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A New Numerical Method to Solve Nonlinear Volterra-Fredholm Integro-Differential Equations

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Abstract. In this paper, a new method combining the simplified reproducing kernel method (SRKM) and the homotopy perturbation method (HPM) to solve the nonlinear Volterra-Fredholm integro-differential equations (V-FIDE) is proposed. Firstly the HPM can convert nonlinear problems into linear problems. After that we use the SRKM to solve the linear problems. Secondly, we prove the uniform convergence of the approximate solution. Finally, some numerical calculations are proposed to verify the effectiveness of the approach.

Keywords: nonlinear Volterra-Fredholm integro-differential equations, simplified reproducing kernel method, homotopy perturbation method.

AMS Subject Classification: 45G10; 45J05.

1 Introduction

This article mainly discusses the nonlinear V-FIDE:

$$Yu(x) + H(u(x)) = y(x), \quad u(a) = \alpha, \tag{1.1}$$

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where

$$Yu(x) = u'(x) + q(x)u(x),$$

$$H(u(x)) = \lambda_1 \int_0^x K_1(x,t)F(u(t))\,dt + \lambda_2 \int_0^1 K_2(x,t)G(u(t))\,dt.$$

The parameters λ_1, λ_2 are constants. $F(u(x))$ and $G(u(x))$ are constant coefficient polynomials of $u(x)$. The V-FIDE has been widely used in physics, biological and engineering [1, 2, 3, 10, 16]. In order to obtain accurate numerical solutions more quickly, many methods for solving such problems have been proposed in recent years. Maleknejad [10] introduced the hybrid functions method. Babolian [2, 3] proposed the triangular functions method and the operational matrix with block-pulse functions. Hybrid Legendre polynomials and block-pulse functions approach were used by Maleknejad [9]. Bakodah [4] discussed the Laplace discrete Adomian decomposition method over the integro-differential equation. Biazar and Ghanbari [5] presented He's homotopy perturbation method. Bildik [6] used the modified decomposition method to obtain the approximate solution of nonlinear V-FIDE. Ghasemi [8] formulated homotopy perturbation method for solving nonlinear equations. In recent years, with the development of reproducing kernel space theory, many scholars have successfully applied reproducing kernel method to solve problems [12,13,14,15,17,18]. But the traditional reproducing kernel method [11] is difficult to deal with the integral term, while the HPM can be effectively dealt with the integral term. Because the traditional reproducing kernel method needs orthogonalization, the calculation method is complex and time-consuming. Our method avoids the Smith orthogonalization process in order to save the calculation time and running memory. This article discusses the nonlinear V-FIDE by using SRKM and HPM in the reproducing kernel space, so that the equation can achieve higher accuracy.

In this paper, we describe the homotopy perturbation theory in Section 2. The reproducing kernel theory will be shown in Sections 3 and 4. The last part presents some numerical examples. In the end, we have the conclusions.

2 Homotopy perturbation method

For Equation (1.1), we first have to solve the nonlinear part. The homotopy perturbation method provides a good theoretical basis, we embed a small parameter p ($p \in [0,1]$) by constructing a homotopy map

$$Yu(x) + pH(u(x)) = y(x), \quad u(a) = \alpha, \tag{2.1}$$

when $p = 0$, the Equation (2.1) is an initial value problem:

$$Yu(x) = y(x), \quad u(a) = \alpha, \tag{2.2}$$

when $p = 1$, Equation (2.1) is the original problem (1.1). The parameter p changes from 0 to 1, then the solution $u(x)$ follows the homotopy path from Equation (2.2) to the original problem Equation (2.1). And the solutions that satisfy the homotopy path can be expanded into a power series of p:

$$u(x,p)=\sum_{n=0}^{\infty}p^n u_n(x).$$

In this way, when $p\to 1$, the approximate solution of the nonlinear operator equation is obtained

$$u(x)=\lim_{p\to 1}u(x,p)=\sum_{n=0}^{\infty}u_n(x).$$

Take the k derivatives of F,G and set $p=0$, then substitute the type into Equation (2.1):

$$\sum_{n=0}^{\infty}p^n Yu_n(x)+p\lambda_1\int_0^x K_1(x,t)\sum_{k=0}^{\infty}A_k p^k dt+p\lambda_2\int_0^1 K_2(x,t)\sum_{k=0}^{\infty}B_k p^k dt=y(x),$$

where

$$\sum_{k=0}^{\infty}A_k p^k=F(\sum_{n=0}^{\infty}p^n u_n(x))=\sum_{k=0}^{\infty}\frac{1}{k!}\frac{d^k}{dp^k}F(\sum_{n=0}^{\infty}p^n u_n(x))\mid_{p=0},$$
$$\sum_{k=0}^{\infty}B_k p^k=G(\sum_{n=0}^{\infty}p^n u_n(x))=\sum_{k=0}^{\infty}\frac{1}{k!}\frac{d^k}{dp^k}G(\sum_{n=0}^{\infty}p^n u_n(x))\mid_{p=0}.$$

Comparing the coefficients of p^i on both sides of equation and setting them equal, we can get for $k=0$,

$$Yu_0(x)=y(x),\quad u_0(a)=\alpha, \tag{2.3}$$

for p^{k+1},

$$\begin{cases}Yu_k(x)=-\lambda_1\int_0^x K_1(x,t)\sum\limits_{k=0}^{\infty}A_k dt-\lambda_2\int_0^1 K_2(x,t)\sum\limits_{k=0}^{\infty}B_k dt,\\ u_k(a)=\alpha.\end{cases} \tag{2.4}$$

Adding the solution u_k of Equations (2.3)–(2.4), we obtain the true solution to nonlinear equations

$$u_n(x)=u_0(x)+u_1(x)+u_2(x)+\dots.$$

3 Reproducing kernel Hilbert space

We will discuss the Equation (1.1) with support of the reproducing kernel space theory.

Definition 1. ([7]) Let H be the Hilbert space, and the elements in H are complex-valued functions on X. If there is a unique function $K_s(t)$ for $\forall s\in X$ that satisfies

$$\langle f,K_s\rangle=f(s),\quad f\in H.$$

Then H is defined as reproducing kernel space, $K(s,t)=K_s(t)$ is defined as reproducing kernel function.

To solve Equation (1.1), we need to introduce two reproducing kernel spaces next. The inner product and the norm of reproducing kernel Hilbert space $W_2^2[a,b]$ and $W_2^1[a,b]$ are defined as

Definition 2. ([7])

$$W_2^2[a,b] = \{u(x) \mid u'(x) \text{ is an absolutely continuous real value function}, \; u''(x) \in L^2[a,b]\}, \quad \langle u,v\rangle_{W_2^2} = u(a)v(a) + u'(a)v'(a) + \int_a^b u''v''dx, \quad u,v \in W_2^2[a,b], \quad \|u\|_{W_2^2} = \sqrt{\langle u(x),u(x)\rangle_{W_2^2}}.$$

Definition 3. ([7])

$$W_2^1[a,b] = \{u(x) \mid u(x) \text{ is an absolutely continuous real value function}, \; u'(x) \in L^2[a,b]\}, \quad \langle u,v\rangle_{W_2^1} = u(a)v(a) + \int_a^b u'v'dx, \quad u,v \in W_2^1[a,b], \quad \|u\|_{W_2^1} = \sqrt{\langle u(x),u(x)\rangle_{W_2^1}}.$$

Theorem 1. *([7]) The reproducing kernel function $R_s(t)$ of $W_2^2[a,b]$ is defined as*

$$R_s(t) = \begin{cases} st + \frac{st^2}{2} - \frac{t^3}{6}, & t \le s, \\ st + \frac{ts^2}{2} - \frac{s^3}{6}, & s \le t. \end{cases}$$

Theorem 2. *([7]) The reproducing kernel function $r_s(t)$ of $W_2^1[a,b]$ is defined as*

$$r_s(t) = \begin{cases} 1 - a + s, t \le s, \\ 1 - a + t, s \le t. \end{cases}$$

4 The combination of HPM and SRKM

We described the HPM for nonlinear equation in Section 2. Equations (2.3) and (2.4) can be considered as follows

$$\begin{cases} Yu_n(x) = f(x), \\ u_n(a) = \alpha, \; n = 0,1,2,\dots. \end{cases} \tag{4.1}$$

Now we introduce reproducing kernel method. Firstly, we define a linear operator $L : W_2^2[a,b] \to W_2^1[a,b]$,

$$Lu(x) = Yu_n(x), \; u(x) \in W_2^2[a,b].$$

Therefore, Equation (4.1) can be expressed as

$$\begin{cases} Lu(x) = f(x), \\ u_n(a) = \alpha, \; n = 0,1,2,\dots. \end{cases} \tag{4.2}$$

It's easy to prove that L is a bounded linear operator. L^* is the adjoint operator of L. Let $\phi_i(x) = L^* r_s(t)(x_i),\ i = 1, 2, \ldots$, where $L^* r_s(t)(x_i) = \langle L^* r_s, R_t\rangle_{W_2^2} = \langle r_s, LR_t\rangle_{W_2^1} = (LR_t)(s) = (LR_s)(t)$, and $\{x_i\}$ is subset on $[a, b]$.

Theorem 3. *$\{x_i\}_{i=1}^{\infty}$ in $[a, b]$ is a set of mutually distinct dense points, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is a complete system on $W_2^2[a, b]$.*

Proof. Assume $\sum_{i=1}^{n} c_i\psi_i(x) = 0$, because L is invertible, and

$$\sum_{i=1}^{n} c_i\psi_i(x) = \sum_{i=1}^{n} c_i L R_{x_i}(x) = L(\sum_{i=1}^{n} c_i R_{x_i}(x)) = 0.$$

In addition, for $u(x) \in W_2^2[a, b]$, if $\langle u(x), \psi_i\rangle_{W_2^2} = u(x_i) = 0,\ i = 1, 2, \ldots$, then $u(x) \equiv 0$. □

Let $\psi_1(x) = R(x, a)$ and $S_{n+1} = span\{\phi_1(x), \phi_2(x), \ldots, \phi_n(x), \psi_1(x)\}$. We can obtain the following conclusions:

Theorem 4. *Define $P_n : W_2^2[a, b] \to S_{n+1}[a, b]$, then $u_n = P_n u$ satisfies:*

$$\langle u_n, \phi_k\rangle = f(x_k, u(x_k)), \quad k = 1, 2, \ldots, n, \quad \langle u_n, \psi_1\rangle = \alpha. \tag{4.3}$$

Proof.

$$\begin{aligned}\langle P_n u, \phi_k\rangle_{W_2^2} &= \langle u, \phi_k\rangle_{W_2^2} = \langle u, L^* r_{x_k}\rangle_{W_2^2} = \langle Lu, r_{x_k}\rangle_{W_2^1} = Lu(x_k)\\ &= f(x_k, u(x_k)), \quad k = 1, 2, \ldots, n.\\ \langle P_n u, \psi_1\rangle_{W_2^2} &= \langle u, P_n\psi_1\rangle_{W_2^2} = \langle u, \psi_1\rangle_{W_2^2} = \langle u, R_a\rangle_{W_2^2} = u(a) = \alpha.\end{aligned}$$

□

Theorem 5. *The approximate solution $u_n(x)$ uniformly converges to $u(x)$ on $[a, b]$.*

Proof. $\|u_n - u\| \to 0$ holds as $n \to \infty$ in $W_2^2[a, b]$. According to the reproducibility of the reproducing kernel function, we have

$$u_n(x) - u(x) = \langle u_n - u, R_x(y)\rangle,$$

thus

$$| u_n(x) - u(x) | = | \langle u_n - u, R_x(y)\rangle_{W_2^2} | \le \|u_n - u\|_{W_2^2} \|R_x(y)\|_{W_2^2},$$

because of the boundedness of the reproducing kernel function, $\|R_x(y)\| \le M_0$, then

$$| u_n(x) - u(x) | \le M_0 \|u_n - u\|_{W_2^2} \to 0.$$

□

Consequently, the exact solution $u_n \in S_{n+1}$ of Equation (4.2) can be developed as follows

$$u_n(x) = \sum_{k=1}^{n} q_k \phi_k(x) + p_1 \psi_1. \tag{4.4}$$

Then applying the form Equation (4.4) to Equation (4.3), we can obtain

$$p_1 \langle \psi_1, \psi_1 \rangle + \sum_{k=1}^{n} q_j \langle \phi_k(x), \psi_1 \rangle = \alpha,$$

$$p_1 \langle \psi_1, \phi_j \rangle + \sum_{k=1}^{n} q_j \langle \phi_k(x), \phi_j \rangle = f(x_j, u(x_j)), \quad j = 1, 2, \ldots, n.$$

Note that

$$A = \begin{pmatrix} \langle \psi_1, \psi_1 \rangle & \langle \phi_1, \psi_1 \rangle & \langle \phi_2, \psi_1 \rangle & \ldots & \langle \phi_n, \psi_1 \rangle \\ \langle \psi_1, \phi_1 \rangle & \langle \phi_1, \phi_1 \rangle & \langle \phi_1, \phi_1 \rangle & \ldots & \langle \phi_n, \phi_1 \rangle \\ \langle \psi_1, \phi_2 \rangle & \langle \phi_1, \phi_2 \rangle & \langle \phi_2, \phi_2 \rangle & \ldots & \langle \phi_n, \phi_2 \rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle \psi_1, \phi_n \rangle & \langle \phi_1, \phi_n \rangle & \langle \phi_2, \phi_n \rangle & \ldots & \langle \phi_n, \phi_n \rangle \end{pmatrix},$$

$$f = \begin{pmatrix} \alpha \\ f(x_1, u(x_1)) \\ f(x_2, u(x_2)) \\ \vdots \\ f(x_n, u(x_n)) \end{pmatrix}.$$

Thus, we just have to compute $(p_1, q_1, q_2, \ldots, q_n)^T = A^{-1} f$.

5 Numerical examples

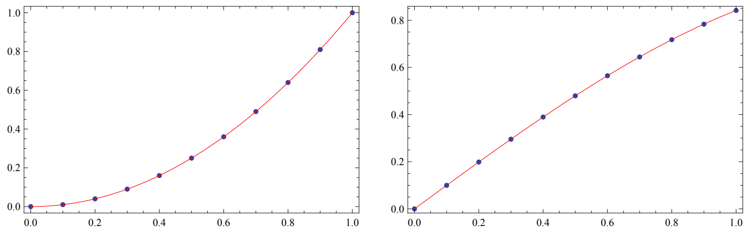
The main methods used in this work have been described in the previous sections, some numerical examples are given to illustrate its effectiveness. Meanwhile, the red lines in the figure represent the approximate solutions and the blue dots represent the exact solutions. The absolute errors of the exact, the approximate solutions and CPU time (seconds) are listed in the tables. We also used the following formula to calculate the convergence rate r:

$$r = \log 2 \frac{\|e_n\|}{\|e_{2n}\|}.$$

Example 1. For the following nonlinear V-FIDE:

$$\begin{cases} u'(x) + u(x) + \frac{1}{2} \int_0^x x u^2(t) dt - \frac{1}{4} \int_0^1 t u^3(t) dt = y(x), \\ u(0) = 0, \end{cases}$$

where $y(x) = 2x + x^2 + \frac{1}{10} x^6 - \frac{1}{32}$, the exact solution is $u(x) = x^2$ (see Figure 1(a)). The comparison of the numerical results and the absolute error are listed in Table 1, we get an exact solution with higher precision than the method of hybrid Legendre polynomials and Block-Pulse functions [9] for $n = 12$.



a) b)

Figure 1. Approximate solutions: a) Example 1, b) Example 2.

Table 1. Numerical result and absolute error for Example 1.

x	Exact solution	Presented method	In [9]	Absolute error in [9]	Absolute error
0.0	0.000000	0.000000	0.000000	0.00000	0.00000
0.1	0.010000	0.010044	0.010917	9.100E-4	4.4259E-5
0.2	0.040000	0.040094	0.041703	1.703E-3	9.3974E-5
0.3	0.090000	0.090146	0.092364	2.364E-3	1.4657E-4
0.4	0.160000	0.160198	0.162911	2.911E-3	1.9813E-4
0.5	0.250000	0.250243	0.253371	3.371E-3	2.4307E-4
0.6	0.360000	0.360289	0.364244	4.244E-3	2.8891E-4
0.7	0.490000	0.490346	0.493830	3.830E-3	3.4553E-4
0.8	0.640000	0.640408	0.642375	2.375E-3	4.0786E-4
0.9	0.810000	0.810468	0.810337	3.370E-4	4.6836E-4
1.0	1.000000	1.000520	0.998506	1.494E-3	5.1647E-4
r		$r_6 = 1.99$		$r_{12} = 1.99$	
CPU			$1.75s$		

Example 2. For the following V-FIDE:

$$\begin{cases} u'(x)+u(x)-2\int_0^x \sin(x)u^2(t)dt=\cos(x)+(1-x)\sin(x)+\cos(x)\sin^2(x), \\ u(0)=0. \end{cases}$$

The exact solution is $u(x)=\sin(x)$ (see Fig. 1 (b))

Table 2 illustrates the numerical results and the absolute error. From the Table 2 results, we can see that our method approximates the exact solution more closely than the hybrid Legendre polynomials and Block-Pulse functions [9] for $n=12$.

Example 3. Consider the nonlinear V-FIDE:

$$\begin{cases} u'(x)+\int_0^x (u^2(t)-2)dt=\frac{1}{5}x^5, \\ u(0)=0. \end{cases}$$

with the exact solution given by $u(x)=x^2$. The comparison of the numerical results and the absolute error are listed in Table 3, our method is more accurate than the method of Laplace discrete adomian decomposition in [4] for $n=4$.

Table 2. Numerical result and absolute error for Example 2.

x	Exact solution	Presented method	In [9]	Absolute error in [9]	Absolute error
0.0	0.000000	0.000000	0.000032	3.2E-5	1.82077E-14
0.1	0.099833	0.099793	0.099435	3.98417E-4	3.99348E-5
0.2	0.198669	0.198584	0.198304	3.65331E-4	8.53956E-5
0.3	0.295520	0.295387	0.295493	2.72067E-4	1.33170E-4
0.4	0.389418	0.389239	0.389688	2.69658E-4	1.79180E-4
0.5	0.479425	0.479207	0.479311	1.14539E-4	2.18684E-4
0.6	0.564642	0.564383	0.562965	1.67747E-3	2.59684E-4
0.7	0.644217	0.643906	0.640005	4.21269E-3	3.11780E-4
0.8	0.717356	0.716984	0.708103	9.25309E-3	3.71839E-4
0.9	0.783326	0.782891	0.764843	1.84839E-2	4.36266E-4
1.0	0.841470	0.840970	0.807845	3.36260E-2	5.01073E-4
r		$r_6 = 2.00$		$r_{12} = 1.99$	
CPU			$0.923s$		

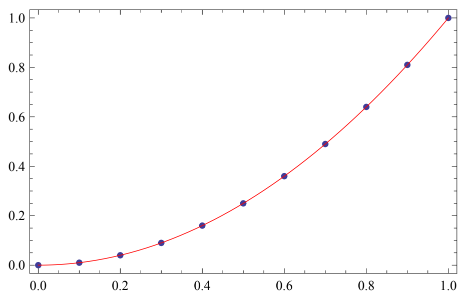


Figure 2. Approximate solution of Example 3.

Table 3. Numerical result and absolute error for Example 3.

x	Exact sol.	Presented method	In [4]	Abs. error in [4]	Abs. error
0.0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
0.2	0.04000000	0.040000000	0.03999986	1.41640E-7	4.68375E-17
0.4	0.16000000	0.160000000	0.15999094	9.05930E-6	2.77556E-17
0.6	0.36000000	0.360000000	0.35989712	1.02879E-4	1.11022E-16
0.8	0.64000000	0.64000000	0.63942742	5.72582E-4	2.22045E-16
1.0	1.00000000	1.00000000	0.99787295	2.12705E-3	1.11022E-15
CPU			$0.579s$		

6 Conclusions

In this article, the SRKM and the HPM were successfully applied to figure out the nonlinear V-FIDE by getting the uniform approximate solution. Besides, compared with the method of Hybrid Legendre polynomials [9], Laplace discrete adomian decomposition method [4], the convergence speed and accuracy of solution were better.

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