(x) - \frac{l(l+2)}{(l+1)(2l+1)}J_l^{(1,1)}(x).$$

Furthermore, (5.3) and (2.1) lead to

$$c_{l,l'} = \begin{cases} -\frac{l(l+2)}{(l+1)(2l+1)}\gamma_l^{(1,1)}, & l=l'-1, \\ \gamma_{l-1}^{(1,1)}, & l=l', \\ -\frac{l}{2l+1}\gamma_{l-2}^{(1,1)}, & l=l'+1, \\ 0, & \text{otherwise.} \end{cases}$$

Formulas (3.116a) and (3.168) of [22] imply

$$J_{l+1}^{(-1,0)}(x) = (1-x)J_l^{(1,0)}(x) = L_l(x) - L_{l+1}(x). \tag{5.4}$$

By (5.4) and (5.2), we deduce that

$$\begin{aligned}(1+x)J_{l+1}^{(-1,0)}(x) &= (1+x)(L_l(x)-L_{l+1}(x)) \\ &= \frac{l}{2l+1}L_{l-1}(x) + \frac{l+2}{2l+3}L_l(x) - \frac{l}{2l+1}L_{l+1}(x) - \frac{l+2}{2l+3}L_{l+2}(x).\end{aligned}$$

Then, we derive that

$$g(l,l') = \begin{cases} -\frac{2(l+2)}{(2l+3)(2l+5)}, & l=l'-2, \\ -\frac{2l}{(2l+1)(2l+3)} + \frac{2(l+2)}{(2l+3)(2l+5)}, & l=l'-1, \\ \frac{2(l+2)+2l}{(2l+1)(2l+3)}, & l=l', \\ \frac{2l}{(2l-1)(2l+1)} - \frac{2(l+2)}{(2l+1)(2l+3)}, & l=l'+1, \\ -\frac{2l}{(2l-1)(2l+1)}, & l=l'+2, \\ 0, & \text{otherwise.} \end{cases}$$

By (6.12) of [22], we verify that

$$\begin{aligned} h_{l,l'} &= \int_{-1}^{1}(1+x)\partial_x J_{l+1}^{(-1,0)}(x)\partial_x J_{l+1}^{(-1,0)}(x)dx \\ &= (l+1)(l'+1)\int_{-1}^{1}(1+x)J_l^{(0,1)}(x)J_{l'}^{(0,1)}(x)dx = (l+1)^2\gamma_l^{(0,1)}\delta_{ll'}.\end{aligned}$$

Next, we denote that

$$\begin{aligned} X_m^{(q)} &= (\tilde{u}_{2,m}^{(q)}(t), \tilde{u}_{3,m}^{(q)}(t), \cdots, \tilde{u}_{N,m}^{(q)}(t))^T, \\ F_m^{(q)} &= (f_{2,m}^{(q)}(t), f_{3,m}^{(q)}(t), \cdots, f_{N,m}^{(q)}(t))^T, \quad q=1,2, \quad 1 \le m \le M \end{aligned}$$

and

$$X^{(3)}=(\tilde{u}_1^{(3)}(t),\tilde{u}_2^{(3)}(t),\cdots,\tilde{u}_N^{(3)}(t))^T,\ F^{(3)}=(f_1^{(3)}(t),f_2^{(3)}(t),\cdots,f_N^{(3)}(t))^T.$$

Then, we have the following compact form of (4.2)

$$\begin{aligned}&\mathbb{A}\partial_t X_m^{(q)}(t)+\nu(\mathbb{B}+m^2\mathbb{C})X_m^{(q)}(t)=2F_m^{(q)}(t),\ m=1,2,\cdots,M,\ q=1,2,\\&\mathbb{G}\partial_t X^{(3)}(t)+\nu\mathbb{H}X^{(3)}(t)=F^{(3)}(t).\end{aligned}$$

They are two ODE systems. We use the Crank-Nicolson method in time t, with the step size τ. For description of the numerical errors, let $\theta_{M,j}=\dfrac{2\pi j}{2M+1}$, $0\le j\le 2M$ be the Fourier interpolation points, and $x_{N,i}$ and $\omega_{N,i}$, $0\le i\le N$ be the zeros and weights of Legendre-Gauss interpolation. The numerical errors are measured by the quantity

$$E_{N,M}(t)=\Big(\frac{2\pi}{2M+1}\sum_{i=0}^{N}\sum_{j=0}^{2M}\big(U(x_{N,i},\theta_{M,j},t)-u_{N,M}(x_{N,i},\theta_{M,j},t)\big)^2\omega_{N,i}\Big)^{\frac12}.$$

Firstly, we take the test function

$$U(x,\theta,t)=(1-x)(1-x^2)e^{(x+\sin\theta+t/10)}.$$

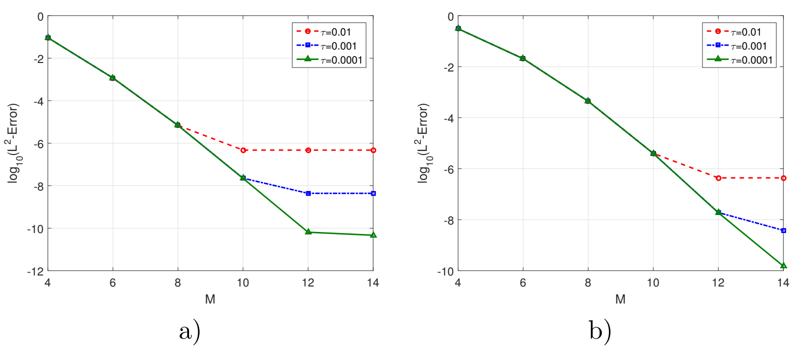


Figure 1. L^2-errors against M, $N(M=N)$ with $t=1$: a) $\nu=1$, b) $\nu=0.001$.

We sketch the L^2-errors with $t=1$, $\tau=0.01,0.001,0.0001$, $\nu=1$ (in Figure 1(a)) and $\nu=0.001$ (in Figure 1(b)). We find that for fixed τ, the errors decay fastly as M and N increase and for fixed N and M the errors decrease as τ decrease. Observing from Figure 1(b) that our algorithm works well for small ν.

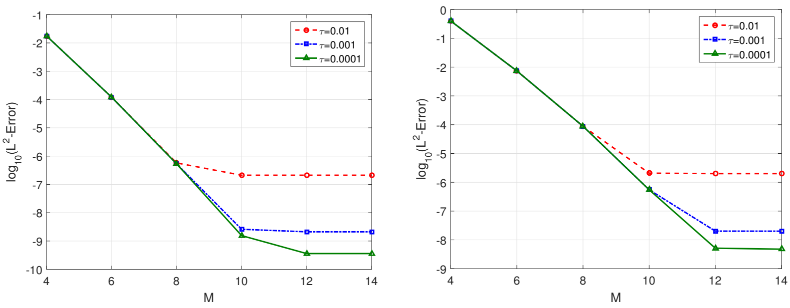
In Table 1, we list the L^2-errors with $\nu=1$, $M=N=10$, $\tau=0.001$ and various t. It implies the stability for long time computation.

Secondly, we take the test function

$$U(x,\theta,t)=(1-x^2)\sin(x+\theta+t/10).$$

Table 1. L^2-errors with $N = M = 10$, $\nu = 1$ and $\tau = 0.001$.

$t = 2$	$t = 4$	$t = 6$	$t = 8$	$t = 10$
2.48e-08	3.01e-08	3.69e-08	4.49e-08	5.45e-08



a) b)

Figure 2. L^2-errors against M, $N(M = N)$ with $t = 1$: a) $\nu = 1$, b) $\nu = 0.001$.

We plot the L^2-errors with $t = 1$, $\tau = 0.01, 0.001, 0.0001$, $\nu = 1$ (in Figure 2(a)) and $\nu = 0.001$ (in Figure 2(b)). The results show that exponential rates of convergence are achieved, and the efficiency of our method.

In Table 2, we tabulate the L^2-errors with $M = N = 10$, $\nu = 1$, $\tau = 0.001$ and various t. The results show the stability for long time computation once again.

Table 2. L^2-errors with $N = M = 10$, $\nu = 1$ and $\tau = 0.001$.

$t = 2$	$t = 4$	$t = 6$	$t = 8$	$t = 10$
2.69e-09	2.71e-09	2.72e-09	2.74e-09	2.77e-09

Thirdly, we take the test function

$$U(x, \theta, t) = (1 - x^2)|x|^{2\pi} \sin(\theta + t/10).$$

In Figures 3(a,b) we depict the solution $U(x, \theta, t)$ at $t = 2$. The solution $U(x, \theta, t)$ has suitable steep spatial gradients.

We plot the L^2-errors with $t = 2$, $\tau = 0.0001$, $M = N$, $\nu = 1$ (in Figure 4). The results show that algebraic rates of convergence are achieved with low-regularity solution.

6 Concluding discussion

In this paper, we recalled some results about Jacobi approximation and Fourier approximation. We constructed some mixed Jacobi-Fourier approximation re-

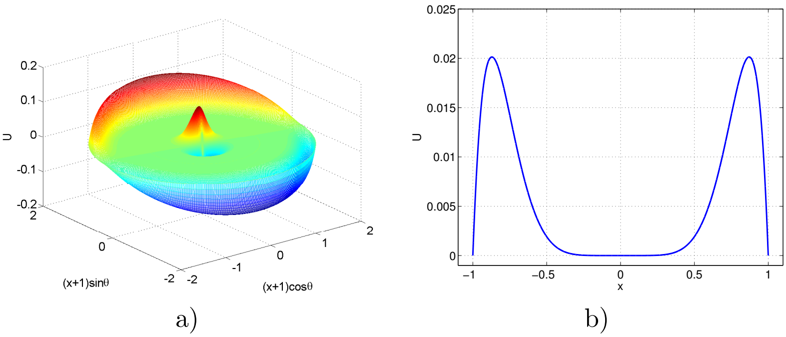


Figure 3. $U(x,\theta,t)$: a) in a disc with $t=2$, b) against x with $\theta=0$ and $t=2$.

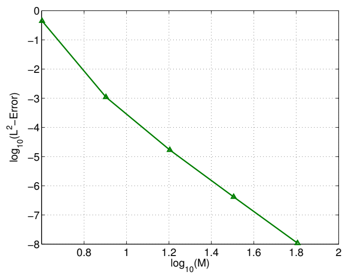


Figure 4. L^2-errors against M, $N(M=N)$ with $\tau=0.001$, $t=2$, $\nu=1$.

sults which play important roles in the theoretical analysis of problem with polar condition in a disc. We proposed a spectral scheme for the Fisher equation in a disc and proved it's generalized stability and convergence. Numerical results demonstrated the efficiency of this new algorithm and coincided well with the theoretical analysis. This approach has several merits: (i) The generalized Jacobi-Fourier approximation with parameters $\alpha=\beta=-1$ and $\alpha=-1,\ \beta=0$ fitted the behaviours of the true solutions well. (ii) The use of the generalized Jacobi-Fourier approximation reduced the difficulty of the theoretical analysis and provided a sparse system which can be solved efficiently. (iii) The numerical solutions possess spectral accuracy in space with smooth solutions. The new approach is good even for solutions with steep gradients (converges with algebraic rate).

Acknowledgements

The authors would like to thank the anonymous referees for their valuable suggestions and comments. The first author is supported in part by NSFC grants (No. 11771299, No. 11371123 and No. 11571151). The second author is supported in part by NSFC grants (No. 11771299, No. 11371123 and No. 11571151). The fourth author is supported by China Postdoctoral Science Foundation funded project (No. 2017M620113).

References

[1] J. Bergh and J. Löfström. *Interpolation Spaces. An Introduction.* Springer-Verlag, Berlin-New York, 1976. https://doi.org/10.1007/978-3-642-66451-9.

[2] Ch. Bernardi and Y. Maday. Polynomial interpolation results in Sobolev spaces. *Journal of Computational and Applied Mathematics*, **43**(1):53–80, 1992. https://doi.org/10.1016/0377-0427(92)90259-Z.

[3] J.P. Boyd. Polynomial series versus sinc expansions for functions with corner or endpoint singularities. *Journal of Computational Physics*, **64**(1):266–270, 1986. https://doi.org/10.1016/0021-9991(86)90031-8.

[4] J.P. Boyd. The asymptotic Chebyshev coefficients for functions with logarithmic endpoint singularities: mappings and singular basis functions. *Applied Mathematics and Computation*, **29**(1):49–67, 1989. https://doi.org/10.1016/0096-3003(89)90039-8.

[5] J.P. Boyd and F. Yu. Comparing seven spectral methods for interpolation and for solving the Poisson equation in a disk: Zernike polynomials, Logan-Shepp ridge polynomials, Chebyshev-Fourier series, cylindrical Robert functions, Bessel-Fourier expansions, square-to-disk conformal mapping and radial basis functions. *Journal of Computational Physics*, **230**(4):1408–1438, 2011. https://doi.org/10.1016/j.jcp.2010.11.011.

[6] N.F. Britton. *Reaction-Diffusion Equations and Their Applications to Biology.* Academic Press, Inc., London, 1986.

[7] R.A. Fisher. The wave of advance of advantageous genes. *Annals of Eugenics*, **7**(4):355–369, 1937. https://doi.org/10.1111/j.1469-1809.1937.tb02153.x.

[8] B.-Y. Guo. Gegenbauer approximation and its applications to differential equations on the whole line. *Journal of Mathematical Analysis and Applications*, **226**(1):180–206, 1998. https://doi.org/10.1006/jmaa.1998.6025.

[9] B.-Y. Guo. *Spectral Methods and Their Applications.* World Scientific, Singapore, 1998.

[10] B.-Y. Guo. Error estimate of Hermite spectral method for nonlinear partial differential equation. *Mathematics of Computation*, **68**(227):1067–1078, 1999. https://doi.org/10.1090/S0025-5718-99-01059-5.

[11] B.-Y. Guo. Gegenbauer approximation in certain Hilbert spaces and its applications to singular differential equations. *SIAM Journal on Numerical Analysis*, **37**(2):621–645, 1999. https://doi.org/10.1137/S0036142998342161.

[12] B.-Y. Guo and Z.-X. Chen. Analytic solutions of the Fisher equation. *Journal of Physics A: Mathematical and General*, **24**(3):645–650, 1991. https://doi.org/10.1088/0305-4470/24/3/023.

[13] B.-Y. Guo and L.-L. Wang. Jacobi interpolation approximations and their applications to singular differential equations. *Advances in Computational Mathematics*, **14**(3):227–276, 2001. https://doi.org/10.1023/A:1016681018268.

[14] B.-Y. Guo and L.-L. Wang. Jacobi approximations in non-uniformly Jacobi-weighted Sobolev spaces. *Journal of Approximation Theory*, **128**(1):1–41, 2004. https://doi.org/10.1016/j.jat.2004.03.008.

[15] R. Jiwari. A Haar wavelet quasilinearization approach for numerical simulation of Burgers' equation. *Computer Physics Communications*, **183**(11):2413–2423, 2012. https://doi.org/10.1016/j.cpc.2012.06.009.

[16] D.S. Jones and B.D. Sleeman. *Differential Equations and Mathematical Biology.* Allen Unwin, London, 1983.

[17] J.D. Logan. *An Introduction to Nonlinear Differential Equations, 2nd edition.* John Wilcy & Sons, Inc., Hoboken, New Jersey, New York, 2008.

[18] T. Matsushima and P.S. Marcus. A spectral method for polar coordinates. *Journal of Computational Physics*, **120**(2):365–374, 1995. https://doi.org/10.1006/jcph.1995.1171.

[19] R.C. Mittal and R. Jiwari. Numerical study of Fisher's equation by using differential quadrature method. *International Journal of Information and Systems Sciiences*, **5**(1):143–160, 2009.

[20] J.D. Murray. *Mathematical Biology ll: Spatial Models and Biomedical Applications.* Springer-Verlag, Berlin Heidelberg, 1993.

[21] D. Olmos and B.D. Shizgal. A pseudospectral method of solution of Fisher's equation. *Journal of Computational and Applied Mathematics*, **193**(1):219–242, 2006. https://doi.org/10.1016/j.cam.2005.06.028.

[22] J. Shen, T. Tang and L.-L. Wang. *Spectral Methods: Algorithms, Analysis and Applications.* Spinger-Verlag, Berlin Heidelberg, 2011.

[23] J. Smoller. *Shock Waves and Reaction-Diffusion Equations*, volume 258 of *Grundlehren der mathematischen Wissenschaften.* Springer, New-York, 1983.

[24] F. Stenger. Numerical methods based on Whittaker cardinal, or sinc functions. *SIAM Review*, **23**(2):165–224, 1981. https://doi.org/10.1137/1023037.

[25] A. Verma, R. Jiwari and M.E. Koksal. Analytic and numerical solutions of nonlinear diffusion equations via symmetry reductions. *Advances in Difference Equations*, **2014**(1):229, 2014. https://doi.org/10.1186/1687-1847-2014-229.

[26] T.-J. Wang. Generalized Laguerre spectral method for Fisher's equation on a semi-infinite interval. *International Journal of Computer Mathematics*, **92**(5):1039–1052, 2015. https://doi.org/10.1080/00207160.2014.920833.

[27] T.-J. Wang and Y.-J. Jiao. A fully discrete pseudospectral method for Fisher's equation on the whole line. *Applied Numerical Mathematics*, **120**(Supplement C):243–256, 2017. https://doi.org/10.1016/j.apnum.2017.06.002.

[28] A.-M. Wazwaz and A. Gorguis. An analytic study of Fisher's equation by using Adomian decomposition method. *Applied Mathematics and Computation*, **154**(3):609–620, 2004. https://doi.org/10.1016/S0096-3003(03)00738-0.

[29] X.-H. Yu and Z.-Q. Wang. Mixed Fourier-Jacobi spectral method for two-dimensional Neumann boundary value problems. *East Asian Journal on Applied Mathematics*, **1**(3):284–296, 2011. https://doi.org/10.4208/eajam.281010.200411a.

[30] X.-H. Yu and Z.-Q. Wang. Jacobi spectral method with essential imposition of Neumann boundary condition. *Applied Numerical Mathematics*, **62**(8):956–974, 2012. https://doi.org/10.1016/j.apnum.2012.03.004.