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Asymptotic Distribution of Eigenvalues and Eigenfunctions of a Nonlocal Boundary Value Problem

Erdoğan Şen[a,b] and Artūras Štikonas[b]

[a] *Tekirdag Namik Kemal University*
Kampus str. 1, TR-59030, Tekirdag, Turkey
[b] *Institute of Applied Mathematics, Vilnius University*
Naugarduko g. 24, LT-03225, Vilnius, Lithuania
E-mail(*corresp.*): esen@nku.edu.tr
E-mail: arturas.stikonas@mif.vu.lt

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Abstract. In this work, we obtain asymptotic formulas for eigenvalues and eigenfunctions of the second order boundary-value problem with a Bitsadze–Samarskii type nonlocal boundary condition.

Keywords: differential equation, nonlocal conditions, asymptotics of eigenvalues and eigenfunctions.

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1 Introduction

Many physical processes, such as the vibration of strings, the interaction of atomic particles, electrodynamics of complex medium, combustion in the chamber of a liquid propellant rocket engine, aerodynamics, polymer rheology or the earths free oscillations yields the second-order eigenvalue problems (see [7, 8, 10, 25]). Using the method of separation of variables to solve some kinds of second order partial differential equations require us to solve equation

$$-u''(t) + q(t)u(t) = \lambda u(t), \tag{1.1}$$

where the real-valued function $q \in C[0,1]$; $\lambda = s^2$ is a complex spectral parameter and $s = x + \imath y$; $x, y \in \mathbb{R}$.

In this study we shall investigate nonlocal eigenvalue problems which consist of Sturm–Liouville equation (1.1) on $[0,1]$, with one classical (local) boundary condition

$$u(0)=0, \tag{1.2}$$

another Bitsadze–Samarskii type nonlocal boundary condition

$$u(1)=\gamma u(\xi), \quad \xi \in (0,1), \tag{1.3}$$

where $\gamma \in \mathbb{R}$.

In some of problems of mathematical physics, biology and biotechnology subsidiary conditions are imposed locally. Asymptotic formulas for eigenvalues and eigenfunctions for these kinds of Boundary-Value Problems (BVPs) (the case $\gamma=0$) are obtained in [1,2,9,13,14,16,19,26,27]. Asymptotic formulas for eigenvalues and eigenfunctions for BVPs which contains a spectral parameter in the local (classic) boundary conditions except from the differential equation obtained in [9,19].

There has been an increasing interest for spectral analysis of nonlocal boundary value problems (NBVPs) in the last decades. NBVPs are widely used for mathematical modelling of various processes of physics, ecology, chemistry and industry, when it is impossible to determine the boundary or initial values of the unknown function. For example, problems with feedback controls such as the steady-states of a thermostat, where a controller at one of its ends adds or removes heat, depending upon the temperature registered in another point, can be interpreted with a second-order ordinary differential equation subject to a nonlocal boundary conditions. The bibliography on the subject of NBVPs is very extensive and we refer to the list of the works in [6,22,23,24]. We should also note that an eigenvalue problem with the nonlocal boundary conditions is closely linked to boundary problems for differential equations with the nonlocal boundary conditions [3,4,5,11,12,18]. However, until this time, there was no work investigating asymptotic properties of eigenvalues and eigenfunctions of the second order nonlocal boundary value problems with potential function $q(x)$ in differential equation.

The paper is organized as follows. In Section 2, notation and definitions used in the paper are stated. Also, we write the general solution of the (1.1) corresponding to the initial conditions and prove the simplicity of eigenvalues. In Section 3, we investigate the distribution of eigenvalues and obtain asymptotic formulas for eigenvalues and eigenfunctions of the boundary-value problem (1.1)–(1.3). Later on, in the same section, we obtain more exact formulas for eigenvalues and eigenfunctions under the condition $q \in C^1[0,1]$. Also, we calculate normalized eigenfunctions for the problem (1.1)–(1.3).

2 Fundamental solutions and simplicity of eigenvalues

In this section, first, we write the initial-value problem (1.1), (2.1) in terms of equivalent integral equation and then construct structure of the solutions of the initial-value problem (1.1), (2.1) and the space of these solutions. We see that any two solutions of the initial-value problem (1.1), (2.1) which are linearly independent on $[0,1]$ form a fundamental system of solutions.

Let $\omega_s(t)$ be a solution of Equation (1.1) satisfying the conditions

$$\omega_s(0)=0, \quad \omega_s'(0)=-1. \tag{2.1}$$

According to [14, Theorem 1.1 in Chapter I], the initial conditions (2.1) determine a unique solution of Equation (1.1) on $[0,1]$. The function $\omega(t,s)=\omega_s(t)$ is an analytic function of s.

Remark 1. In this article $s\in\mathbb{C}_s:=\mathbb{R}_s\cup\mathbb{C}_s^+\cup\mathbb{C}_s^-$, where $\mathbb{R}_s:=\mathbb{R}_s^-\cup\mathbb{R}_s^+\cup\mathbb{R}_s^0$, $\mathbb{R}_s^-:=\{s=x+\imath y\in\mathbb{C}: x=0, y>0\}$, $\mathbb{R}_s^+:=\{s=x+\imath y\in\mathbb{C}: x>0, y=0\}$, $\mathbb{R}_s^0:=\{s=0\}$, $\mathbb{C}_s^+:=\{s=x+\imath y\in\mathbb{C}: x>0, y>0\}$ and $\mathbb{C}_s^-:=\{s=x+\imath y\in\mathbb{C}:\ x>0,\ y<0\}$. Then a map $\lambda=s^2$ is the bijection between $\mathbb{C}_s$ and $\mathbb{C}_\lambda:=\mathbb{C}$ [24].

Lemma 1. *Let $\omega_s(t)$ be a solution of Equation* (1.1) *with the initial conditions* (2.1)*. Then the following integral equation holds:*

$$\omega_s(t)=-\frac{1}{s}\sin(st)+\frac{1}{s}\int_0^t \sin\big(s(t-\tau)\big)q(\tau)\omega_s(\tau)d\tau. \tag{2.2}$$

Proof. Following [14], it is enough to substitute $s^2\omega_s(\tau)+(\omega_s)''(\tau)$ instead of $q(\tau)\omega_s(\tau)$ in the integral in (2.2) and integrate by parts twice. □

Lemma 2. *Let $s\in\mathbb{C}_s$. Then there exists $q_0>0$ such that for $|s|\geq 2q_0$ one has the estimate*

$$\omega_s(t)=\mathcal{O}(s^{-1}e^{|y|t}) \tag{2.3}$$

and more precisely

$$\omega_s(t)=-s^{-1}\sin(st)+\mathcal{O}(s^{-2}e^{|y|t}). \tag{2.4}$$

These estimates hold uniformly for $0\leq t\leq 1$.

Proof. Put $\omega_s(t)=e^{|y|t}F(t)$. Then from (2.2) we obtain

$$F(t)=-\frac{1}{s}\sin(st)e^{-|y|t}+\frac{1}{s}\int_0^t \sin\big(s(t-\tau)\big)e^{-|y|(t-\tau)}q(\tau)F(\tau)d\tau. \tag{2.5}$$

Let $\mu=\max_{0\leq\tau\leq 1}|F(\tau)|$ and $q_0:=\int_0^1|q(\tau)|d\tau$. Then it follows from (2.5) that

$$\mu\leq|s|^{-1}+\mu|s|^{-1}q_0.$$

So, we get

$$\mu\leq|s|^{-1}/(1-|s|^{-1}q_0) \tag{2.6}$$

under the condition that the denominator is positive. Namely, if $|s|\geq 2q_0$ then we get (2.3). Now, substituting (2.3) into the integral of (2.2) we obtain the estimate (2.4). □

Lemma 3. *Let $s \in \mathbb{C}_s$. Then there exists $q_0 > 0$ such that for $|s| \geq 2q_0$ one has the estimate*

$$\omega_s'(t) = \mathcal{O}(e^{|y|t}) \tag{2.7}$$

and more precisely

$$\omega_s'(t) = -\cos(st) + \mathcal{O}(s^{-1}e^{|y|t}). \tag{2.8}$$

These estimates hold uniformly for $0 \leq t \leq 1$.

Proof. Taking derivative with respect to t in (2.2) we get

$$\omega_s'(t) = -\cos(st) + \int_0^t \cos\big(s(t-\tau)\big)q(\tau)\omega_s(\tau)d\tau. \tag{2.9}$$

Let us take $q_0 = \int_0^1 |q(\tau)|d\tau$ as in the previous lemma. If $|s| \geq 2q_0$ then $1 - |s|^{-1}q_0 \geq |s|^{-1}q_0$, and we have (see (2.6))

$$e^{-|y|\tau}|\omega_s(\tau)| \leq q_0^{-1}.$$

Then from (2.9) we obtain

$$e^{-|y|t}\omega_s'(t) = -e^{-|y|t}\cos(st) + \int_0^t \cos\big(s(t-\tau)\big)e^{-|y|(t-\tau)}q(\tau)e^{-|y|\tau}\omega_s(\tau)d\tau = \mathcal{O}(1).$$

Namely, if $|s| \geq 2q_0$ then we get (2.7). Now, substituting (2.3) into the integral of (2.9) we obtain the estimate (2.8). □

Theorem 1. [see, Theorem II.2.2 in [17]] *In order that the functions $\omega_s^1(t)$ and $\omega_s^2(t)$, solutions of the initial-value problem* (1.1), (2.1) *be linearly dependent on* $[0,1]$ *it is necessary and sufficient that*

$$W[\omega_s^1, \omega_s^2](0) = \begin{vmatrix} \omega_s^1(0) & \omega_s^2(0) \\ (\omega_s^1)'(0) & (\omega_s^2)'(0) \end{vmatrix} = 0.$$

The algebraic multiplicity of the eigenvalue is its multiplicity as a root of the characteristic polynomial.

We will say that geometric multiplicity of an eigenvalue of a boundary value problem is the maximum number of linearly independent eigenfunctions associated with the related eigenvalue. By the definition of eigenvalues and eigenfunctions, geometric multiplicity of an eigenvalue is equal or greater than one because each eigenvalue has at least one eigenfunction.

Theorem 2. *The geometric multiplicity of eigenvalues of the problem* (1.1)–(1.3) *is one.*

Proof. Let $\lambda = s^2$ be an eigenvalue of the problem (1.1)–(1.3) and $v_s(t)$ be corresponding eigenfunction. From (2.1) we have

$$W[v_s,\omega_s](0) = \begin{vmatrix} v_s(0) & \omega_s(0) \\ v_s'(0) & \omega_s'(0) \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ v_s'(0) & -1 \end{vmatrix} = 0.$$

Thus, according to Theorem 1, the functions $v_s(t)$ and $\omega_s(t)$ are linearly dependent on $[0,1]$. Hence $\omega_s(t)$ is an eigenfunction for (1.1)–(1.3), too. □

3 Spectral asymptotics for eigenvalues and eigenfunctions

In the case $q(t) \equiv 0$, the spectrum of the Sturm–Liouville problem (1.1)–(1.3) has countably many eigenvalues and all eigenvalues are positive for $|\gamma| \le 1$. If $|\gamma| < 1$ then all eigenvalues are algebraically simple. A unique negative eigenvalue exists for $\gamma > \frac{1}{\xi}$ and $\lambda = 0$ is eigenvalue if and only if $\gamma = \frac{1}{\xi}$. Complex eigenvalues may exist for $|\gamma| > 1$ [21].

Substituting $\omega_s(t)$ into (1.3), we get the characteristic equation

$$\omega_s(1) - \gamma\omega_s(\xi) = 0. \tag{3.1}$$

We introduce a function

$$H(s) := -s\big(\omega_s(1) - \gamma\omega_s(\xi)\big). \tag{3.2}$$

The set of eigenvalues of boundary-value problem (1.1)–(1.3) coincides with the set $\{\lambda\colon \lambda = s^2, -H(s)/s = \omega_s(1) - \gamma\omega_s(\xi) = 0\}$. The function $H(s)$ is actually an analytic function of s. Substituting (2.2) into (3.2) we get

$$\begin{aligned} H(s) =& \sin s - \int_0^1 \sin\big((1-\tau)s\big)q(\tau)\omega_s(\tau)d\tau \\ & - \gamma\Big(\sin(\xi s) - \int_0^\xi \sin\big((\xi-\tau)s\big)q(\tau)\omega_s(\tau)d\tau\Big). \end{aligned} \tag{3.3}$$

So, we have

$$H(s) = \sin s - \gamma\sin(\xi s) + \mathcal{O}(s^{-1}e^{|y|}). \tag{3.4}$$

Remark 2. The formula

$$H'(s) = \cos s - \gamma\xi\cos(\xi s) + \mathcal{O}(s^{-1}e^{|y|})$$

is valid.

Theorem 3. *The real eigenvalues of the problem* (1.1)–(1.3) *are bounded below.*

Proof. Set $\tilde{H}(\lambda) := \imath^3 H(s)$. Let $s = \imath y, y > 0$. Then

$$\widetilde{H}(-y^2) = e^y\big(1 - e^{-2y} - \gamma e^{(\xi-1)y} + \gamma e^{-(\xi+1)y}\big)/2 + \mathcal{O}(y^{-1}e^y).$$

It is clear that $\lim_{y\to\infty}\tilde{H}(-y^2) = \infty$. Then there exists a $y_0 > 0$ such that $\tilde{H}(-y^2) \ne 0$ for $y > y_0$. Therefore we get $\tilde{H}(\lambda) \ne 0$ for $\lambda < -y_0^2$. Accordingly, $\lambda > -y_0^2$. □

Corollary 1. The number of negative eigenvalues of the problem (1.1)–(1.3) is finite (maybe zero).

Theorem 4. *The problem* (1.1)–(1.3) *has infinitely many (countable) positive eigenvalues for* $|\gamma| < 1$*.*

Proof. We consider $s = x$, $0 < x \in \mathbb{R}$. We have $|\gamma \sin(\xi x) + \mathcal{O}(x^{-1})| < 1$ for large x. The function $\sin x$ takes its local maximum points at $M_k = (2k-3/2)\pi$, $k \in \mathbb{N}$, and its local minimum points at $m_k = (2k - 1/2)\pi$, $k \in \mathbb{N}$. Thus, from Intermediate value theorem at least one root of the function $H(x)$ lies in each interval $\big((k - 1/2)\pi, (k + 1/2)\pi\big)$, $K < k \in \mathbb{N}$ for large K. So, we have infinite (countable) number roots of equation $H(x) = 0$. □

Corollary 2. The function H has at least one positive root in the interval $\big((k - 1/2)\pi, (k + 1/2)\pi\big)$.

Remark 3. The function h has the same property, but the root in the interval $\big((k - 1/2)\pi, (k + 1/2)\pi\big)$ is unique for $|\gamma| < 1$ [24]. We can write the roots of (3.6) as $x_k = x_k(\gamma) = \pi k + f_k(\gamma)$ where the analytic function $f_k(\gamma)$ is bounded ($|f_k(\gamma)| \le \pi/2$) and $f_k(0) = 0$.

Let us consider the equation

$$-u''(t) = \lambda u(t) \tag{3.5}$$

with the boundary conditions (1.2)–(1.3). The characteristic equation of Equation (3.5) is

$$h(s) := \sin s - \gamma \sin(\xi s) = 0, \tag{3.6}$$

where $\lambda = s^2$, $s \in \mathbb{C}_s$. This equation was investigated in [15, 21]. For $|\gamma| < 1$, Equation (3.6) has infinite (countable) number positive simple roots x_k, $k \in \mathbb{N}$. These roots we can find by solving equation

$$h(x) = \sin x - \gamma \sin(\xi x) = 0, \quad x \in \mathbb{R}. \tag{3.7}$$

Between two roots of this equation there exists point $\tilde{x}_k \in (x_k, x_{k+1})$ such that $h'(\tilde{x}_k) = 0$, i.e. the root of equation $\cos x - \gamma\xi \cos(\xi x)$.

Lemma 4. *Let* $|\gamma| < 1$, $0 < \xi < 1$, $\beta \ge 0$*. If* $\sin x - \gamma\xi^\beta \sin(\xi x) = 0$ *then there exists* $\kappa > 0$ *such that* $|\cos x| - |\gamma||\cos(\xi x)| \ge \kappa > 0$*.*

Proof. Let take μ such that $0 < \mu < 1$ (for example, we can take $\mu = 1/2$). If $\alpha := \sqrt{1 - \mu^2}$, then $0 < \alpha < 1$.

We will consider three possible cases:

1) $\sin x = 0$;

2) $|\sin x| \geq \mu$ or equivalently $|\cos x| \leq \alpha$;

3) $0 < |\sin x| < \mu$ or equivalently $|\cos x| > \alpha$.

Case 1. If $\sin x = 0$ then we have $|\cos x| = 1$. Thus we obtain that $|\cos x| - |\gamma||\cos(\xi x)| \geq 1 - |\gamma| =: \kappa_1 > 0$.

Now let us assume that $\sin x \neq 0$. Then we have $\sin(\xi x) \neq 0$ and $\gamma \neq 0$. So, we have $0 < |\sin x / \sin(\xi x)| = |\gamma|\xi^\beta = 1 - \varepsilon$, where $\varepsilon := 1 - |\gamma|\xi^\beta$, $0 < \varepsilon < 1$. Therefore, we obtain $\sin^2 x = \sin^2(\xi x)(1-\varepsilon)^2$ or

$$\cos^2 x = \cos^2(\xi x) + \varepsilon(2-\varepsilon)\sin^2(\xi x) \geq \cos^2(\xi x) + \varepsilon \sin^2(\xi x). \tag{3.8}$$

By (3.8), it follows that $|\cos x| > |\cos(\xi x)|$ and the following inequalities

$$0 < |\cos x| - |\cos(\xi x)| \leq |\cos x| - |\gamma||\cos(\xi x)| \tag{3.9}$$

are valid for Case 2 and Case 3.

Case 2. Let us take $\kappa_2 := \varepsilon\mu^2/3 > 0$. Since $0 < \kappa_2 < 1$ and $|\sin(\xi x)| > |\sin x|$, then we get

$$\varepsilon \sin^2(\xi x) > \varepsilon \sin^2 x \geq \varepsilon\mu^2 = 3\kappa_2 \geq 2\kappa_2 + \kappa_2^2 \geq 2\kappa_2|\cos(\xi x)| + \kappa_2^2. \tag{3.10}$$

Thus, by (3.8) and (3.10), it follows that

$$\cos^2 x \geq \cos^2(\xi x) + 2\kappa_2|\cos(\xi x)| + \kappa_2^2 = (|\cos(\xi x)| + \kappa_2)^2$$

or $|\cos x| \geq |\cos(\xi x)| + \kappa_2$. So, we have $|\cos x| - |\cos(\xi x)| \geq \kappa_2 > 0$. Thus, by (3.9), we prove $|\cos x| - |\gamma||\cos(\xi x)| \geq \kappa_2 > 0$.

Case 3. Let us denote $\kappa_3 := (1 - |\gamma|)\alpha$. By (3.9), it follows that

$$|\cos x| - |\gamma||\cos(\xi x)| \geq (1 - |\gamma|)|\cos x| \geq \kappa_3 > 0.$$

Consequently, if we choose $\kappa = \min\{\kappa_1, \kappa_2, \kappa_3\} > 0$ then we have $|\cos x| - |\gamma||\cos(\xi x)| \geq \kappa > 0$. □

Since $|\cos x_k - \gamma\xi\cos(\xi x_k)| \geq |\cos x_k| - |\gamma||\cos(\xi x_k)|$ we get the following corollary.

Corollary 3. Let x_k be a root of Equation (3.7). Then there exists $\kappa > 0$ such that $|\cos x_k - \gamma\xi\cos(\xi x_k)| \geq \kappa > 0$ for all $k \in \mathbb{N}$.

Lemma 5. *Let $|\gamma| < 1$, $0 < \xi < 1$, $\beta \geq 0$. If $\cos x - \gamma\xi^\beta\cos(\xi x) = 0$ then there exists $\tilde{\kappa} > 0$ such that $|\sin x| - |\gamma||\sin(\xi x)| \geq \tilde{\kappa} > 0$.*

The proof of this lemma is similar to the proof of Lemma 4. From inequality $|\sin x - \gamma\sin(\xi x)| \geq |\sin x| - |\gamma||\sin(\xi x)|$ we get the following corollary.

Corollary 4. Let $\tilde{x}_k$ be a root of equation $\cos x - \gamma\xi\cos(\xi x) = 0$. Then there exists $\tilde{\kappa} > 0$ such that $|\sin\tilde{x}_k - \gamma\sin(\xi\tilde{x}_k)| \geq \tilde{\kappa} > 0$ for all k.

Remark 4. Lemma 4 and Lemma 5 are valid for $\beta = \infty$ (in this case $\xi^\beta = 0$).

Corollary 5. Let $x = a_k := (k+1/2)\pi$, $k \in \mathbb{N}$ (in this case $\cos a_k = 0$). Then there exists $\tilde{\kappa} > 0$ such that $|\sin a_k| - |\gamma||\sin(\xi a_k)| \geq \tilde{\kappa} > 0$ for all k.

Let us denote $D_k = \{s : |x| \leq a_k = (k+1/2)\pi, |y| \leq a_k\}$, $D_k^s = D_k \cap \mathbb{C}_s$, $k \in \mathbb{N}$, and a contour $\Gamma_k^s = \partial D_k \cap \mathbb{C}_s$. Then we have $|s| \geq 3\pi/2$ on Γ_k^s, $k \in \mathbb{N}$. The corresponding contour Γ_k^λ in the the plane $\mathbb{C}_\lambda$ will be the boundary of the domain D_k^λ.

Lemma 6. *Let* $|\gamma| < 1$. *Then there exists* $q_1 > 0$ *such that all eigenvalues of the problem* (1.1)–(1.3) *in the domain* $\{s \in \mathbb{C}_s \colon |s| > q_1\}$ *are positive.*

Proof. On the vertical part of contour $s = a_k + \imath y$, $y \in [-a_k, a_k]$, $\operatorname{Re} h(s) = \sin a_k \cosh y - \gamma \sin(\xi a_k) \cosh(\xi y)$. We estimate

$$\begin{aligned}|h(s)| \geq& |\operatorname{Re} h(s)| \geq |\sin a_k| \cosh y - |\gamma|\, |\sin(\xi a_k)| \cosh(\xi y)\\ \geq& (|\sin a_k| - |\gamma|\, |\sin(\xi a_k)|) \cosh y.\end{aligned}$$

Using Corollary 5 we get $|h(s)| \geq \tilde{\kappa} \cosh y \geq A_1 \mathrm{e}^{|y|}$, where $A_1 > 0$. On the remaining part of contour $y = \pm a_k$, $0 \leq x \leq a_k$. From formulas

$$|\sin s| = \sqrt{\sinh^2 y + \sin^2 x} = \sqrt{\cosh^2 y - \cos^2 x}$$

we have

$$|\sin s| \geq \sinh |y|, \quad |\sinh(\xi s)| \leq \cosh(\xi y).$$

So,

$$|h(s)| \geq \sinh |y| - |\gamma|\, \cosh(\xi y) \geq \sinh |y| - \cosh(\xi y).$$

Consider a function $f(y) := (\sinh |y| - \cosh(\xi y))e^{-y}$, for $y \in [0, +\infty)$. It easy to see, that exist $y_*(\xi) > 0$ such that $f(y) \geq 1/4$. So, $|h(s)| \geq \mathrm{e}^{|y|}/4$ for $|y| > y_*$. Finally, taking $A = \min\{A_1, 1/4\}$, we have $|h(s)| \geq A\mathrm{e}^{|y|}$ on Γ_k for sufficiently large k.

From formula (3.4) $H(s) = h(s) + h_0(s)$ where $h_0(s) = \mathcal{O}(s^{-1}e^{|y|})$. Hence, we have $|h_0(s)| \leq c_1|s|^{-1}\mathrm{e}^{|y|} < A\mathrm{e}^{|y|} \leq |h(s)|$ on the contours Γ_k for sufficiently large k. Therefore, by Rouché theorem it follows that the number of zeros of $H = h + h_0$ and h are the same inside Γ_k for sufficiently large k.

In the domain between contours Γ_{k-1} and Γ_k there is exactly one positive root of the function h (see Remark 3). The function H has one root in this domain for sufficiently large k. But interval $\big((k-1/2)\pi, (k+1/2)\pi\big)$ belongs to this domain. So, the single root of H in this domain is positive (see Corollary 2). □

We can enumerate the zeros of H as s_k, $k \in \mathbb{N}$. The first zeros can be complex numbers or not simple. From Lemma 6 we have that s_k are positive for sufficiently large k. Now we will investigate the distribution of these positive eigenvalues of the problem (1.1)–(1.3) and we leave out the note about sufficiently large k.

Now we consider only real positive $s = x > 0$. In this case formula (3.4) may be rewritten in the form

$$H(s) = h(s) + \mathcal{O}(s^{-1}), \quad h(s) = \sin s - \gamma \sin(\xi s) \tag{3.11}$$

and

$$H'(s) = \cos s - \gamma\xi\cos(\xi s) + \mathcal{O}(s^{-1}). \tag{3.12}$$

Since x_k, $s_k \in \big((k-1/2)\pi, (k+1/2)\pi\big)$, we have

$$s_k \sim x_k \sim \pi k \text{ (as } k \to \infty).$$

Let us denote $\delta_k = s_k - x_k$. The functions H and h are analytic. So, from (3.12) and $H(s) = 0$ we have

$$s_k = x_k + o(1) \quad \text{or} \quad \delta_k = o(1) \quad (\text{as } k \to \infty).$$

From (2.4) we get the equality

$$\omega_s(t) = -\frac{\sin(st)}{s} + \mathcal{O}(s^{-2}). \tag{3.13}$$

Theorem 5. *Let $q \in C[0,1]$ and $|\gamma| < 1$. For eigenvalues $\lambda_k = s_k^2$ and eigenfunctions u_k of the problem* (1.1)–(1.3) *the asymptotic formulas*

$$s_k = x_k + \mathcal{O}(k^{-1}), \quad u_k(t) = -\frac{\sin(x_k t)}{x_k} + \mathcal{O}(k^{-2})$$

are valid for sufficiently large k.

Proof. Substituting $s_k = x_k + \delta_k$ into (3.12) we obtain

$$\sin x_k - \gamma\sin(\xi x_k) + \big(\cos x_k - \gamma\xi\cos(\xi x_k)\big)\delta_k + \mathcal{O}(\delta_k^2) = \mathcal{O}(k^{-1}).$$

Since $\sin x_k - \gamma\sin(\xi x_k) = 0$ we rewrite this equality as

$$\big(\cos x_k - \gamma\xi\cos(\xi x_k) + \mathcal{O}(\delta_k)\big)\delta_k = \mathcal{O}(k^{-1}).$$

Thus, by Corollary 3 we get $\delta_k = \mathcal{O}(k^{-1})$.

Substituting $s_k = x_k + \delta_k$ into equality (3.13), we find the asymptotic formula

$$\begin{aligned} u_k(t) = \omega_{\lambda_k}(t) &= -\frac{\sin\big((x_k+\delta_k)t\big)}{x_k+\delta_k} + \mathcal{O}(k^{-2}) \\ &= -\frac{\sin(x_k t)}{x_k} - \frac{x_k t\cos(x_k t) - \sin(x_k t)}{x_k^2}\delta_k + \mathcal{O}(\delta_k^2) + \mathcal{O}(k^{-2}) \\ &= -\frac{\sin(x_k t)}{x_k} + \mathcal{O}(k^{-2}). \end{aligned}$$

□

Remark 5. Normalized eigenfunctions are

$$v_k(t) = \sqrt{2}\sin(x_k t) + \mathcal{O}(k^{-1}).$$

Under the condition that $q \in C^1[0,1]$ the more exact asymptotic formulas may be obtained. In this case the following formulas

$$\int_0^t q(\tau)\cos(2s\tau)d\tau = \mathcal{O}(s^{-1}), \quad \int_0^t q(\tau)\sin(2s\tau)d\tau = \mathcal{O}(s^{-1}) \tag{3.14}$$

are valid for $t \in [0,1]$ [14]. Let $Q(t) = \frac{1}{2}\int_0^t q(\tau)d\tau$. It is obvious that the function $Q(t)$ is bounded for $0 \le t \le 1$.

Substituting the expression (3.13) into the integrals in (2.2) we have

$$\begin{aligned}\omega_s(t) = &-\frac{1}{s}\sin(st) + \frac{Q(t)\cos(st)}{s^2}\\ &- \frac{\cos(st)}{s^2}\int_0^t \cos(2s\tau)q(t)d\tau - \frac{\sin(st)}{s^2}\int_0^t \sin(2s\tau)q(t)d\tau + \mathcal{O}(s^{-3}).\end{aligned}$$

Using the formula (3.14) formulas we derive

$$\omega_s(t) = -\frac{1}{s}\sin(st) + \frac{Q(t)\cos(st)}{s^2} + \mathcal{O}(s^{-3}). \tag{3.15}$$

Then we have asymptotic formula

$$H(s) = \sin s - \gamma\sin(s\xi) - \frac{Q(1)\cos s - \gamma Q(\xi)\cos(\xi s)}{s} + \mathcal{O}(s^{-2}). \tag{3.16}$$

Let us denote

$$Q_1(s) = Q_1(s;\gamma,\xi) := \frac{Q(1)\cos s - \gamma Q(\xi)\cos(\xi s)}{\cos s - \gamma\xi\cos(\xi s)}.$$

Theorem 6. *Let $q \in C^1[0,1]$ and $|\gamma| < 1$. For eigenvalues $\lambda_k = s_k^2$ and eigenfunctions u_k of the problem (1.1)–(1.3) the asymptotic formulas*

$$\begin{aligned} s_k &= x_k + Q_1(x_k)x_k^{-1} + \mathcal{O}(k^{-2}),\\ u_k(t) &= -\frac{\sin(x_k t)}{x_k} + \big(Q(t) - tQ_1(x_k)\big)\frac{\cos(x_k t)}{x_k^2} + \mathcal{O}(k^{-3}) \end{aligned} \tag{3.17}$$

are valid for sufficiently large k.

Proof. Substituting $s_k = x_k + \delta_k$ into (3.16), we have

$$\begin{aligned}&\sin x_k - \gamma\sin(\xi x_k) - \frac{Q(1)\cos x_k - \gamma Q(\xi)\cos(\xi x_k)}{x_k}\\ &+ \big(\cos x_k - \gamma\xi\cos(\xi x_k) + \big(Q(1)\sin x_k - \gamma\xi Q(\xi)\sin(\xi x_k)\big)x_k^{-1}\big)\delta_k\\ &+ \mathcal{O}(\delta_k^2) = \mathcal{O}(k^{-2}).\end{aligned}$$

Since $\sin x_k - \gamma\sin(\xi x_k) = 0$ we rewrite this equality as

$$\big(\cos x_k - \gamma\xi\cos(\xi x_k) + \mathcal{O}(k^{-1})\big)\delta_k = \frac{Q(1)\cos x_k - \gamma Q(\xi)\cos(\xi x_k)}{x_k} + \mathcal{O}(k^{-2})$$

or

$$\delta_k = \frac{Q(1)\cos x_k - \gamma Q(\xi)\cos(\xi x_k)}{x_k\big(\cos x_k - \gamma\xi\cos(\xi x_k)\big)} + \mathcal{O}(k^{-2}) = \frac{Q_1(x_k)}{x_k} + \mathcal{O}(k^{-2}).$$

Now, we are ready to obtain a sharper asymptotic formula for the eigenfunctions. Substituting $s_k = x_k + \delta_k$ into (3.15), we have

$$u_k(t) = -\frac{\sin(x_k t)}{x_k} + \frac{Q(t)\cos(x_k t)}{x_k^2} - \frac{t\cos(x_k t)}{x_k}\delta_k + \mathcal{O}(k^{-3}).$$

Since $\delta_k = Q_1(x_k)x_k^{-1} + \mathcal{O}(k^{-2})$ we derive (3.17). □

Remark 6. To obtain this asymptotic expansion for the normalized eigenfunctions $v_k(t) = -\alpha_k^{-1}u_k(t)$ let us consider the integral

$$\begin{aligned}\alpha_k^2 = &\int_0^1 u_k^2(t)dt = \frac{1}{x_k^2}\int_0^1 \sin^2(x_k t)dt\\ &+ \frac{1}{x_k^4}\int_0^1 \cos^2(x_k t)(Q_1(x_k))^2 t^2 dt + \frac{1}{x_k^3}\int_0^1 Q_1(x_k)t\sin(2x_k t)dt + \mathcal{O}(k^{-4})\\ =&\frac{1}{2x_k^2}\big(1 - \frac{1}{2x_k}\sin(2x_k) + \mathcal{O}(k^{-2})\big).\end{aligned}$$

Thus, the normalizing coefficients

$$\alpha_k^{-1} = \sqrt{2}x_k\big(1 + \frac{1}{4x_k}\sin(2x_k) + \mathcal{O}(k^{-2})\big).$$

Then we have

$$\begin{aligned}v_k(t) =&\sqrt{2}\big(\sin(x_k t) - \big(Q(t) - tQ_1(x_k)\big)\frac{\cos(x_k t)}{x_k} + \mathcal{O}(k^{-2})\big)\\ &\times\big(1 + \frac{\sin(2x_k)}{4x_k} + \mathcal{O}(k^{-2})\big).\end{aligned}$$

So, normalized eigenfunctions are

$$\begin{aligned}v_k(t) =&\sqrt{2}\sin(x_k t)\\ &+\sqrt{2}\,\frac{0.25\sin(2x_k)\sin(x_k t) - \big(Q(t) - tQ_1(x_k)\big)\cos(x_k t)}{x_k} + \mathcal{O}(k^{-2}).\end{aligned}$$

Remark 7. Assuming that $q \in C^2[0,1]$, one can prove a more precise asymptotic formula

$$s_k = x_k + Q_1(x_k)x_k^{-1} + Q_2(x_k)x_k^{-2} + \mathcal{O}(k^{-3}).$$

Here Q_2 is a bounded function.

Remark 8. In the case of $\gamma = 0$ we get the classical case and for sufficiently large k, again, under the condition that $q \in C[0,1]$ we have the formulas

$$s_k = k\pi + \mathcal{O}(k^{-1}), \quad u_k(t) = -\frac{\sin(k\pi t)}{k\pi} + \mathcal{O}(k^{-2}),$$

and under the condition that $q \in C^1[0,1]$ we have the formulas

$$s_k = k\pi + \frac{Q(1)}{k\pi} + \mathcal{O}(k^{-2}),$$
$$u_k(t) = -\frac{\sin(k\pi t)}{k\pi} + \big(Q(t) - tQ(1)\big)\frac{\cos(k\pi t)}{k^2\pi^2} + \mathcal{O}(k^{-3})$$

for eigenvalues and eigenfunctions, respectively [14].

4 Conclusions

In this paper, the spectrum and asymptotic formulas of eigenfunctions for Sturm–Liouville problem with one Bitsadze–Samarskii type nonlocal boundary condition was investigated. The results obtained in this work can be extended to differential equations with retarded argument [20]. Furthermore, asymptotics of eigenvalues and eigenfunctions of the same differential equation but with different boundary conditions such as integral boundary conditions can be also investigated.

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