New Recursive Approximations for Variable-Order Fractional Operators with Applications

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Abstract. To broaden the range of applicability of variable-order fractional differential models, reliable numerical approaches are needed to solve the model equation. In this paper, we develop Laguerre spectral collocation methods for solving variable-order fractional initial value problems on the half line. Specifically, we derive three-term recurrence relations to efficiently calculate the variable-order fractional integrals and derivatives of the modified generalized Laguerre polynomials, which lead to the corresponding fractional differentiation matrices that will be used to construct the collocation methods. Comparison with other existing methods shows the superior accuracy of the proposed spectral collocation methods.

1   Introduction

The variable-order fractional (VO-F) operators [8,17], which are generalizations of constant-order fractional operators [23], open up new possibilities for robust mathematical modeling and simulation of diverse physical problems in science and engineering, such as modeling of diffusive-convective effects on the oscillatory flows [15], linear and nonlinear oscillators with viscoelastic damping [8], processing of geographical data using VO-F derivatives [9], constitutive laws in viscoelastic continuum mechanics [16], signature verification through variable/adaptive fractional order differentiators [21], anomalous diffusion problems [10, 24] and chloride ions sub-diffusion in concrete structures [22]. The VO-F operators can be employed to depict the variable memory of systems [12].

The VO-F operators are nonlocal with singular kernels, which makes the VO-F models complicated. Hence, the solution of VO-F models are also more complicated. Numerical computation of the VO-F operators is the key to understand the behavior and physical meaning of the VO-F models. Fu et al. [10] applied the method of approximate particular solutions to VO-F diffusion models. Cao and Qiu [7] proposed a second order numerical approximation via the VO-F weighted and shifted Grünwald-Letnikov formula to VO-F Riemann-Liouville derivative, and used it to solve VO-F ordinary differential equations. Zayernouri and Karniadakis [25] introduced fractional spectral collocation methods for linear and nonlinear VO-F differential equations. Atangana et al. [3] developed the Crank-Nicholson scheme to handle the time VO-F telegraph equation. Bhrawy et al. proposed accurate spectral collocation methods for VO-F differential equations such as Schrödinger equation [6], Galilei invariant advection diffusion equations [1], diffusion equation [24] and cable equation [5]. Moghaddam et al [13, 14] developed accurate and robust algorithms for approximating VO-F derivatives and integrals. Tayebi et al [20] proposed an accurate and robust meshless method based on the moving least squares approximation and the finite difference scheme for the numerical solution of VO-F advection-diffusion equation on two-dimensional arbitrary domains.

In this paper, we focus on the computation of the VO-F integrals and derivatives of the modified generalized Laguerre polynomials. Applications of the constructed computations are illustrated to compute the VO-F Caputo derivative. Besides, using the modified generalized Laguerre polynomials as the basis functions, we develop Laguerre-Gauss collocation methods to solve fractional differential equations of variable and constant orders on the half line.

This paper is organized as follows. Section 2 presents the fundamentals of VO-F operators and properties of the modified generalized Laguerre polynomials. Numerical algorithms for calculating the VO-F integral and the Caputo derivative are presented in Sections 3 and 4, respectively. The applications of the algorithms are illustrated in Section 5. Numerical examples are presented in Section 6, and the conclusion is drawn in the last section.

2   Preliminaries and fundamentals

In this section, we concisely point out some definitions of the VO-F operators [8, 12,17]. We then collect some important properties of the modified generalized

Laguerre polynomials [4]. Assume that $u(x) = 0$ for $x < 0$.

1. The following VO-F integral operator was proposed in [17]

$$ {}_0I_x^{\varrho(x)}[u] := x \mapsto \frac{1}{\Gamma(\varrho(x))} \int_0^x (x-r)^{\varrho(x)-1} u(r) dr, \quad x \geq 0. \qquad (2.1) $$

2. In [12] several definitions were proposed. The first is identical to (2.1). The next one, is

$$ {}_0I_x^{\varrho(x)}[u] := x \mapsto \int_0^x \frac{(x-r)^{\varrho(r)-1}}{\Gamma(\varrho(r))} u(r) dr, \quad x \geq 0. $$

3. The following operator was introduced in [12]

$$ {}_0I_x^{\varrho(x)}[u] := x \mapsto \int_0^x \frac{(x-r)^{\varrho(x-r)-1}}{\Gamma(\varrho(x-r))} u(r) dr \quad x \geq 0. \qquad (2.2) $$

The VO-F Caputo derivative could now be defined, as in the case of constant order [17], as follows

$$ {}_0^C D_x^{\varrho(x)} := {}_0I_x^{n-q(x,r)} \circ \frac{d^n}{dx^n}, $$

where $q(x,r) = \varrho(x)$, $q(x,r) = \varrho(r)$ and $q(x,r) = \varrho(x-r)$, in cases (2.1)–(2.2). Thus, we obtain, respectively:

1. *The type I: left Caputo fractional derivative of order* $\varrho(x)$

$$ {}_0^C\mathcal{D}_x^{\varrho(x)} u(x) = \frac{1}{\Gamma(n-\varrho(x))} \int_0^x \frac{u^{(n)}(r) dr}{(x-r)^{\varrho(x)-n+1}}. \qquad (2.3) $$

2. *The type II: left Caputo fractional derivative of order* $\varrho(x)$

$$ {}_0^C\mathfrak{D}_x^{\varrho(x)} u(x) = \int_0^x \frac{1}{\Gamma(n-\varrho(r))} \frac{u^{(n)}(r) dr}{(x-s)^{\varrho(r)-n+1}}. $$

3. *The type III: left Caputo fractional derivative of order* $\varrho(x)$

$$ {}_0^C\mathbb{D}_x^{\varrho(x)} u(x) = \int_0^x \frac{1}{\Gamma(n-\varrho(x-r))} \frac{u^{(n)}(r) dr}{(x-r)^{\varrho(x-r)-n+1}}, $$

where $n - 1 < \varrho(\cdot) < n \in \mathbb{N}$. Such operators have been used by researchers, for examples, Coimbra et al. [8, 18] employed the first type in the modeling of viscous-viscoelastic oscillator. Ingman and Suzdalnitsky [11] used the second type in the modeling of viscoelastic deformation process. Atanackovic and Pilipovic [2] used the third type in generalization of Hamilton's principle. Sun et al. [19] introduced a comparative investigation of constant-order fractional derivative and the first two types of VO-F derivatives in characterizing the memory property of systems. However, the differences between the three types

in applications are still not clear. There are other definitions of VO-F derivatives [2]. In this paper, we will focus our attention on the first type of VO-F integrals and derivatives, i.e. $q(x,s) = \varrho(x)$.

The operator $^C_0\mathcal{D}^{\varrho(x)}_x$ satisfies the following property ($n-1 < \varrho(x) \leq n \in \mathbb{N}$)

$$^C_0\mathcal{D}^{\varrho(x)}_x x^\gamma = \begin{cases} 0, & \gamma = 0, \ldots, n-1, \\ \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\varrho(x))} x^{\gamma-\varrho(x)}, & \gamma = n, n+1, \ldots. \end{cases}$$

Next, let us introduce some properties of the modified generalized Laguerre polynomials [4]. Let $\Lambda = \{x \mid 0 < x < \infty\}$ and $\chi(x)$ be a certain weight function on Λ in the usual sense. Define

$$L^2_\chi(\Lambda) = \{y \mid y \text{ is measurable } \& \ ||y||_\chi < \infty\},$$

which is a Hilbert space, equipped with the following inner product and norm:

$$(u,y)_\chi = \int_\Lambda \chi(x)u(x)y(x) \ dx, \qquad ||y||_\chi = (y,y)^{\frac{1}{2}}_\chi.$$

In principle, the generalized Laguerre polynomials are suitable for the approximation of functions with growth at infinity. We now recall some properties of the generalized Laguerre polynomials $\mathcal{L}^{(\theta,\beta)}_i(x)$.

Let $\chi^{(\theta,\beta)} = x^\theta e^{-\beta x}$, $\theta > -1$, and $\beta > 0$, $\partial_x v(x) = \frac{\partial}{\partial x}v(x)$. The corresponding generalized Laguerre polynomials of degree i are defined by

$$\mathcal{L}^{(\theta,\beta)}_i(x) = \frac{1}{i!}x^{-\theta}e^{\beta x}\partial^i_x(x^{i+\theta}e^{-\beta x}), \qquad i = 0, 1, 2, \ldots.$$

They are the eigenfunctions of the Sturm-Liouville problem

$$\partial_x(\chi^{(\theta+1,\beta)}(x)\partial_x v(x)) + \lambda^{(\beta)}_i \chi^{(\theta,\beta)}(x)v(x) = 0, \qquad 0 < x < \infty$$

with the corresponding eigenvalues $\lambda^{(\beta)}_i = i\beta$. They fulfill the following three-term recurrence relations:

$$\mathcal{L}^{(\theta,\beta)}_0(x) = 1, \qquad \mathcal{L}^{(\theta,\beta)}_1(x) = -\beta x + \frac{\Gamma(\theta+2)}{\Gamma(\theta+1)},$$

$$\mathcal{L}^{(\theta,\beta)}_{i+1}(x) = \frac{2i+\theta+1-\beta x}{i+1}\mathcal{L}^{(\theta,\beta)}_i(x) - \frac{i+\theta}{i+1}\mathcal{L}^{(\theta,\beta)}_{i-1}(x), \qquad i \geq 1 \tag{2.4}$$

and

$$\mathcal{L}^{(\theta,\beta)}_i(x) = \mathcal{L}^{(\theta+1,\beta)}_i(x) - \mathcal{L}^{(\theta+1,\beta)}_{i-1}(x) = \frac{1}{\beta}(\partial_x\mathcal{L}^{(\theta,\beta)}_i(x) - \partial_x\mathcal{L}^{(\theta,\beta)}_{i+1}(x)). \tag{2.5}$$

The m-th derivative of a generalized Laguerre polynomial satisfies the relation

$$\partial^m_x\mathcal{L}^{(\theta,\beta)}_i(x) = (-\beta)^m\mathcal{L}^{(\theta+m,\beta)}_{i-m}(x).$$

The generalized Laguerre polynomials form a complete $L^2_{\chi^{\theta,\beta}}(\Lambda)$-orthogonal system,

$$\left(\mathcal{L}^{(\theta,\beta)}_i(x), \mathcal{L}^{(\theta,\beta)}_j(x)\right)_{\chi^{(\theta,\beta)}} = \gamma^{(\theta,\beta)}_i \delta_{i,j}, \qquad \gamma^{(\theta,\beta)}_i = \frac{\Gamma(i+\theta+1)}{\beta^{\theta+1}\Gamma(i+1)},$$

where $\delta_{i,j}$ is the Kronecker symbol.

3   Approximation to the variable-order fractional integral

The main goal of this section is to develop a new algorithm to approximate the VO-F integral of a given function.

Let $u \in L^2_{\chi^{\theta,\beta}}(\Lambda)$ and N be a positive integer, then, we can expand it in terms of generalized Laguerre polynomials as

$$u(x) \simeq u_N(x) = \sum_{i=0}^{N} \ell_i^{(\theta,\beta)} \mathcal{L}_i^{(\theta,\beta)}(x).$$

If $u_N(x)$ is an orthogonal projection of $u(x)$, then $\ell_i^{(\theta,\beta)}$ can be determined by the orthogonality of $\{\mathcal{L}_i^{(\theta,\beta)}(x)\}$. If $u_N(x)$ is the interpolation of $u(x)$ on the generalized Laguerre-Gauss points $\{x_{N,i}^{\theta,\beta}\}_{i=0}^{N}$, then $\ell_i^{(\theta,\beta)}$ can be determined by

$$\ell_i^{(\theta,\beta)} = \frac{1}{\gamma_i^{(\theta,\beta)}} \sum_{j=0}^{N} u(x_{N,j}^{(\theta,\beta)}) \mathcal{L}_i^{(\theta,\beta)}(x_{N,j}^{(\theta,\beta)}) \chi_{N,j}^{(\theta,\beta)}, \qquad (3.1)$$

where $\chi_{N,j}^{(\theta,\beta)}$ are the corresponding weights. In this paper, we assume that $u_N(x)$ is the interpolation of $u(x)$.

Therefore, for any $n-1 < \varrho_{min} < \varrho(x) < \varrho_{max} < n$, the VO-F integral $_0I_x^{\varrho(x)} u(x)$ can be approximated by

$$_0I_x^{\varrho(x)} u(x) \approx {}_0I_x^{\varrho(x)} u_N(x) = \frac{1}{\Gamma(\varrho(x))} \int_0^x (x-t)^{\varrho(x)-1} u_N(t)dt = \frac{1}{\Gamma(\varrho(x))}$$

$$\times \sum_{i=0}^{N} \ell_i^{(\theta,\beta)} \int_0^x (x-t)^{\varrho(x)-1} \mathcal{L}_i^{(\theta,\beta)}(t)dt = \sum_{i=0}^{N} \ell_i^{(\theta,\beta)} \hat{\mathcal{L}}_i^{(\varrho(x),\theta,\beta)}(x),$$

where

$$\hat{\mathcal{L}}_i^{(\varrho(x),\theta,\beta)}(x) = \frac{1}{\Gamma(\varrho(x))} \int_0^x (x-t)^{\varrho(x)-1} \mathcal{L}_i^{(\theta,\beta)}(t)dt.$$

Next, we develop a recurrence formula to calculate the $\varrho(x)$th-order fractional integral of $\mathcal{L}_i^{(\theta,\beta)}$. From (2.4), we can easily get

$$\hat{\mathcal{L}}_0^{(\varrho(x),\theta,\beta)}(x) = \frac{x^{\varrho(x)}}{\Gamma(\varrho(x)+1)}, \quad \hat{\mathcal{L}}_1^{(\varrho(x),\theta,\beta)}(x) = \frac{x^{\varrho(x)}}{\Gamma(\varrho(x)+1)} - \frac{\beta x^{\varrho(x)+1}}{\Gamma(\varrho(x)+2)}.$$

For $i \geq 1$, we have

$$\hat{\mathcal{L}}_{i+1}^{(\varrho(x),\theta,\beta)}(x) = \frac{1}{\Gamma(\varrho(x))} \int_0^x (x-t)^{\varrho(x)-1} \mathcal{L}_{i+1}^{(\theta,\beta)}(t)dt = \frac{1}{\Gamma(\varrho(x))} \frac{1}{i+1}$$

$$\times \int_0^x (x-t)^{\varrho(x)-1} \times \quad \left[(2i+\theta+1-\beta t)\mathcal{L}_i^{(\theta,\beta)}(t) - (i+\theta)\mathcal{L}_{i-1}^{(\theta,\beta)}(t) \right] dt$$

$$= \frac{1}{i+1} \Big\{ (2i+\theta+1-\beta x)\hat{\mathcal{L}}_i^{(\varrho(x),\theta,\beta)}(x) - (i+\theta)\hat{\mathcal{L}}_{i-1}^{(\varrho(x),\theta,\beta)}(x)$$

$$+ \frac{\beta}{\Gamma(\varrho(x))} \int_0^x (x-t)^{\varrho(x)} \mathcal{L}_i^{(\theta,\beta)}(t)dt \Big\}.$$

We can verify from formula (2.5) that for $i \geq 1$,

$$
\begin{aligned}
\hat{\mathcal{L}}_{i+1}^{(\varrho(x),\theta,\beta)}(x) =& \frac{1}{i+1}\Big\{(2i+\theta+1-\beta x)\hat{\mathcal{L}}_i^{(\varrho(x),\theta,\beta)}(x) - (i+\theta)\hat{\mathcal{L}}_{i-1}^{(\varrho(x),\theta,\beta)}(x) \\
&+ \frac{1}{\Gamma(\varrho(x))}\int_0^x (x-t)^{\varrho(x)}\big(\partial_t\mathcal{L}_i^{(\theta,\beta)}(t) - \partial_t\mathcal{L}_{i+1}^{(\theta,\beta)}(t)\big)dt\Big\} \\
=& \frac{1}{i+1}\Big\{(2i+\theta+1-\beta x)\hat{\mathcal{L}}_i^{(\varrho(x),\theta,\beta)}(x) - (i+\theta)\hat{\mathcal{L}}_{i-1}^{(\varrho(x),\theta,\beta)}(x) \\
&+ \frac{1}{\Gamma(\varrho(x))}\Big[(x-t)^{\varrho(x)}\big(\mathcal{L}_i^{(\theta,\beta)}(t) - \mathcal{L}_{i+1}^{(\theta,\beta)}(t)\big)\Big]_0^x \\
&+ \varrho(x)\big(\hat{\mathcal{L}}_i^{(\varrho(x),\theta,\beta)}(x) - \hat{\mathcal{L}}_{i+1}^{(\varrho(x),\theta,\beta)}(x)\big)\Big\} \\
=& \frac{1}{(i+\varrho(x)+1)}\Big\{(2i+\theta+\varrho(x)+1-\beta x)\hat{\mathcal{L}}_i^{(\varrho(x),\theta,\beta)}(x) \\
&- (i+\theta)\hat{\mathcal{L}}_{i-1}^{(\varrho(x),\theta,\beta)}(x) - \frac{x^{\varrho(x)}}{\Gamma(\varrho(x))}\big(\mathcal{L}_i^{(\theta,\beta)}(0) - \mathcal{L}_{i+1}^{(\theta,\beta)}(0)\big)\Big\}.
\end{aligned}
$$

Hence, for $i \geq 1$, we get the following recurrence relation

$$
\begin{aligned}
\hat{\mathcal{L}}_{i+1}^{(\varrho(x),\theta,\beta)}(x) =& \frac{1}{(i+\varrho(x)+1)}\Big\{(2i+\theta+\varrho(x)+1-\beta x)\hat{\mathcal{L}}_i^{(\varrho(x),\theta,\beta)}(x) \\
&- (i+\theta)\hat{\mathcal{L}}_{i-1}^{(\varrho(x),\theta,\beta)}(x) - \frac{x^{\varrho(x)}}{\Gamma(\varrho(x))}\big(\mathcal{L}_i^{(\theta,\beta)}(0) - \mathcal{L}_{i+1}^{(\theta,\beta)}(0)\big)\Big\}.
\end{aligned}
$$

So, $\hat{\mathcal{L}}_i^{(\varrho(x),\theta,\beta)}(x)$ can be calculated by the following formula

$$
\hat{\mathcal{L}}_0^{(\varrho(x),\theta,\beta)}(x) = \frac{x^{\varrho(x)}}{\Gamma(\varrho(x)+1)}, \quad \hat{\mathcal{L}}_1^{(\varrho(x),\theta,\beta)}(x) = \frac{x^{\varrho(x)}}{\Gamma(\varrho(x)+1)} - \frac{\beta x^{\varrho(x)+1}}{\Gamma(\varrho(x)+2)},
$$

$$
\begin{aligned}
\hat{\mathcal{L}}_{i+1}^{(\varrho(x),\theta,\beta)}(x) =& \frac{1}{(i+\varrho(x)+1)}\Big\{(2i+\theta+\varrho(x)+1-\beta x)\hat{\mathcal{L}}_i^{(\varrho(x),\theta,\beta)}(x) \\
&- (i+\theta)\hat{\mathcal{L}}_{i-1}^{(\varrho(x),\theta,\beta)}(x) - \frac{x^{\varrho(x)}}{\Gamma(\varrho(x))}\big(\mathcal{L}_i^{(\theta,\beta)}(0) - \mathcal{L}_{i+1}^{(\theta,\beta)}(0)\big)\Big\}, \quad i \geq 1. \quad (3.2)
\end{aligned}
$$

Therefore, $_0I_x^{\varrho(x)}u(x)$ can be approximated by

$$
_0I_x^{\varrho(x)}u(x) \approx {}_0I_x^{\varrho(x)}u_N(x) = \sum_{i=0}^N \ell_i^{(\theta,\beta)}\hat{\mathcal{L}}_i^{(\varrho(x),\theta,\beta)}(x), \quad (3.3)
$$

where $\hat{\mathcal{L}}_i^{(\varrho(x),\theta,\beta)}(x)$ is given by (3.2), and $\ell_i^{(\theta,\beta)}$ is given by (3.1).

Remark 1. When $\varrho(x) = \varrho = constant$, then the operator of VO-F integral is reduced to corresponding integral of constant order and the approximation relation (3.3) reduces to

$$
_0I_x^\varrho u(x) \approx {}_0I_x^\varrho u_N(x) = \sum_{i=0}^N \ell_i^{(\theta,\beta)}\hat{\mathcal{L}}_i^{(\varrho,\theta,\beta)}(x).
$$

4 Approximation to the variable-order fractional Caputo derivative

In this section, we describe how to use the generalized Laguerre polynomials to numerically approximate the VO-F derivative in the Caputo sense of a given function $u(x), x \in \Lambda$. The computerized mathematical algorithm is based on the numerical approximation of the VO-F integral derived in the previous section.

Suppose that $u_N(x)$ is the approximate polynomial of $u(x)$, which can be expressed by

$$u_N(x) = \sum_{i=0}^{N} \ell_i^{(\theta,\beta)} \mathcal{L}_i^{(\theta,\beta)}(x), \quad x \in \Lambda.$$

Let $n-1 < \varrho_{min} < \varrho(x) < \varrho_{max} < n \in \mathbb{N}$, we approximate the VO-F derivative $_0^C\mathcal{D}_x^{\varrho(x)} u(x)$ by the generalized Laguerre polynomials as

$$_0^C\mathcal{D}_x^{\varrho(x)} u_N(x) = \frac{1}{\Gamma(n-\varrho(x))} \int_0^x (x-t)^{n-\varrho(x)-1} \partial_t^n u_N(t) dt$$

$$= \frac{1}{\Gamma(n-\varrho(x))} \sum_{i=0}^{N} \ell_i^{(\theta,\beta)} \int_0^x (x-t)^{n-\varrho(x)-1} \partial_t^n \mathcal{L}_i^{(\theta,\beta)}(t) dt$$

$$= \sum_{i=0}^{N} \ell_i^{(\theta,\beta)} (-\beta)^n \left(\frac{1}{\Gamma(n-\varrho(x))} \int_0^x (x-t)^{n-\varrho(x)-1} \mathcal{L}_{i-n}^{(\theta+n,\beta)}(t) dt \right)$$

$$= \sum_{i=0}^{N} \ell_i^{(\theta,\beta)} (-\beta)^n \hat{\mathcal{L}}_{i-n}^{(n-\varrho(x),\theta+n,\beta)}(x),$$

where $\hat{\mathcal{L}}_{i-n}^{(n-\varrho(x),\theta+n,\beta)}(x) = 0$ for $0 \le i \le n-1$.

Therefore, the VO-F derivative of $u(x)$ can be approximated by

$$_0^C\mathcal{D}_x^{\varrho(x)} u_N(x) = \sum_{i=0}^{N} \ell_i^{(\theta,\beta)} (-\beta)^n \hat{\mathcal{L}}_{i-n}^{(n-\varrho(x),\theta+n,\beta)}(x) = \sum_{i=0}^{N} \ell_i^{(\theta,\beta)} D_{i,n,\theta,\beta}^{(\varrho(x))}(x),$$

which alternatively may be written in the matrix form:

$$_0^C\mathcal{D}_x^{\varrho(x)} u_N(x) = \mathcal{L}^{(\theta,\beta)} \mathbf{D}_{n,\theta,\beta}^{(\varrho(x))}(x) \tag{4.1}$$

with $\mathcal{L}^{(\theta,\beta)} = \left[\ell_0^{(\theta,\beta)}, \ell_1^{(\theta,\beta)}, \ldots, \ell_N^{(\theta,\beta)} \right]$,

$$\mathbf{D}_{n,\theta,\beta}^{(\varrho(x))}(x) = \left[D_{0,n,\theta,\beta}^{(\varrho(x))}(x), D_{1,n,\theta,\beta}^{(\varrho(x))}(x), \ldots, D_{N,n,\theta,\beta}^{(\varrho(x))}(x) \right]^T,$$

where $n-1 < \varrho_{min} < \varrho(x) < \varrho_{max} < n \in \mathbb{N}$ and

$$D_{i,n,\theta,\beta}^{(\varrho(x))}(x) = \frac{1}{\Gamma(n-\varrho(x))} \int_0^x (x-t)^{n-\varrho(x)-1} \partial_t^n \mathcal{L}_i^{(\theta,\beta)}(t) dt$$

$$= (-\beta)^n \hat{\mathcal{L}}_{i-n}^{(n-\varrho(x),\theta+n,\beta)}, \quad i \ge n$$

and $D_{i,n,\theta,\beta}^{(\varrho(x))}(x) = 0$ for $0 \le i \le n-1$.

5   Applications of the algorithms

After the construction of the VO-F differentiation matrix of Caputo type (4.1), we now use the generalized Laguerre spectral collocation method together with the generalized Laguerre operational matrix of VO-F derivative to solve the following VO-F differential equation:

$$
\begin{aligned}
a(x)u'(x) + b(x)\,{}^C_0\mathcal{D}_x^{\varrho(x)}u(x) + c(x)u(x) &= f(x), \qquad x \in \Lambda, \\
u(0) &= u_0,
\end{aligned}
\tag{5.1}
$$

where $0 < \varrho_{min} < \varrho(x) < \varrho_{max} < 1$, $a(x)$, $b(x)$, $c(x)$ and $f(x)$ are real-valued functions. A special case of (5.1) occurs when $\varrho(x) = \varrho = \frac{1}{2}$, corresponding to the fractional Basset equation. This model represents a classical problem in fluid dynamics in the scope of an unsteady motion of a particle that accelerates in a viscous fluid under the action of gravity.

Suppose that $u_N(x)$ is the approximate solution of $u(x)$ and x_i $(0 \leq i \leq N)$ is the generalized Laguerre-Gauss nodes of $\mathcal{L}_{N+1}^{(\theta,\beta)}(x)$. Now, using (4.1) then it is easy to write

$$
a(x_i)\mathcal{L}^{(\theta,\beta)}\mathbf{D}_{1,\theta,\beta}^{(1)}(x_i) + b(x_i)\mathcal{L}^{(\theta,\beta)}\mathbf{D}_{1,\theta,\beta}^{(\varrho(x_i))}(x_i) + c(x_i)\mathcal{L}^{(\theta,\beta)}\mathbf{D}_{0,\theta,\beta}^{(0)}(x_i) = f(x_i),
$$

$$
u_N(0) = u_0, \quad i = 0,1,\ldots,N-1.
\tag{5.2}
$$

Let us denote $F = [f(x_0),\ldots,f(x_{N-1}),u_0]$. Then (5.2) is equivalent to the following matrix equation

$$
\mathcal{L}^{(\theta,\beta)}E = F,
$$

where

$$
E = (e_{ij}) =
\begin{cases}
a(x_j)D_{i,1,\theta,\beta}^{(1)}(x_j) + b(x_j)D_{i,1,\theta,\beta}^{(\varrho(x_j))}(x_j), +c(x_j)D_{i,0,\theta,\beta}^{(0)}(x_j), \\
\qquad 0 \leq i \leq N,\ 0 \leq j \leq N-1, \\
\dfrac{\Gamma(i+\theta+1)}{\Gamma(1+\theta)i!}, \quad 0 \leq i \leq N,\ j = N.
\end{cases}
$$

For the following fractional initial value problem

$$
\begin{aligned}
a(x)u^{(m)}(x) + b(x)\,{}^C_0\mathcal{D}_x^{\varrho(x)}u(x) + c(x)u(x) &= f(x), \qquad x \in \Lambda, \\
u(0) = u_0, \quad u'(0) &= v_0,
\end{aligned}
\tag{5.3}
$$

where $1 < \varrho_{min} < \varrho(x) < \varrho_{max} < 2$, $m = 1$ or 2, we can also get the algebraic equation of the form like (5.2), where $F = [f(x_0),\ldots,f(x_{N-2}),u_0,v_0]$ and

$$
E = (e_{ij}) =
\begin{cases}
a(x_j)D_{i,m,\theta,\beta}^{(m)}(x_j) + b(x_j)D_{i,2,\theta,\beta}^{(\varrho(x_j))}(x_j), +c(x_j)D_{i,0,\theta,\beta}^{(0)}(x_j), \\
\qquad 0 \leq i \leq N,\ 0 \leq j \leq N-2, \\
\dfrac{\Gamma(i+\theta+1)}{\Gamma(1+\theta)i!}, \qquad 0 \leq i \leq N,\ j = N-1, \\
D_{i,1,\theta,\beta}^{(1)}(0), \qquad 0 \leq i \leq N,\ j = N.
\end{cases}
$$

In our implementation, these systems have been solved using the Mathematica function FindRoot with zero initial approximation.

A special case of (5.3) occurs when $m = 2$ and $\varrho(x) = \varrho = \frac{3}{2}$, corresponding to the fractional Bagley-Torvik equation. The fractional Bagley-Torvik equation is important for modeling the motion of a thin rigid plate immersed in a Newtonian fluid.

6 Numerical examples

In this section we present three numerical examples to illustrate the accuracy and efficiency of the algorithms presented in the previous sections. The first one is introduced to test the accuracy of the formula (4.1).

Example 1. Let $u(x) = e^x$, $x \in [0, 1]$. Now we numerically calculate the VO-F derivative ${}^{C}_{0}\mathcal{D}^{\varrho(x)}_{x}u(x)$, $n - 1 \leq \varrho(x) \leq n$.

The analytical form of the VO-F derivative of $u(x)$ is given by

$$
{}^{C}_{0}\mathcal{D}^{\varrho(x)}_{x}e^x = e^x \left(1 - \frac{\Gamma(n - \varrho(x), x)}{\Gamma(n - \varrho(x))} \right).
$$

In Tables 1 and 2, we list the maximum absolute errors (AEs) between the exact solution $u(x)$ and the approximate solutions u_N at different constant- and variable-orders, respectively.

Table 1. The maximum AEs at different constant fractional orders for Example 1.

N	(θ, β)	$\varrho = 0.2$	$\varrho = 0.5$	$\varrho = 0.8$	$\varrho = 1.2$	$\varrho = 1.5$	$\varrho = 1.8$
10	(1,3)	7.93−3	1.46−2	3.36−2	1.10−1	1.78−1	3.62−1
20		1.53−5	3.45−5	9.72−5	4.10−4	8.24−4	2.07−3
40		3.07−11	8.49−11	2.93−10	1.61−9	4.00−9	1.25−8
80		4.88−15	6.21−15	5.77−15	7.32−15	1.55−14	1.37−14
10	(2,6)	2.93−6	6.04−6	1.55−5	5.76−5	1.06−4	2.48−4
20		1.09−12	2.73−12	8.64−12	4.13−11	9.46−11	2.73−10
40		1.33−15	2.66−15	2.67−15	1.77−15	3.10−15	2.66−15
80		1.33−15	2.66−15	2.67−15	1.77−15	3.10−15	2.66−15

Example 2. Consider the following VO-F Bagley-Torvik equation

$$
u''(x) + {}^{C}_{0}\mathcal{D}^{\varrho(x)}_{x}u(x) + u(x) = f(x), \qquad x \in (0, L],
$$
$$
u(0) = 0, \qquad u'(0) = 1.
$$

The exact solution is $u(x) = \sin x$.

Table 3 displays the maximum AEs of the proposed method at $\varrho(x) = \frac{3}{2}$ and $\varrho(x) = \frac{9 + \sin(x - 10)}{5}$ for different chooses of θ, β and N. Figure 1 shows the logarithmic graphs of the AEs at $\varrho(x) = \frac{9 + \sin(x - 10)}{5}$, $N = 25$ and different chooses of $\theta = \beta$.

Table 2.   The maximum AEs at different variable orders for Example 1.

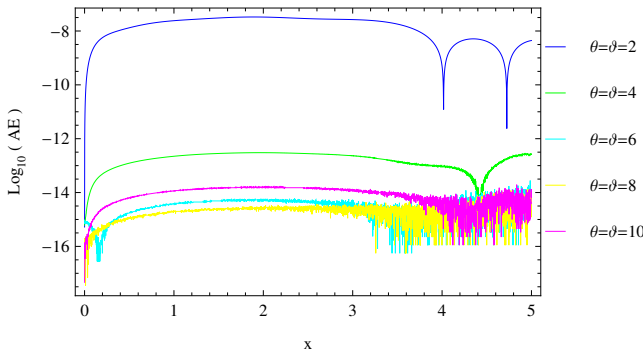| (θ, β)	N	$\varrho(x) = \frac{9+\sin t}{10}$	$\varrho(x) = \frac{3+\tanh t}{2}$
(2,4)	10	4.648×10^{-3}	1.833×10^{-2}
	20	4.556×10^{-7}	2.598×10^{-6}
	30	2.282×10^{-11}	1.625×10^{-10}
	40	5.329×10^{-15}	7.688×10^{-15}
(3,6)	10	9.862×10^{-5}	4.228×10^{-4}
	20	1.013×10^{-10}	6.287×10^{-10}
	30	3.997×10^{-15}	3.552×10^{-15}
	40	3.997×10^{-15}	3.552×10^{-15}

Figure 1.   Logarithmic graphs of the AE at $\varrho(x) = \frac{9+\sin(x-10)}{5}$, $N = 25$ and various choices of $\theta = \beta$ for Example 2.

Example 3. Consider the following VO-F Bagley-Torvik equation

$$u''(x) + {}_0^C\mathcal{D}_x^{\varrho(x)}u(x) + u(x) = \frac{\Gamma(4)}{\Gamma(4 - \varrho(x))}x^{3-\varrho(x)} + x^3 + 7x + 1, \ x \in (0, \frac{\pi}{2}],$$

$$u(0) = u'(0) = 1.$$

The exact solution is $u(x) = x^3 + x + 1$.

In this example, we consider two cases, $\varrho(x) = 1.5$ and $\varrho(x) = 1 + 0.5\,|\sin x|$, $x \in [0, \frac{\pi}{2}]$. Table 4 displays a comparison between the M- and IM-algorithms [13, 14] by means of the maximum AEs.

7   Conclusions

In this paper, an efficient three-term recurrence relation to calculate both VO-F integrals and derivatives of the modified generalized Laguerre polynomials was developed. Spectral collocation methods were developed to solve VO-F differential equations. The results of this paper expand the application of

Table 3. The maximum AEs at $L = 1$ for Example 2.

$\varrho(x)$	N	$\theta = 0, \beta = 1$	$\theta = 2, \beta = 4$	$\theta = 3, \beta = 6$
$\frac{3}{2}$	5	5.546×10^{-3}	2.916×10^{-4}	1.427×10^{-4}
	10	4.485×10^{-4}	1.431×10^{-7}	9.038×10^{-9}
	15	8.845×10^{-6}	3.675×10^{-11}	9.313×10^{-12}
	20	8.133×10^{-6}	2.166×10^{-13}	2.220×10^{-15}
$\frac{9+\sin(x-10)}{5}$	5	8.318×10^{-3}	2.666×10^{-3}	1.231×10^{-3}
	10	2.515×10^{-3}	3.854×10^{-6}	1.179×10^{-7}
	15	1.771×10^{-4}	2.721×10^{-9}	7.242×10^{-11}
	20	5.418×10^{-6}	1.0522×10^{-12}	2.742×10^{-14}

Table 4. Comparison of the maximum AEs for Example 3 with $\theta = \beta = 10$ and two cases of $\varrho(x)$, $x \in [0, \frac{\pi}{2}]$.

$\varrho(x)$		M-algorithm	IM-algorithm		Collocation method
1.5	$h=0.02$	8.16×10^{-3}	1.15×10^{-3}	$N=3$	5.77×10^{-15}
	$h=0.01$	4.27×10^{-3}	4.29×10^{-4}	$N=4$	4.57×10^{-15}
	$h=0.005$	2.18×10^{-3}	1.56×10^{-4}	$N=5$	4.44×10^{-15}
$1+0.5\,|\sin x|$	$h=0.02$	7.22×10^{-3}	1.53×10^{-4}	$N=3$	4.88×10^{-15}
	$h = 0.01$	3.70×10^{-3}	5.57×10^{-5}	$N=4$	3.10×10^{-15}
	$h=0.005$	1.86×10^{-3}	1.93×10^{-5}	$N=5$	2.77×10^{-15}

the Laguerre-Gauss collocation methods to VO-F problems. The suggested algorithms can be used also for solving VO-F fractional partial differential equations. Hence, the method is promising for VO-F differential equations. However, the analysis of the scheme is a challenging problem deserving further study.

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