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# On Singular Solutions of the Stationary Navier-Stokes System in Power Cusp Domains

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**Abstract.** The boundary value problem for the steady Navier–Stokes system is considered in a $2D$ bounded domain with the boundary having a power cusp singularity at the point $O$. The case of a boundary value with a nonzero flow rate is studied. In this case there is a source/sink in $O$ and the solution necessarily has an infinite Dirichlet integral. The formal asymptotic expansion of the solution near the singular point is constructed and the existence of a solution having this asymptotic decomposition is proved.

**Keywords:** stationary Navier-Stokes problem, power cusp domain, singular solutions, asymptotic expansion.

**AMS Subject Classification:** 35Q30; 35A20; 76M45; 76D03.

## 1 Introduction

In the paper we consider the boundary value problem for the steady Navier-Stokes system [2]

$$\begin{cases} -\nu \Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, \quad x \in \Omega, \\ \operatorname{div} \mathbf{u} = 0, \\ \mathbf{u}|_{\partial\Omega} = \mathbf{a}(x), \end{cases} \tag{1.1}$$

in a two-dimensional[1] bounded domain $\Omega = \Omega_0 \cup G_H$ (see Figure 1), where $G_H = \big\{x \in \mathbb{R}^2 : |x_1| < \gamma_0 x_2^\lambda,\, x_2 \in (0, H]\big\}$ for some $\gamma_0 > 0$ and $\lambda > 1$, $\gamma_0 =$

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[1] The proposed algorithm of constructing an asymptotic representation of a solution does not work in $3D$ case; the singularity of the solution depending on the cusp power $\lambda$ deteriorates with each new term of the asymptotic decomposition.

const, $\lambda > 1$, and $\partial\Omega \cap \partial\Omega_0$ is $C^2$. In (1.1), $\mathbf{u} = (u_1, u_2)$ stands for the velocity field, $p$ stands for the pressure and $\nu > 0$ is a coefficient of the kinematic viscosity.

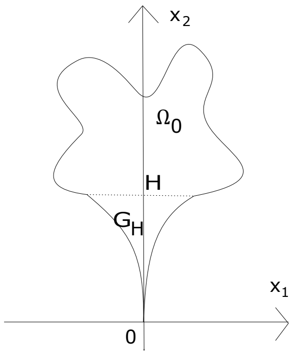


**Figure 1.** Domain $\Omega$.

We assume that the support of the boundary value $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ is separated from the cusp point $O$, i.e. $\operatorname{supp}\mathbf{a} \subset \Lambda \subset \partial\Omega_0 \cap \partial\Omega$, where $\Lambda$ is a connected set, and that

$$\int_{\Lambda} \mathbf{a}\cdot\mathbf{n}\, dS = F, \quad F \neq 0.$$

Then the velocity part $\mathbf{u}$ of the solution to problem (1.1) necessarily has the nonzero flux $-F$:

$$\int_{\sigma(R)} \mathbf{u}\cdot\mathbf{n}\, dS = -F, \tag{1.2}$$

where the interval $\sigma(R) = (-\varphi(R), \varphi(R))$ is a cross section of $G_H$ by the straight line $x_2 = R$ and we use the notation $\gamma_0 x_2^\lambda = \varphi(x_2)$. So the compatibility condition

$$\int_{\sigma(R)} \mathbf{u}\cdot\mathbf{n}\, dS + \int_{\Lambda} \mathbf{a}\cdot\mathbf{n}\, dx = 0$$

holds. Notice that a vector field $\mathbf{u}$ satisfying (1.2) necessarily has infinite Dirichlet integral $\int\limits_{\Omega} |\nabla\mathbf{u}(x)|^2 dx = \infty$. Indeed, using the Cauchy and Poincaré inequalities, we have

$$|F|^2 = \Big| \int\limits_{\sigma(x_2)} \mathbf{u}\cdot\mathbf{n}\, dS\Big|^2 \leq |\sigma(x_2)| \int\limits_{\sigma(x_2)} |\mathbf{u}|^2\, dx_1 \leq c\varphi^2(x_2) \int\limits_{\sigma(x_2)} |\nabla\mathbf{u}|^2\, dx_1.$$

From this it follows that

$$|F|^2 \int_{\varepsilon}^{H} \frac{dx_2}{\varphi^2(x_2)} \leq c \int_{\varepsilon}^{H} \int_{\sigma(x_2)} |\nabla\mathbf{u}|^2\, dx_1 dx_2.$$

The left-hand side of the last inequality tends to infinity as $\varepsilon \to 0$ and so $\int_{G_H} |\nabla\mathbf{u}|^2\, dx = \infty$.

The point source/sink approach is widely used in physics and astronomy. Fluid point sources/sinks also are commonly used in fluid and aerodynamics. Point source-sink pairs are often used as simple models for driving flow through a gap in a wall, the use of localized suction to control vortices around aerofoil sections also is one of such problems. In oceanography, it is common to use point sources to model the influx of fluid from channels and holes. There are also many others applications of point source/sink models.

The asymptotic behaviour of solutions to the Stokes and Navier–Stokes equations in singularly perturbed domains became of growing interest during the last fifty years. There is an extensive literature concerning these issues for various elliptic problems, e.g., [2, 5, 16, 17, 18, 19, 20, 21, 22, 23, 24, 26, 27]. In particular, the steady Navier–Stokes equations are studied in a punctured domain $\Omega = \Omega_0 \setminus \{O\}$ with $O \in \Omega_0$ assuming that the point $O$ is a sink or source of the fluid [10, 30, 31] (see also [11] for the review of these results). We also mention the papers [7, 8, 9] where the existence of a solution (with an infinite Dirichlet integral) to the Navier–Stokes problem with a sink or source in the cusp point $O$ was proved for arbitrary data and the papers [12, 13, 14] where the asymptotics of a solution to the nonstationary Stokes problem is studied in domains with conical points and conical outlets to infinity.

In recent papers the authors have studied existence of singular solutions to the time-periodic and initial boundary value problems for the linear Stokes equations [3, 4] and an initial boundary value problem for the Navier–Stokes equations [28, 29] in domains having a power-cusp (peak type) singular point on the boundary. The case when the flux of the boundary value is nonzero was considered. Therefore, there is a sink or source in the cusp point $O$ and a solution is necessarily singular. In constructing the formal asymptotic decomposition of the solution the considered problems with singular data were reduced to ones with regular right-hand sides and then the well known methods of proving the solvability were applied. In constructing the asymptotic representation we followed the ideas proposed in the paper [25] where the asymptotic behaviour of solutions to the stationary Stokes and Navier–Stokes problems was studied in unbounded domains with paraboloidal outlets to infinity. In turn, the method used in [25] was a variant of the algorithm of constructing the asymptotic representation of solutions to elliptic equations in slender domains (see [16, 20] for arbitrary elliptic problems; [21, 24] for the stationary Stokes and Navier–Stokes equations).

In [8] the existence of a generic stationary solution with infinite Dirichlet integral is proved. So, the behaviour of this solution near the cusp point is not specified (and in general is not known). In this paper we construct the solution $\mathbf{u}$ to problem (1.1) which has the special structure: it is represented as a sum of the formal asymptotics near the cusp point and a vector field belonging to a suitable weighted second order Sobolev space $V^{2,2}(\Omega)$ (its precise definition will be given in Section 3.1). Since the constructed formal asymptotic decomposition has an explicit form and it is regular in $\Omega \setminus O$, the solution is understood almost everywhere in $\Omega$. Thus, we have proved the following result[2].

[2] For more details see Theorem 4.1.

**Theorem.** *Let* $\mathbf{f} \in L^2(\Omega)$, $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$. *There exist a constant* $\kappa_o > 0$ *such that if*

$$\|\mathbf{f}\|^2_{L^2(\Omega)} + \|\mathbf{a}\|^2_{W^{3/2,2}(\partial\Omega)} + \|\mathbf{a}\|^4_{W^{3/2,2}(\partial\Omega)} \le \kappa_o,$$

*then problem* (1.1) *has a unique solution* $(\mathbf{u}, p)$ *admitting the representation* $\mathbf{u} = \mathbf{V} + \mathbf{v}, \quad p = q + Q$, *where the pair* $(\mathbf{V}, Q)$ *coincides near the cusp point* $O$ *with the formal asymptotic decomposition of the solution,* $\mathbf{v} \in V^{2,2}(\Omega)$, $\nabla q \in L^2(\Omega)$ *and the following estimate*

$$\begin{aligned}&\|\mathbf{v}\|^2_{V^{2,2}(\Omega)} + \|\nabla q\|^2_{L^2(\Omega)} \\ &\qquad \le c\big(\|\mathbf{f}\|^2_{L^2(\Omega)} + \|\mathbf{a}\|^2_{W^{3/2,2}(\partial\Omega)} + \|\mathbf{a}\|^4_{W^{3/2,2}(\partial\Omega)}\big)\end{aligned}$$

*holds with the constant* $c$ *depending only on* $\lambda$, $\gamma_0$ *and the geometry of* $\Omega_0$.

Let $D$ be a bounded domain in $\mathbb{R}^n$. In this article, we use usual notations of functional spaces (e.g., [1]). By $L^p(D)$ and $W^{m,p}(D)$, $1 \le p < \infty$, we denote the usual Lebesgue and Sobolev spaces, respectively. The norms in $L^p(D)$ and $W^{m,p}(D)$ are indicated by $\|\cdot\|_{L^p(D)}$ and $\|\cdot\|_{W^{m,p}(D)}$. We denote by $C^\infty(D)$ the set of all infinitely differentiable functions defined on $D$ and by $C_0^\infty(D)$ the subset of all functions from $C^\infty(D)$ with compact supports in $\Omega$. By $\mathring{W}^{1,2}(D)$ we denote the completion of $C_0^\infty(D)$ in the $\|\cdot\|_{W^{1,2}}$ norm.

## 2 Steady Navier-Stokes problem. Formal asymptotic decomposition

Let us consider a solution $(\mathbf{u}, p)$ of problem (1.1) in a neighbourhood of the cuspidal point, i.e. in the domain $G_H$. Recall that $\mathbf{u}|_{\partial G_H \cap \partial\Omega} = 0$. Changing the variables $(x_1, x_2) \longrightarrow \left(\frac{x_1}{x_2^\lambda}, x_2\right) := (y_1, y_2)$ we rewrite the problem (1.1) in the following form:[3]

$$\begin{cases} -\nu(y_2^{-2\lambda}\partial_1^2 + \mathfrak{D}^2)u_1 + (\mathbf{u}\cdot\mathfrak{N})u_1 + y_2^{-\lambda}\partial_1 p = 0, \ y \in \Pi, \\ -\nu(y_2^{-2\lambda}\partial_1^2 + \mathfrak{D}^2)u_2 + (\mathbf{u}\cdot\mathfrak{N})u_2 + \mathfrak{D}p = 0, \ y \in \Pi, \\ y_2^{-\lambda}\partial_1 u_1 + \mathfrak{D}u_2 = 0, \\ \mathbf{u}|_{\partial\Pi} = 0, \end{cases} \tag{2.1}$$

where $\Pi = \big\{y \in \mathbb{R}^2 : |y_1| < \gamma_0, \ y_2 \in (0, H)\big\}$, $\partial_k = \dfrac{\partial}{\partial y_k}$, $k = 1, 2$, $\mathfrak{D} = \partial_2 - \lambda y_2^{-1} y_1 \cdot \partial_1$, $\quad \mathfrak{N} = \begin{pmatrix} y_2^{-\lambda}\cdot\partial_1 \\ \mathfrak{D}\end{pmatrix}$.

### 2.1 The leading term of the asymptotic expansion

The main term of the asymptotic expansion of the solution $(\mathbf{u}, p)$ is the same as that for the linear (Stokes) problem (see [3]): the leading term $\mathbf{U}_{\mu_0} =$

[3] Note that the algorithms used below are similar to those used in [25] for constructing the asymptotics of solutions to the stationary Navier–Stokes problem in unbounded domains with paraboloidal outlets to infinity.

$(U_{1,\mu_0}, U_{2,\mu_0})$, $P_{\mu_0}$ is

$$U_{1,\mu_0}(y_1,y_2) = y_2^{\mu_0+3\lambda-2}\mathcal{U}_{1,\mu_0}(y_1), \quad U_{2,\mu_0}(y_1,y_2) = \frac{F}{\kappa_0} y_2^{\mu_0+2\lambda-1}\Phi(y_1),$$
$$P_{\mu_0}(y_1,y_2) = \frac{F}{\kappa_0\mu_0} y_2^{\mu_0} + y_2^{\mu_0+2\lambda-2}\mathcal{Q}_{\mu_0}(y_1),$$

where

$$\mu_0 = 1 - 3\lambda, \tag{2.2}$$

the function $\Phi$ is the solution to

$$\begin{cases} \nu\partial_1^2\Phi(y_1) = 1, & |y_1| < \gamma_0, \\ \Phi(y_1)|_{\partial\omega} = 0, & |y_1| = \gamma_0, \end{cases} \tag{2.3}$$

i.e.,

$$\Phi(y_1) = \frac{1}{2\nu}\left(|y_1|^2 - \gamma_0^2\right), \tag{2.4}$$

where $\omega = \{y_1 \in \mathbb{R} : |y_1| < \gamma_0\}$; the constant $\kappa_0$ is given by

$$\kappa_0 = \int_{-\gamma_0}^{\gamma_0} \Phi(y_1)\, dy_1 = -\frac{2}{3\nu}\gamma_0^3 < 0 \tag{2.5}$$

and $(\mathcal{U}_{1,\mu_0}, \mathcal{Q}_{\mu_0})$ is the solution of

$$\begin{cases} -\nu\partial_1^2\mathcal{U}_{1,\mu_0}(y_1) + \partial_1\mathcal{Q}_{\mu_0}(y_1) = 0, & |y_1| < \gamma_0, \\ \partial_1\mathcal{U}_{1,\mu_0}(y_1) = \mathcal{G}_0(y_1), \\ \mathcal{U}_{1,\mu_0}(y_1)|_{|y_1|=\gamma_0} = 0, \end{cases} \tag{2.6}$$

with $\mathcal{G}_0(y_1) = \lambda\kappa_0^{-1}F(1+y_1\cdot\partial_1)\Phi(y_1)$. Moreover, by construction, the following compatibility condition for problem (2.6)

$$\int_{-\gamma_0}^{\gamma_0} \mathcal{G}_0(y_1)\, dy_1 = 0$$

is valid. The system of ordinary differential equations (2.6) is of Stokes type, and obviously, it has a unique (up to the additive constant in the pressure component $\mathcal{Q}_{\mu_0}$) smooth solution.

Functions $(\mathbf{U}_{\mu_0}, Q_{\mu_0})$ leave in equations $(2.1)_1$, $(2.1)_2$ the discrepancies $H_{1,\mu_0}(y_1,y_2)$, $H_{2,\mu_0}(y_1,y_2)$:

$$\begin{aligned} H_{1,\mu_0}(y_1,y_2) &= \nu\mathfrak{D}^2U_{1,\mu_0}(y_1,y_2) - (\mathbf{U}_{\mu_0}\cdot\mathfrak{N})U_{1,\mu_0}(y_1,y_2) \\ &= y_2^{\mu_0+3\lambda-4}\mathcal{F}_{1,\mu_0}(y_1) + y_2^{\mu_0+2\lambda-3}\mathcal{N}_{1,\mu_0}(y_1), \\ H_{2,\mu_0}(y_1,y_2) &= \nu\mathfrak{D}^2U_{2,\mu_0}(y_1,y_2) - (\mathbf{U}_{\mu_0}\cdot\mathfrak{N})U_{2,\mu_0}(y_1,y_2) \\ &- \mathfrak{D}\left(y_2^{\mu_0+2\lambda-2}\mathcal{Q}_{\mu_0}(y_1)\right) = y_2^{\mu_0+2\lambda-3}\mathcal{F}_{2,\mu_0}(y_1) + y_2^{\mu_0+\lambda-2}\mathcal{N}_{2,\mu_0}(y_1). \end{aligned} \tag{2.7}$$

Discrepancies (2.7) are represented as the sums containing the terms denoted by $\mathcal{F}$ plus the terms denoted by $\mathcal{N}$, where $\mathcal{N}$ are discrepancies arising from the nonlinear term in equation $(1.1)_1$, while $\mathcal{F}$ are discrepancies arising from the linear part of the same equation $(1.1)_1$. Note, that the latter discrepancies are the same as that for the Stokes problem (see [3]).

### 2.2 Formal asymptotic decomposition

Now we construct higher-order terms of the asymptotic expansion. Consider Equations (2.1)

$$\begin{cases} -\nu(y_2^{-2\lambda}\partial_1^2+\mathfrak{D}^2)u_1+(\mathbf{u}\cdot\mathfrak{N})u_1+y_2^{-\lambda}\partial_1 p=Z_1(\varphi_1,\varphi_2),\\ -\nu(y_2^{-2\lambda}\partial_1^2+\mathfrak{D}^2)u_2+(\mathbf{u}\cdot\mathfrak{N})u_2+\mathfrak{D}p=Z_2(\varphi_1,\varphi_2),\\ y_2^{-\lambda}\partial_1 u_1+\mathfrak{D}u_2=0,\\ \mathbf{u}|_{\partial\Pi}=0, \end{cases} \tag{2.8}$$

with the right-hand sides $(Z_1(\varphi_1,\varphi_2),Z_2(\varphi_1,\varphi_2))$ having the form of one of the following expressions

$$(Z_1(\varphi_1,\varphi_2),Z_2(\varphi_1,\varphi_2))=\begin{cases} \left(\nu\mathfrak{D}^2\varphi_1,\nu\mathfrak{D}^2\varphi_2-\mathfrak{D}p_\varphi\right),\\ \text{or}\\ -\left((\boldsymbol{\varphi}\cdot\mathfrak{N})\varphi_1,(\boldsymbol{\varphi}\cdot\mathfrak{N})\varphi_2\right), \end{cases} \tag{2.9}$$

where the functions $\boldsymbol{\varphi}=(\varphi_1,\varphi_2)$ and $p_\varphi$ are specified below.

In order to prove the existence of the solution to problem (1.1), we first construct the formal asymptotic decomposition of it such that discrepancies belong to $L^2$-space. Obviously, this is not true for the discrepancies left by the leading asymptotic term. Therefore, we have to compensate the appeared singular terms in the corresponding to the leading term discrepancies (2.7), i.e., we have to construct functions $(\mathbf{U}_\mu,Q_\mu)$ which satisfy equations (2.8) with the right-hand sides $(Z_1(U_{1,\mu_0},U_{2,\mu_0}),Z_2(U_{1,\mu_0},U_{2,\mu_0}))$ coming from (2.7). Functions $(\mathbf{U}_\mu,Q_\mu)$ leave some new discrepancies which may also be singular. So, we have to compensate them by the same manner and we continue this process until the discrepancies belong to $L^2$-space.

Note that at each step of the construction we obtain the same equations with the right-hand sides having similar structure. In order to construct more asymptotical terms, we should be able to define which part of the discrepancies we are compensating at each step. In other words, we have to describe a rule for the power exponents of the discrepancies, in such a way that we can find which terms are most singular.

Consider first the terms related to the linear part of the Navier–Stokes equations. The linear Stokes problem was investigated by authors earlier [3] and it was shown that the most singular part of the discrepancies left by the anzatz $(\mathbf{U}_\mu,Q_\mu)$ have to be compensated by the anzats $(\mathbf{U}_{\mu+2\lambda-2},Q_{\mu+2\lambda-2})$. Therefore, the rule $M_L$ describing the changes in power exponents of terms arising from the linear part is defined by

$$M_L:\mu\in M\Rightarrow\mu+2\lambda-2\in M. \tag{2.10}$$

Discrepancies contain also terms arising from the nonlinearity of the equations. As it is shown in the calculations below, the rule $M_N$ describing the changes in power exponents of terms arising from the nonlinear part is

$$M_N:\mu_1,\mu_2\in M\Rightarrow\mu_1+\mu_2+4\lambda-2\in M, \tag{2.11}$$

(recall that the nonlinear terms contain products of the functions $\mathbf{U}_{\mu_1}$ and $\mathbf{U}_{\mu_2}$).

The following lemma describes the narrowest set of numbers $M$ obeying both rules $M_L$ and $M_N$.

**Lemma 2.1.**

$$M = \{1 - 3\lambda + k(\lambda - 1) : k = 0, 1, \dots\}.$$

*Proof.* The elements of the set $M$ obey the rules (2.10) and (2.11) with the starting point $\mu_0 = 1 - 3\lambda \in M$ (see (2.2)). Taking $\mu_1 = \mu_2 = \mu_0$, from (2.11) follows the relation

$$\mu_0 + \mu_0 + 4\lambda - 2 = \mu_0 + \lambda - 1 = (1 - 3\lambda) + \lambda - 1.$$

If $\mu_1 = 1 - 3\lambda + k(\lambda - 1)$, $\mu_2 = 1 - 3\lambda + j(\lambda - 1)$, then (2.10), (2.11) yield

$$\begin{aligned} &\mu_1 + 2\lambda - 2 = 1 - 3\lambda + (k+2)(\lambda - 1), \\ &\mu_1 + \mu_2 + 4\lambda - 2 = 1 - 3\lambda + (k + j + 1)(\lambda - 1). \end{aligned}$$

□

Let

$$\begin{aligned} &\varphi_{1,\mu}(y_1, y_2) = y_2^{\mu+3\lambda-2}\mathcal{U}_{1,\mu}(y_1), \quad \varphi_{2,\mu}(y_1, y_2) = y_2^{\mu+2\lambda-1}\mathcal{U}_{n,\mu}(y_1), \\ &p_{\varphi,\mu}(y_1, y_2) = y_2^{\mu} C_\mu + y_2^{\mu+2\lambda-2}\mathcal{Q}_\mu(y_1), \end{aligned}$$

$C_\mu = \text{const}$, $i = 1, 2$ and $\mu$ belongs to the set $M$. Substituting these expressions into (2.9), we obtain

$$\begin{pmatrix} Z_1(\varphi_{1,\mu}, \varphi_{2,\mu}) \\ Z_2(\varphi_{1,\mu}, \varphi_{2,\mu}) \end{pmatrix} = \begin{pmatrix} \nu\mathfrak{D}^2\varphi_{1,\mu} \\ \nu\mathfrak{D}^2\varphi_{2,\mu} - \mathfrak{D}p_{\varphi,\mu} \end{pmatrix} \sim \begin{pmatrix} y_2^{\mu+3\lambda-4}\mathcal{F}_{1,\mu}(y_1) \\ y_2^{\mu+2\lambda-3}\mathcal{F}_{2,\mu}(y_1) \end{pmatrix}$$

in the case when the linear term is the most singular one (the rule $M_L$), and

$$\begin{pmatrix} Z_1(\varphi_{1,\mu_1}, \varphi_{2,\mu_2}) \\ Z_2(\varphi_{1,\mu_1}, \varphi_{2,\mu_2}) \end{pmatrix} = \begin{pmatrix} -(\boldsymbol{\varphi}_{\mu_1} \cdot \mathfrak{N})\varphi_{1,\mu_2} \\ -(\boldsymbol{\varphi}_{\mu_1} \cdot \mathfrak{N})\varphi_{2,\mu_2} \end{pmatrix} \sim \begin{pmatrix} y_2^{\mu_1+\mu_2+5\lambda-4}\mathcal{N}_{1,\mu}(y_1) \\ y_2^{\mu_1+\mu_2+4\lambda-3}\mathcal{N}_{2,\mu}(y_1) \end{pmatrix}$$

in the case when the most singular term in the right-hand side is produced by the nonlinearity (the rule $M_N$).

In the case when power exponents obtained by rules $M_L$ and $M_N$ are the same, we compensate the sum of all terms containing these exponents.

Suppose that the approximate solution $(\mathbf{U}^{[M]}, P^{[M]})$ is represented in the form[4]

$$\begin{aligned} &U_1^{[M]}(y_1, y_2) = \sum_{\mu\in M} y_2^{\mu+3\lambda-2}\mathcal{U}_{1,\mu}(y_1), \quad U_2^{[M]}(y_1, y_2) = \sum_{\mu\in M} y_2^{\mu+2\lambda-1}\mathcal{U}_{2,\mu}(y_1), \\ &P^{[M]}(y_1, y_2) = \sum_{\mu\in M} y_2^{\mu} C_\mu + y_2^{\mu+2\lambda-2}\mathcal{Q}_\mu(y_1), \end{aligned} \tag{2.12}$$

[4] In the paper we use the notation $\mathbf{U}(y_1, y_2) = y_2^{\gamma}\mathcal{U}(y_1)$, i.e. functions marked as calligraphic letters depend only on $y_1$.

where $M$ is the set of indices described in Lemma 2.1; the pair of functions $(\mathcal{U}_{1,\mu}, \mathcal{Q}_\mu)$ is the solution of

$$\begin{cases} -\nu\partial_1^2\mathcal{U}_{1,\mu}(y_1) + \partial_1\mathcal{Q}_\mu(y_1) = Z_1(\mathcal{U}_{1,\bar\mu}, \mathcal{U}_{2,\bar\mu}), \quad |y_1| < \gamma_0, \\ \partial_1\mathcal{U}_{1,\mu}(y_1) = -A(\mu)\mathcal{U}_{2,\mu}(y_1), \\ \mathcal{U}_{1,\mu}(y_1)|_{|y_1|=\gamma_0} = 0, \end{cases} \tag{2.13}$$

where $\mu$, $\bar\mu \in M$,[5]

$$A(\mu) = \mu + 2\lambda - 1 - \lambda y_1\partial_1, \quad \mathcal{U}_{2,\mu}(y_1) = C_\mu\mu\varPhi(y_1) + \mathcal{U}^*_{2,\mu}(y_1), \tag{2.14}$$

the function $\varPhi$ is the solution of problem (2.3) (see (2.4)), the functions $\mathcal{U}^*_{2,\mu}$ satisfy the equations

$$\begin{cases} -\nu\partial_1^2\mathcal{U}^*_{2,\mu}(y_1) = Z_2(\mathcal{U}_{1,\bar\mu}, \mathcal{U}_{2,\bar\mu}), \quad |y_1| < \gamma_0,\ \mu,\ \bar\mu \in M, \\ \mathcal{U}^*_{2,\mu}|_{|y_1|=\gamma_0}(y_1) = 0. \end{cases}$$

The constants $C_\mu$ are uniquely determined from the following solvability condition for problem (2.13)

$$\int_{-\gamma_0}^{\gamma_0} A(\mu)\mathcal{U}_{2,\mu}(y_1)\, dy_1 = 0. \tag{2.15}$$

Indeed, using (2.5) and the equality

$$\int_{-\gamma_0}^{\gamma_0} y_1 \cdot \partial_1\varPhi(y_1)\, dy_1 = -\int_{-\gamma_0}^{\gamma_0} \varPhi(y_1)\, dy_1 = -\kappa_0,$$

we rewrite (2.15) as follows

$$C_\mu\mu\kappa_0(\mu + 3\lambda - 1) = -\int_{-\gamma_0}^{\gamma_0} A(\mu)\mathcal{U}^*_{2,\mu}(y_1)\, dy_1.$$

Thus, if $\mu \neq 0$ and $\mu \neq \mu_0$,

$$C_\mu = -\frac{1}{\mu\kappa_0}\int_{-\gamma_0}^{\gamma_0} \mathcal{U}^*_{2,\mu}(y_1)\, dy_1.$$

If $\mu = \mu_0$, then $C_{\mu_0} = F/(\mu_0\kappa_0)$ (see Section 2.1 and [3]).

Finally, from (2.12) and Lemma 2.1 we get

$$\begin{aligned} U_1^{[J]}(y_1, y_2) = \quad & y_2^{-1}\mathcal{U}_{1,0}(y_1) + \sum_{k=1}^J y_2^{-1+k(\lambda-1)}\mathcal{U}_{1,k}(y_1), \\ U_2^{[J]}(y_1, y_2) = \quad & \frac{F}{\kappa_0}y_2^{-\lambda}\varPhi(y_1) + \sum_{k=1}^J y_2^{-\lambda+k(\lambda-1)}\mathcal{U}_{2,k}(y_1), \\ P^{[J]}(y_1, y_2) = \quad & \frac{F}{\kappa_0(1-3\lambda)}y_2^{1-3\lambda} + y_2^{-1-\lambda}\mathcal{Q}_0(y_1) \\ & + \sum_{k=1}^J y_2^{1-3\lambda+k(\lambda-1)}C_k + y_2^{-1-\lambda+k(\lambda-1)}\mathcal{Q}_k(y_1), \end{aligned}$$

[5] Numbers $\mu$, $\bar\mu$ are different.

where $\mathbf{U}_0 = \mathbf{U}_{\mu_0}$, the pair $(\mathcal{U}_{1,0}, \mathcal{Q}_0)$ solves problem (2.6), the function $\Phi$ is solution to (2.3) and is described by (2.4), the pair $(\mathcal{U}_{1,k}, \mathcal{Q}_k)$ solves the problem

$$\begin{cases} -\nu\partial_1^2\mathcal{U}_{1,k}(y_1) + \partial_1\mathcal{Q}_k(y_1) = \mathcal{Z}_{1,k}(y_1), \quad |y_1| < \gamma_0, \\ \partial_1\mathcal{U}_{1,k}(y_1) = -A(1-3\lambda+k(\lambda-1))\mathcal{U}_{2,k}(y_1), \\ \mathcal{U}_{1,k}(y_1)|_{|y_1|=\gamma_0} = 0, \end{cases} \tag{2.16}$$

where $A$ is given by (2.14),

$$\mathcal{U}_{2,k}(y_1) = C_k(1-3\lambda+k(\lambda-1))\Phi(y_1) + \mathcal{U}^*_{2,k}(y_1),$$

the functions $\mathcal{U}^*_{2,k}$ satisfy the equations

$$\begin{cases} -\nu\partial_1^2\mathcal{U}^*_{2,k}(y_1) = \mathcal{Z}_{2,k}(y_1), \quad |y_1| < \gamma_0, \\ \mathcal{U}^*_{2,k}(y_1)|_{|y_1|=\gamma_0} = 0, \end{cases} \tag{2.17}$$

where $\mathbf{Z}_k(y_1) = (\mathcal{Z}_{1,k}(y_1), \mathcal{Z}_{2,k}(y_1))$ is specified below. The constants $C_k$ are uniquely determined from the following solvability condition for problem (2.16):

$$\int_{-\gamma_0}^{\gamma_0} A(1-3\lambda+k(\lambda-1))\mathcal{U}_{2,k}(y_1)\,dy_1 = 0.$$

Then, similarly as before, we get

$$C_k = -\frac{1}{\kappa_0(1-3\lambda+k(\lambda-1))}\int_{-\gamma_0}^{\gamma_0}\mathcal{U}^*_{2,k}(y_1)\,dy_1,$$

$k = 1, 2, \ldots$. Moreover, $C_0 = \frac{F}{\kappa_0(1-3\lambda)}$, (see Section 2.1 or [3]). Note that $1-3\lambda+k(\lambda-1) \neq 0$ due to the assumption that $\mu \neq 0$.

The pair of functions $(\mathbf{U}_k, Q_k)$ leaves in Equations (2.8) the discrepancies $\mathbf{H}_k = (H_{1,k}, H_{2,k})$,

$$\mathbf{H}_k(y_1, y_2) = \mathbf{F}_{k-1}(y_1, y_2) + \mathbf{F}_k(y_1, y_2) + \mathbf{N}_k(y_1, y_2), \tag{2.18}$$

where $k = 0, 1, \ldots$, $\mathbf{F}_{-1}(y_1, y_2) = 0$,

$$\begin{aligned} \mathbf{F}_k(y_1, y_2) &= \left(y_2^{-3+k(\lambda-1)}\mathcal{F}_{1,k}(y_1), y_2^{-\lambda-2+k(\lambda-1)}\mathcal{F}_{2,k}(y_1)\right), \\ \mathbf{N}_k(y_1, y_2) &= \left(y_2^{-\lambda-2+k(\lambda-1)}\mathcal{N}_{1,k}(y_1), y_2^{-2\lambda-1+k(\lambda-1)}\mathcal{N}_{2,k}(y_1)\right), \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} \mathcal{F}_{1,k}(y_1) &= y_2^{-(-3+k(\lambda-1))}\nu\mathfrak{D}^2 U_{1,k}(y_1, y_2), \\ \mathcal{F}_{2,k}(y_1) &= y_2^{-(-\lambda-2+k(\lambda-1))}\left[\nu\mathfrak{D}^2 U_{2,k}(y_1, y_2) - \mathfrak{D}\left(y_2^{-\lambda-1+k(\lambda-1)}\mathcal{Q}_k(y_1)\right)\right], \\ \mathcal{N}_{1,k}(y_1) &= -y_2^{-(-\lambda-2+k(\lambda-1))}\sum_{i+j=k}(\mathbf{U}_i\cdot\mathfrak{N})U_{1,j}(y_1, y_2), \\ \mathcal{N}_{2,k}(y_1) &= -y_2^{-(-2\lambda-1+k(\lambda-1))}\sum_{i+j=k}(\mathbf{U}_i\cdot\mathfrak{N})U_{2,j}(y_1, y_2), \end{aligned}$$

$k$, $i$, $j = 0, 1, 2, \ldots$, and $\mathbf{N}_k = (\mathcal{N}_{1,k}, \mathcal{N}_{2,k})$, $\boldsymbol{\mathfrak{F}}_k = (\boldsymbol{\mathfrak{F}}_{k,1}, \boldsymbol{\mathfrak{F}}_{k,2})$.

The functions $\mathbf{Z}_k = (\mathcal{Z}_{1,k}, \mathcal{Z}_{2,k})$ contain the most singular terms which we compensate at step $k = 1, 2, \ldots,$. One can deduce that the function $\mathbf{Z}_k$ is described by the following rule

$$\boldsymbol{\mathcal{N}}_0 \to \boldsymbol{\mathcal{F}}_0 + \boldsymbol{\mathcal{N}}_1 \to \boldsymbol{\mathcal{F}}_1 + \boldsymbol{\mathcal{N}}_2 \to \cdots \to \boldsymbol{\mathcal{F}}_{j-1} + \boldsymbol{\mathcal{N}}_j \to \ldots$$

$j = 0, 1, 2, \ldots$, i.e., the functions $(\mathcal{U}_{1,1}, \mathcal{Q}_1)$, $\mathcal{U}^*_{2,1}$ solve problems (2.16)–(2.17) with the right-hand side $\boldsymbol{\mathcal{N}}_0$; the functions $(\mathcal{U}_{1,2}, \mathcal{Q}_2)$, $\mathcal{U}^*_{2,2}$ solve problems (2.16)–(2.17) with the right-hand side $\boldsymbol{\mathcal{F}}_0 + \boldsymbol{\mathcal{N}}_1$ and so on.

Above we have supposed that $\mu \neq 0$. If $\mu = 0$, i.e. $1 - 3\lambda + \bar{k}(\lambda - 1) = 0$, we look for $(\mathbf{U}_{\bar{k}}, P_{\bar{k}})$ in the form

$$\mathbf{U}_{\bar{k}}(y_1, y_2) = \mathbf{U}_k(y_1, y_2), \quad P_{\bar{k}}(y_1, y_2) = C_{\bar{k}} \ln\, y_2 + y_2^{-1-\lambda+\bar{k}(\lambda-1)} \mathcal{Q}_{\bar{k}}(y_1).$$

For $\mathcal{U}_{2,\bar{k}}(y_1) = C_{\bar{k}} \Phi(y_1) + \mathcal{U}^*_{2,\bar{k}}(y_1)$ and $(\boldsymbol{\mathcal{U}}_{1,\bar{k}}(y_1), \mathcal{Q}_{\bar{k}}(y_1))$ we get the same Equations (2.16)–(2.17); the solvability condition for problem (2.16) is changed into

$$C_{\bar{k}} \bar{k}(\lambda - 1)\kappa_0 = -\int_{-\gamma_0}^{\gamma_0} A(1 - 3\lambda + k(\lambda - 1))\mathcal{U}^*_{2,\bar{k}}(y_1)dy_1.$$

Finally, we get the following expression for $\mathbf{U}^{[J]}, P^{[J]}$:

$$\begin{aligned}
&\mathbf{U}_1^{[J]}(y_1, y_2) = y_2^{-1}\mathcal{U}_{1,0}(y_1) + \sum_{k=1}^{J} y_2^{-1+k(\lambda-1)} \boldsymbol{\mathcal{U}}_{1,k}(y_1),\\
&U_2^{[J]}(y_1, y_2) = \tfrac{F}{\kappa_0} y_2^{-\lambda}\Phi(y_1) + \sum_{k=1}^{J} y_2^{-\lambda+k(\lambda-1)}\mathcal{U}_{2,k}(y_1),\\
&P^{[J]}(y_1, y_2) = \frac{F}{\kappa_0(1-3\lambda)} y_2^{1-3\lambda} + y_2^{-1-\lambda}\mathcal{Q}_0(y_1)\\
&+ \sum_{k=1}^{J} \big[C_k y_2^{1-3\lambda+k(\lambda-1)}[1 + \delta_k^{\bar{k}}(y_2^{-1+3\lambda-k(\lambda-1)} \ln y_2 - 1)]\\
&+ y_2^{-1-\lambda+k(\lambda-1)}\mathcal{Q}_k(y_1)\big],
\end{aligned}$$

where $J \in \mathbb{N}$. All other steps of the construction remain the same as for the case $\mu \neq 0$.

### 2.3 Estimates of the asymptotic decomposition and discrepancies

All boundary value problems for ordinary differential equations which appear by constructing the asymptotic decomposition have smooth solutions and it is straightforward to see that there hold the estimates

$$\Big|\frac{\partial^l U_1^{[J]}(y_1, y_2)}{\partial y_1^l}\Big| \le c\frac{|F|}{y_2^{1+l}}, \quad \Big|\frac{\partial^l U_2^{[J]}(y_1, y_2)}{\partial y_1^l}\Big| \le c\frac{|F|}{y_2^{\lambda+l}}, \quad l = 0, 1, \ldots. \qquad (2.20)$$

From (2.18)–(2.19) it follows that the discrepancy $\mathbf{H}^{[J]}$ left by asymptotic decomposition $\big(\mathbf{U}^{[J]}, P^{[J]}\big)$ in Equations (1.1) admits the estimate

$$\big|\mathbf{H}^{[J]}(y_1, y_2)\big| \le c\big(|F| + |F|^2\big)/y_2^{2\lambda+1-J(\lambda-1)}.$$

Therefore, if $J > (4\lambda + 1)/\big(2(\lambda - 1)\big)$, then $\mathbf{H}^{[J]} \in L^2(G_H)$ and

$$\|\mathbf{H}^{[J]}\|_{L^2(G_H)} \le c\Big(|F| + |F|^2\Big).$$

# 3 Preliminary results

## 3.1 Some notation

Let us consider the cusp domain $\Omega$. Let $h_0 = H, h_k = h_{k-1} - \frac{\varphi(h_{k-1})}{2L}$, $k = 1, 2, \dots$, and $L$ be a Lipschitz constant of the function $\varphi(x_2)$ in the domain $G_H$. Recall that $\varphi(x_2) = \gamma_0 x_2^\lambda$. The sequence $\{h_k\}$ is decreasing and bounded from below. Assume that the limit of this sequence is $a_0 \neq 0$. From the definition it follows that $a_0 = a_0 - \frac{\varphi(a_0)}{2L}$. Then $\varphi(a_0) = 0$. However, $\varphi(a_0) \neq 0$ for $a_0 \neq 0$ and, hence the limit $a_0 = 0$. Since the sequence is decreasing and the limit is equal to 0, all its elements are positive.

Denote $\omega_l = \{x \in \mathbb{R}^2 : |x_1| < \varphi(x_2), x_2 \in (h_l, h_{l-1})\}$, $l = 1, \dots$. Note that

$$\tfrac{1}{2}\varphi(h_l) \leq \varphi(t) \leq \tfrac{3}{2}\varphi(h_l), \quad t \in [h_{l+1}, h_l]. \tag{3.1}$$

Define the transformation $y = \mathcal{P}_l x$ by the formulas

$$y_1 = \frac{2Lx_1}{\varphi(x_2)}, \quad y_2 = \frac{2L(x_2 - h_{l-1})}{\varphi(h_{l-1})}$$

and the domain

$$G_0 = \{y \in \mathbb{R}^2 : |y_1| < 2L, -1 < y_2 < 0\}.$$

Obviously the transformation $\mathcal{P}_l^{-1}$ maps $G_0$ onto $\omega_l$. It is easy to see that

$$\Omega = \Omega_0 \cup \Big(\bigcup_{l=1}^{\infty} \omega_l\Big).$$

Denote

$$\Omega_l^\sharp = \Omega_0 \cup \Big(\bigcup_{l=1}^{l} \omega_j\Big), \quad l = 1, 2, \dots.$$

Below we also will need the space $V^{2,2}(\Omega)$ consisting of functions having the finite norm

$$\|u\|^2_{V^{2,2}(\Omega)} = \|u\|^2_{W^{2,2}(\Omega_0)} + \int\limits_{\Omega\setminus\Omega_0} \varphi^{-4}(x_2)|u|^2 dx + \int\limits_{\Omega\setminus\Omega_0} \varphi^{-2}(x_2)|\nabla u|^2 dx + \int_{\Omega\setminus\Omega_0} |\nabla^2 u|^2 dx.$$

## 3.2 Two inequalities

**Lemma 3.1.** *(Poincaré type inequality). Let $u \in W^{1,2}_{loc}(\bar\Omega \setminus O)$, $u|_{\partial\Omega} = 0$ and $\kappa \in \mathbb{R}$ is arbitrary. If $\int\limits_{G_H} |\varphi(x_2)|^\kappa |\nabla u(x)|^2 dx < \infty$, then the integral $\int\limits_{G_H} |\varphi(x_2)|^{\kappa-2}|u(x)|^2 dx$ is finite and the following inequality*

$$\int\limits_0^H \int\limits_{-\varphi(x_2)}^{\varphi(x_2)} |\varphi(x_2)|^{\kappa-2}|u(x)|^2 dx \leq \tfrac{4}{\pi^2} \int\limits_0^H \int\limits_{-\varphi(x_2)}^{\varphi(x_2)} |\varphi(x_2)|^{\kappa}|\nabla u(x)|^2 dx$$

*holds.*

The proof of this lemma can be found in [28] (see Lemma 2.1).

**Lemma 3.2.** *Let $u \in V^{2,2}(\Omega)$. Then the following inequality*

$$\|u\|^2_{L^\infty(\Omega_0)} + \|\varphi^{-1}u\|^2_{L^\infty(G_H)} \le c\|u\|^2_{V^{2,2}(\Omega)} \tag{3.2}$$

*holds with the constant $c$ depending only on $\lambda$, $\gamma_0$ and the geometry of $\Omega_0$.*

*Proof.* Consider a function $u \in V^{2,2}(\Omega)$ in the domain $\omega_l$. After the transformation $\mathcal{P}_l$, $\omega_l$ is transformed into the domain $G_0 = \{y : \ |y_1| < 1, -1 < y_2 < 0\}$ which is independent of $l$. In $G_0$ holds the inequality (see [1])

$$\|u\|^2_{L^\infty(G_0)} \le c\|u\|^2_{W^{2,2}(G_0)} = c\int\limits_{G_0} \big(|u(y)|^2 + |\nabla_y u(y)|^2 + |\nabla_y^2 u(y)|^2\big)dy.$$

Passing in the last inequality to the variables $x$, using (3.1) and the relations $|x_1| \le \varphi(x_2) = \gamma_0 x_2^\lambda$, $\lambda > 1$, $|\varphi'(x_2)| \le c$ and $|\varphi''(x_2)| \le \frac{c}{\varphi(x_2)}$ we obtain

$$\|u\|^2_{L^\infty(\omega_l)} \le c\int\limits_{\omega_l} \Big(\frac{|u(x)|^2}{\varphi^2(h_{l-1})} + |\nabla_x u(x)|^2 + \varphi^2(h_{l-1})|\nabla_x^2 u(x)|^2\Big)dx.$$

Dividing both sides by $\varphi^2(h_{l-1})$ and applying again (3.1) gives

$$\begin{aligned}\|\varphi^{-1}u\|^2_{L^\infty(\omega_l)} \le & c\int\limits_{\omega_l} \Big(\frac{|u(x)|^2}{\varphi^4(x_2)} + \frac{|\nabla u(x)|^2}{\varphi^2(x_2)} + |\nabla_x^2 u(x)|^2\Big)dx \\ \le & c\|u\|^2_{V^{2,2}(\Omega)}\end{aligned} \tag{3.3}$$

with a constant $c$ independent of $l$. Taking in (3.3) supremum over all $l$ yields

$$\|\varphi^{-1}u\|^2_{L^\infty(G_H)} \le c\|u\|^2_{V^{2,2}(\Omega)}. \tag{3.4}$$

(3.4) and the estimate

$$\|u\|^2_{L^\infty(\Omega_0)} \le c\|u\|^2_{W^{2,2}(\Omega_0)} \le c\|u\|^2_{V^{2,2}(\Omega)}$$

imply (3.2) . □

### 3.3 Estimates of solutions to the Stokes problem

Consider in $\Omega$ the Dirichlet boundary value problem for the Stokes system

$$\begin{cases} -\nu\Delta\mathbf{v} + \nabla p = \mathbf{f}, \\ \operatorname{div}\mathbf{v} = 0, \\ \mathbf{v}\big|_{\partial\Omega} = 0. \end{cases} \tag{3.5}$$

The following result was proved in [28] (see Lemmas 3.1–3.3).

**Lemma 3.3.** *Let $\mathbf{f} \in L^2(\Omega)$. Then there exists a unique solution $(\mathbf{v}, p)$ of problem (3.5) such that $\mathbf{v} \in V^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega)$, $\nabla p \in L^2(\Omega)$ and the estimate*

$$\|\mathbf{v}\|_{V^{2,2}(\Omega)} + \|\nabla p\|_{L^2(\Omega)} \le c\|\mathbf{f}\|_{L^2(\Omega)}$$

*holds with the constant $c$ depending only on $\lambda$, $\gamma_0$ and the geometry of $\Omega_0$.*

# 4 Solvability of problem (1.1)

## 4.1 Construction of the extension of boundary data

Suppose that $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$, $\operatorname{supp}\mathbf{a} \subset \partial\Omega_0 \cap \partial\Omega \subset \partial\Omega_1^\sharp$. Consider the linear extension operator $E$ in the domain $\Omega_3^\sharp$, $E: \ W^{3/2,2}(\partial\Omega_3^\sharp) \longmapsto W^{2,2}(\Omega_3^\sharp)$ given by $E\mathbf{a} = \mathbf{w}^{(1)}$, where $\mathbf{w}^{(1)}|_{\partial\Omega_3^\sharp} = \mathbf{a}$. Since the boundary $\partial\Omega \cap \partial\Omega_3^\sharp$ is $C^2$ and $\operatorname{supp}\mathbf{a} \subset \partial\Omega_0 \cap \partial\Omega \subset \partial\Omega_1^\sharp$, the linear operator $E$ is bounded:

$$\|E\mathbf{a}\|^2_{W^{2,2}(\Omega_3^\sharp)} = \|\mathbf{w}^{(1)}\|^2_{W^{2,2}(\Omega_3^\sharp)} \le c\|\mathbf{a}\|^2_{W^{3/2,2}(\partial\Omega)}. \tag{4.1}$$

Moreover, $\mathbf{w}^{(1)}$ can be constructed in such a way that $\operatorname{supp}\mathbf{w}^{(1)} \subset \overline{\Omega_2^\sharp}$ (see, e.g., [1]).

Let $\mathbf{U}^{[J]}\Big(\frac{x_1}{x_2^\lambda}, x_2\Big)$ be the formal asymptotic decomposition of the velocity component near the cusp point $O$ constructed in Section 2. Consider the function $\mathbf{B} = \mathbf{w}^{(1)} + \zeta\mathbf{U}^{[J]}$, where $\zeta = \zeta(x_2)$ is a smooth cut-off function equal to one in $\Omega \setminus \Omega_2^\sharp$, equal to zero in $\Omega_1^\sharp$ and $0 < \zeta(x_2) < 1$ in $\Omega_2^\sharp \setminus \Omega_1^\sharp$. Obviously, $\mathbf{B}|_{\partial\Omega} = \mathbf{a}$, however, $\mathbf{B}$ is not solenoidal, $\operatorname{div}\mathbf{B} = \operatorname{div}\mathbf{w}^{(1)} + \nabla\zeta \cdot \mathbf{U}^{[J]} := h$. Notice that

$$\begin{aligned}\int\limits_{\Omega_2^\sharp} h dx = \int\limits_{\partial\Omega_2^\sharp} (\mathbf{w}^{(1)} + \zeta\mathbf{U}^{[J]})\cdot\mathbf{n}dS = &\int\limits_{\partial\Omega_0\cap\partial\Omega} \mathbf{a}\cdot\mathbf{n}dS + \int\limits_{\partial\Omega_2^\sharp\setminus\partial\Omega} \mathbf{U}^{[J]}\cdot\mathbf{n}dS\\ =&F - F = 0.\end{aligned}$$

Since $\operatorname{supp} h \subset \overline{\Omega_2^\sharp}$ and the boundary $\partial\Omega_3^\sharp \cap \partial\Omega$ is smooth, there exist a function $\mathbf{w}^{(2)} \in W^{2,2}(\Omega_3^\sharp)$ such that $\operatorname{supp}\mathbf{w}^{(2)} \subset \overline{\Omega_3^\sharp}$, $\mathbf{w}^{(2)} = 0$ in the neighbourhood of $\partial\Omega_3^\sharp \setminus \partial\Omega$ and

$$\begin{cases} \operatorname{div}\mathbf{w}^{(2)} = h \ \text{ in } \ \Omega_3^\sharp, \\ \mathbf{w}|^{(2)}_{\partial\Omega_3^\sharp} = 0. \end{cases}$$

Moreover,

$$\|\mathbf{w}^{(2)}\|^2_{W^{2,2}(\Omega_3^\sharp)} \le c\|h\|^2_{W^{1,2}(\Omega_3^\sharp)},$$

(see [6]). From this inequality and (4.1), (2.20) it follows that

$$\begin{aligned}\|\mathbf{w}^{(2)}\|^2_{W^{2,2}(\Omega_3^\sharp)} &\le c\Big(\|\mathbf{w}^{(1)}\|^2_{W^{2,2}(\Omega_3^\sharp)} + \|\mathbf{U}^{[J]}\|^2_{W^{1,2}(\Omega_3^\sharp)}\Big)\\ &\le c\Big(\|\mathbf{a}\|^2_{W^{3/2,2}(\partial\Omega)} + |F|^2\Big) \le c\|\mathbf{a}\|^2_{W^{3/2,2}(\partial\Omega)}.\end{aligned} \tag{4.2}$$

Define

$$\mathbf{W} = \mathbf{w}^{(1)} + \mathbf{w}^{(2)}, \quad \mathbf{V} = \mathbf{W} + \zeta\mathbf{U}^{[J]},$$

where $\zeta$ is a smooth cut-off function defined above. By construction $\operatorname{div}\mathbf{V} = 0$, $\mathbf{V}|_{\partial\Omega} = \mathbf{a}$ and $\mathbf{V} = \mathbf{U}^{[J]}$ for $x \in \Omega \setminus \Omega_3^\sharp$. Therefore, for $x \in \Omega \setminus \Omega_3^\sharp$ the function $\mathbf{V}$ satisfies estimates (2.20), while from (4.1), (4.2) and (2.20) it follows that

$$\|\mathbf{V}\|_{W^{2,2}(\Omega_3^\sharp)} \le c\Big(\|\mathbf{a}\|^2_{W^{3/2,2}(\partial\Omega)} + |F|^2\Big) \le c\|\mathbf{a}\|^2_{W^{3/2,2}(\partial\Omega)}. \tag{4.3}$$

We look for the solution $(\mathbf{u}, p)$ of problem (1.1) in the form

$$\mathbf{u} = \mathbf{v} + \mathbf{V}, \quad p = q + \zeta P^{[J]}.$$

Then for $(\mathbf{v}, q)$ we obtain the following problem

$$\begin{cases} -\nu \Delta \mathbf{v} + \nabla q = -(\mathbf{v}\cdot\nabla)\mathbf{v} - (\mathbf{V}\cdot\nabla)\mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{V} + \widehat{\mathbf{f}}, \\ \operatorname{div}\mathbf{v} = 0, \\ \mathbf{v}|_{\partial\Omega\setminus O} = 0, \end{cases} \tag{4.4}$$

where $\widehat{\mathbf{f}} = \mathbf{f} - \mathbf{f}_1$, $\mathbf{f}_1 = -\nu\Delta\mathbf{V} + (\mathbf{V}\cdot\nabla)(\mathbf{V}) + \nabla(\zeta P^{[J]})$. Recall that the number $J$ was chosen in such a way that

$$-\nu\Delta\mathbf{U}^{[J]} + (\mathbf{U}^{[J]}\cdot\nabla)\mathbf{U}^{[J]} + \nabla P^{[J]} = \mathbf{H}^{[J]} \in L^2(G_H).$$

Therefore, taking into account that $\mathbf{W}$ has compact support in $\overline{\Omega}_3^\sharp$, we conclude $\widehat{\mathbf{f}} \in L^2(\Omega)$. Moreover, using (4.3) we obtain

$$\|\widehat{\mathbf{f}}\|^2_{L^2(\Omega)} \le c\Big(\|\mathbf{f}\|^2_{L^2(\Omega)} + \|\mathbf{a}\|^2_{W^{3/2,2}(\partial\Omega)} + \|\mathbf{a}\|^4_{W^{3/2,2}(\partial\Omega)}\Big).$$

## 4.2 Existence of the strong solution

**Theorem 4.1.** *Let* $\mathbf{f} \in L^2(\Omega)$, $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$. *There exist a constant* $\kappa_o > 0$ *such that if*

$$\|\mathbf{f}\|^2_{L^2(\Omega)} + \|\mathbf{a}\|^2_{W^{3/2,2}(\partial\Omega)} + \|\mathbf{a}\|^4_{W^{3/2,2}(\partial\Omega)} \le \kappa_o,$$

*then problem* (4.4) *admits a unique solution* $(\mathbf{v}, q)$ *with* $\mathbf{v} \in V^{2,2}(\Omega)$, $\nabla q \in L^2(\Omega)$ *and the following estimate*

$$\begin{aligned} &\|\mathbf{v}\|^2_{V^{2,2}(\Omega)} + \|\nabla q\|^2_{L^2(\Omega)} \\ &\qquad \le c\Big(\|\mathbf{f}\|^2_{L^2(\Omega)} + \|\mathbf{a}\|^2_{W^{3/2,2}(\partial\Omega)} + \|\mathbf{a}\|^4_{W^{3/2,2}(\partial\Omega)}\Big) \end{aligned} \tag{4.5}$$

*holds with the constant* $c$ *depending only on* $\lambda$, $\gamma_0$ *and the geometry of* $\Omega_0$.

*Proof.* Let $\mathbf{z} \in V^{2,2}(\Omega)$. Then $(\mathbf{z}\cdot\nabla)\mathbf{z} \in L^2(\Omega)$. Indeed, using (3.2) (see Lemma 3.2) we obtain

$$\begin{aligned} \|(\mathbf{z}\cdot\nabla)\mathbf{z}\|^2_{L^2(\Omega)} \le &\int_{\Omega_0} |\mathbf{z}|^2|\nabla\mathbf{z}|^2 dx + \int_{G_H} |\mathbf{z}|^2|\nabla\mathbf{z}|^2 dx \\ \le &\|\mathbf{z}\|^2_{L^\infty(\Omega_0)} \int_{\Omega_0} |\nabla\mathbf{z}|^2 dx + \|\varphi^{-1}\mathbf{z}\|^2_{L^\infty(G_H)} \int_{G_H} \varphi^2|\nabla\mathbf{z}|^2 dx \\ \le &c\|\mathbf{z}\|^2_{V^{2,2}(\Omega)}\Big(\int_{\Omega_0} |\nabla\mathbf{z}|^2 dx + \int_{G_H} \varphi^2|\nabla\mathbf{z}|^2 dx\Big) \le c\|\mathbf{z}\|^4_{V^{2,2}(\Omega)}. \end{aligned}$$

Further,

$$\begin{aligned} \int_\Omega |\mathbf{V}|^2|\nabla\mathbf{z}|^2 dx + \int_\Omega |\mathbf{z}|^2|\nabla\mathbf{V}|^2 dx \le &\int_{\Omega_3^\sharp} \Big(|\mathbf{V}|^2|\nabla\mathbf{z}|^2 + |\mathbf{z}|^2|\nabla\mathbf{V}|^2\Big) dx \\ &+ \int_{\Omega\setminus\Omega_3^\sharp} \Big(|\mathbf{V}|^2|\nabla\mathbf{z}|^2 + |\mathbf{z}|^2|\nabla\mathbf{V}|^2\Big) dx = J_1 + J_2. \end{aligned}$$

By (2.20),

$$J_2 \le c|F|^2 \int\limits_{\Omega\setminus\Omega_3^\sharp} \Big(\varphi^{-4}|\mathbf{z}|^2 + \varphi^{-2}|\nabla\mathbf{z}|^2\Big)dx \le c\|\mathbf{a}\|^2_{W^{3/2,2}(\partial\Omega)}\|\mathbf{z}\|^2_{V^{2,2}(\Omega)}.$$

Using (4.1), (4.2), (2.20) and (3.2), for the term $J_1$ we obtain the estimate

$$\begin{aligned}J_1 &\le \|\mathbf{W}+\zeta\mathbf{U}^{[J]}\|^2_{L^\infty(\Omega_3^\sharp)}\int\limits_{\Omega_3^\sharp}|\nabla\mathbf{z}|^2dx+\|\mathbf{z}\|^2_{L^\infty(\Omega_3^\sharp)}\int\limits_{\Omega_3^\sharp}\Big(|\nabla\mathbf{W}|^2+|\nabla(\zeta\mathbf{U}^{[J]})|^2\Big)dx\\ &\le c\Big(\|\mathbf{a}\|^2_{W^{3/2,2}(\partial\Omega)}+|F|^2\Big)\|\mathbf{z}\|^2_{V^{2,2}(\Omega)} \le c\|\mathbf{a}\|^2_{W^{3/2,2}(\partial\Omega)}\|\mathbf{z}\|^2_{V^{2,2}(\Omega)}.\end{aligned}$$

Assume that $\mathbf{z}\in V^{2,2}(\Omega)$, $\mathbf{f}\in L^2(\Omega)$ are given functions and consider the Stokes problem

$$\begin{cases} -\nu\Delta\mathbf{v}+\nabla q = -(\mathbf{z}\cdot\nabla)\mathbf{z}-(\mathbf{V}\cdot\nabla)\mathbf{z}-(\mathbf{z}\cdot\nabla)\mathbf{V}+\widehat{\mathbf{f}},\\ \operatorname{div}\mathbf{v}=0,\\ \mathbf{v}|_{\partial\Omega}=0.\end{cases} \tag{4.6}$$

According to the above inequalities the right hand side of (4.6) belongs to $L^2(\Omega)$, and hence, problem (4.6) admits a unique solution $(\mathbf{v},q)$ satisfying the estimate

$$\begin{aligned}\|\mathbf{v}\|^2_{V^{2,2}(\Omega)}+\|\nabla q\|^2_{L^2(\Omega)} &\le c\Big(\|(\mathbf{z}\cdot\nabla)\mathbf{z}\|^2_{L^2(\Omega)}+\|(\mathbf{V}\cdot\nabla)\mathbf{z}\|^2_{L^2(\Omega)}\\ &\quad+\|(\mathbf{z}\cdot\nabla)\mathbf{V}\|^2_{L^2(\Omega)}+\|\widehat{\mathbf{f}}\|^2_{L^2(\Omega)}\Big)\\ &\le c_1\Big(\|\mathbf{z}\|^4_{V^{2,2}(\Omega)}+\|\mathbf{a}\|^2_{W^{3/2,2}(\partial\Omega)}\|\mathbf{z}\|^2_{V^{2,2}(\Omega)}\Big)\\ &\quad+c_2\Big(\|\mathbf{f}\|^2_{L^2(\Omega)}+\|\mathbf{a}\|^2_{W^{3/2,2}(\partial\Omega)}+\|\mathbf{a}\|^4_{W^{3/2,2}(\partial\Omega)}\Big)\end{aligned} \tag{4.7}$$

(see [15]). Thus, (4.6) is equivalent to an operator equation in the space $V^{2,2}(\Omega)$:

$$\mathbf{v}=\mathfrak{A}\mathbf{z}, \tag{4.8}$$

where $\mathfrak{A}\mathbf{z}=\mathfrak{L}^{-1}(-(\mathbf{z}\cdot\nabla)\mathbf{z}-(\mathbf{V}\cdot\nabla)\mathbf{z}-(\mathbf{z}\cdot\nabla)\mathbf{V}+\widehat{\mathbf{f}})$ and $\mathfrak{L}^{-1}$ is the inverse operator (bounded) of Stokes problem (4.6).

Denote $\|\mathbf{f}\|^2_{L^2(\Omega)}+\|\mathbf{a}\|^2_{W^{3/2,2}(\partial\Omega)}+\|\mathbf{a}\|^4_{W^{3/2,2}(\partial\Omega)}=\mu_0^2$ and suppose that $\|\mathbf{z}\|^2_{V^{2,2}(\Omega)}\le R_0^2$, where $R_0^2=2c_2\mu_0^2$. Then (4.7) yields

$$\|\mathbf{v}\|^2_{V^{2,2}(\Omega)}\le c_1(4c_2^2+2c_2)\mu_0^4+c_2\mu_0^2.$$

Thus if

$$\mu_0^2\le\frac{1}{2c_1(2c_2+1)}, \tag{4.9}$$

then the operator $\mathfrak{A}$ maps the ball $\|\mathbf{z}\|^2_{V^{2,2}(\Omega)}\le R_0^2$ into itself.

Let us show that $\mathfrak{A}$ is a contraction. Assume $\mathbf{z}_1,\mathbf{z}_2\in V^{2,2}(\Omega)$, $\|\mathbf{z}_i\|^2_{V^{2,2}(\Omega)}\le R_0^2$ and let $\mathbf{v}_1=\mathfrak{A}\mathbf{z}_1$, $\mathbf{v}_2=\mathfrak{A}\mathbf{z}_2$. The difference $\mathbf{A}=\mathbf{v}_1-\mathbf{v}_2$ is the solution of the problem

$$\begin{cases} -\nu\Delta\mathbf{A} + \nabla Q = -(\mathbf{Z}\cdot\nabla)\mathbf{z}_1 + (\mathbf{z}_2\cdot\nabla)\mathbf{Z} - (\mathbf{V}\cdot\nabla)\mathbf{Z} - (\mathbf{Z}\cdot\nabla)\mathbf{V}, \\ \operatorname{div}\mathbf{A} = 0, \\ \mathbf{A}|_{\partial\Omega} = 0, \end{cases} \tag{4.10}$$

where $\mathbf{Z} = \mathbf{z}_1 - \mathbf{z}_2$ and $Q = q_1 - q_2$, $q_i$ are corresponding to $\mathbf{v}_i$, $i = 1, 2$, pressure functions. The same argument as above gives us the estimates

$$\|(\mathbf{Z}\cdot\nabla)\mathbf{z}_1\|^2_{L^2(\Omega)} + \|(\mathbf{z}_2\cdot\nabla)\mathbf{Z}\|^2_{L^2(\Omega)} \le c\Big(\|\mathbf{z}_1\|^2_{V^{2,2}(\Omega)} + \|\mathbf{z}_2\|^2_{V^{2,2}(\Omega)}\Big)\|\mathbf{Z}\|^2_{V^{2,2}(\Omega)};$$
$$\|(\mathbf{V}\cdot\nabla)\mathbf{Z}\|^2_{L^2(\Omega)} + \|(\mathbf{Z}\cdot\nabla)\mathbf{V}\|^2_{L^2(\Omega)} \le c\|\mathbf{a}\|^2_{W^{3/2,2}(\partial\Omega)}\|\mathbf{Z}\|^2_{V^{2,2}(\Omega)}.$$

Therefore, the solution $\mathbf{A} = \mathfrak{A}\mathbf{z}_1 - \mathfrak{A}\mathbf{z}_2$ of problem (4.10) admits the estimate

$$\begin{aligned} \|\mathfrak{A}\mathbf{z}_1 - \mathfrak{A}\mathbf{z}_2\|^2_{V^{2,2}(\Omega)} &\le c\Big(2R_0^2 + \|\mathbf{a}\|^2_{W^{3/2,2}(\partial\Omega)}\Big)\|\mathbf{z}_1 - \mathbf{z}_2\|^2_{V^{2,2}(\Omega)} \\ &\le c_3\mu_0^2\|\mathbf{z}_1 - \mathbf{z}_2\|^2_{V^{2,2}(\Omega)}. \end{aligned}$$

If $\gamma_0$ satisfies condition (4.9) and $\mu_0^2 < 1/c_3$, then the operator $\mathfrak{A}$ is a contraction in the ball $B_{R_0} = \{z \in V^{2,2}(\Omega) : \|\mathbf{z}_1\|^2_{V^{2,2}(\Omega)} \le R_0\}$. Thus, by the Banach fixed point theorem, operator equation (4.8) has a unique fixed point in $B_{R_0}$. Equivalently, problem (4.4) has a unique solution $(\mathbf{v}, q)$ such that $\mathbf{v} \in V^{2,2}(\Omega)$, $\nabla q \in L^2(\Omega)$ and estimate (4.5) is valid. $\square$

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