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Implementing Reproducing Kernel Method to Solve Singularly Perturbed Convection-Diffusion Parabolic Problems

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Abstract. In the present paper, reproducing kernel method (RKM) is introduced, which is employed to solve singularly perturbed convection-diffusion parabolic problems (SPCDPPs). It is noteworthy to mention that regarding very serve singularities, there are regular boundary layers in SPCDPPs. On the other hand, getting a reliable approximate solution could be difficult due to the layer behavior of SPCDPPs. The strategy developed in our method is dividing the problem region into two regions, so that one of them would contain a boundary layer behavior. For more illustrations of the method, certain linear and nonlinear SPCDPP are solved.

Keywords: reproducing kernel method, singularly perturbed parabolic problems, error analysis, boundary layer behavior.

AMS Subject Classification: 35K20; 35K67; 35K55.

1 Introduction

Let us consider the following singularly perturbed convection-diffusion parabolic problem,

$$
\begin{cases}
\mathcal{L}(y(x,t)) + \mathcal{N}(y(x,t)) = f(x,t), \\
(x,t) \in D \equiv [0,1] \times [0,1], \\
y(0,t) = y(1,t) = 0, \quad t \in [0,1], \\
y(x,0) = y_0(x), \quad x \in [0,1],
\end{cases}
\tag{1.1}
$$

where $\mathcal{L}y \equiv -\varepsilon \partial_x^2 y(x,t) + p(x,t)\partial_x y(x,t) + q(x,t)y(x,t) + \partial_t y(x,t)$ and $0 < \varepsilon \ll 1$, is perturbation parameter and $p(x,t), q(x,t), f(x,t)$ are sufficiently smooth functions such that, $p(x,t) \geq \alpha > 0, q(x,t) \geq \beta \geq 0$, and $\mathcal{N}(y(x,t))$ is nonlinear differential operator. Problem (1.1) with the above-mentioned conditions is of a unique solution $y(x,t)$ with boundary layer behavior, in which the boundary layer width is $O(\varepsilon\,ln(\frac{1}{\varepsilon}))$ at neighborhood $x = 1$, see [22]. SPCDPPs have several applications in various fields of science; for example, the fluid dynamics, electromagnetic field problems, semiconductor device modeling, and meteorological, biological and chemical applications [6, 12, 18, 20, 23, 24, 30]. Some applications of the RKM and singularly perturbed problems are introduced in [14, 16, 26, 28]. In addition, we can observe applications of the RKM method to solve fractional problems in [1, 2, 3, 4, 5, 9, 27, 29].

In this study, the RKM is utilized without the Gram-Schmidt orthogonalization process, which was primarily introduced by Wang et al., [32, 33, 34]. The strategy developed in order to solve these problems with layer behavior is explained in three steps. The first step is splitting the region into two regions, so that one of them would contain a boundary layer behavior; the second step is shifting the layer region to another region, and as the final step, a proper set of collocation points is also required for the boundary layer and regular regions. Since the RKM is a powerful numerical method, if these three steps are properly applied to the problem, this technique will be able to provide an appropriate approximation of the solution, even with severe singularities.

Remark 1. Problem (1.1) with two regions, such that one of the regions contains a boundary layer behavior, therefore $D = D_1 \cup D_2$, where $D_1 \equiv [0, 1-\mu] \times [0,1]$ and $D_2 \equiv [1-\mu, 1] \times [0,1]$. Since, the RKM does not provide an approximate solution for the right layer region, it is essential to shift region D_2 to another region, such as $D_3 \equiv [-1, 0] \times [0,1]$. Further details are provided in [34].

2 Main idea

2.1 Preliminaries and notations

Definition 1. $\mathcal{W}_2^m[a,b] = \{u(x)|u^{(m-1)}(x) \text{ is absolutely continuous}, u^{(m)}(x) \in L^2[a,b], u(a) = u(b) = 0\}$. *The inner product and norm in* $\mathcal{W}_2^m[a,b]$ *are given*

as follows,

$$< u_1(x), u_2(x) >_{\mathcal{W}_2^m[a,b]} = \sum_{i=0}^{m-1} u_1^{(i)}(a) u_2^{(i)}(a) + \int_a^b u_1^{(m)}(x) u_2^{(m)}(x) dx,$$

$$\|u(x)\|_{\mathcal{W}_2^m[a,b]} = \sqrt{< u, u >_{\mathcal{W}_2^m[a,b]}}, \qquad u_1(x), u_2(x) \in \mathcal{W}_2^m[a,b].$$

Definition 2. $\mathcal{W}_2^{(m,n)}(\mathcal{D}) = \mathcal{W}_2^m[a,b] \otimes \mathcal{W}_2^n[c,d] = \left\{ u(x,t) | \frac{\partial^{m+n-2} u(x,t)}{\partial x^{m-1} \partial t^{n-1}} \text{ is completely continuous in } \mathcal{D}, \frac{\partial^{m+n} u(x,t)}{\partial x^m \partial t^n} \in L^2(\mathcal{D}), u(a,t) = u(b,t) = u(x,c) = 0 \right\}.$
The inner product and norm in $\mathcal{W}_2^{(m,n)}(\mathcal{D})$ are given as follows,

$$< u_1(x,t), u_2(x,t) >_{\mathcal{W}_2^{(m,n)}(\mathcal{D})} = \sum_{i=0}^{m-1} \int_c^d \left[\frac{\partial^n}{\partial t^n} \frac{\partial^i}{\partial x^i} u_1(a,t) \frac{\partial^n}{\partial t^n} \frac{\partial^i}{\partial x^i} u_2(a,t) \right] dt$$

$$+ \sum_{j=0}^{n-1} \left\langle \frac{\partial^j}{\partial t^j} u_1(x,c), \frac{\partial^j}{\partial t^j} u_2(x,c) \right\rangle_{\mathcal{W}_2^m[a,b]}$$

$$+ \int_a^b \int_c^d \frac{\partial^m}{\partial x^m} \frac{\partial^n}{\partial t^n} u_1(x,t) \frac{\partial^m}{\partial x^m} \frac{\partial^n}{\partial t^n} u_2(x,t) dx dt,$$

$$\|u(x,t)\|_{\mathcal{W}_2^{(m,n)}(\mathcal{D})} = \sqrt{< u, u >_{\mathcal{W}_2^{(m,n)}(\mathcal{D})}}, \quad u_1(x,t), u_2(x,t) \in \mathcal{W}_2^{(m,n)}(\mathcal{D}).$$

Remark 2. Reproducing kernel for spaces $\mathcal{W}_2^3[0, 1-\mu]$, $\mathcal{W}_2^2[0,1]$ and $\mathcal{W}_2^3[-1,0]$ are given as follow,

$$\dot{\mathcal{R}}_\eta(x) = \begin{cases} \dot{\mathcal{R}}(x,\eta), & x \leq \eta, \\ \dot{\mathcal{R}}(\eta,x), & \eta > x, \end{cases} \quad \dot{\mathcal{R}}(x,\eta) = \frac{x^5}{120} - \frac{\eta x^4}{24} + \frac{\eta^2 x^3}{12} + \frac{\eta^2 x^2}{4} + \eta x,$$

$$\mathring{\mathcal{R}}_\xi(t) = \begin{cases} \mathring{\mathcal{R}}(t,\xi), & t \leq \xi, \\ \mathring{\mathcal{R}}(\xi,t), & \xi > t, \end{cases} \quad \mathring{\mathcal{R}}(t,\xi) = -\frac{t^3}{6} + \frac{\xi t^2}{2} + \xi t,$$

$$\ddot{\mathcal{R}}_\eta(x) = \begin{cases} \ddot{\mathcal{R}}(x,\eta), & x \leq \eta, \\ \ddot{\mathcal{R}}(\eta,x), & \eta > x, \end{cases} \quad \ddot{\mathcal{R}}(x,\eta) = -\frac{\eta^5}{120} - \frac{\eta^5 x^5}{18720} - \frac{\eta^2 x^5}{468} - \frac{\eta x^5}{96}$$

$$- \frac{\eta x^4}{24} + \frac{\eta^2 x^3}{12} - \frac{\eta^5 x^2}{468} + \frac{97\eta^2 x^2}{234} + \frac{\eta x^2}{3} - \frac{\eta^5 x}{96} + \frac{\eta^2 x}{3} + \frac{29\eta x}{96}.$$

2.1.1 Reproducing kernel spaces for regular region $\mathbf{D_1} \equiv [\mathbf{0}, \mathbf{1} - \mu] \times [\mathbf{0}, \mathbf{1}]$

Consider reproducing kernel spaces $\mathcal{W}_2^{(3,2)}(D_1)$ and $\mathcal{W}_2^{(1,1)}(D_1)$ where their reproducing kernels are given as $\dot{\mathcal{K}}_{\eta,\xi}(x,t) = \dot{\mathcal{R}}_\eta(x) \mathring{\mathcal{R}}_\xi(t)$ and $\dot{k}_{\eta,\xi}(x,t) = \dot{r}_\eta(x) \mathring{r}_\xi(t)$, respectively.

2.1.2 Reproducing kernel spaces for shifted boundary-layer region $\mathbf{D_3} \equiv [-\mathbf{1}, \mathbf{0}] \times [\mathbf{0}, \mathbf{1}]$

Consider reproducing kernel spaces $\mathcal{W}_2^{(3,2)}(D_3)$ and $\mathcal{W}_2^{(1,1)}(D_3)$ where their reproducing kernels are given as, $\ddot{\mathcal{K}}_{\eta,\xi}(x,t) = \ddot{\mathcal{R}}_\eta(x) \mathring{\mathcal{R}}_\xi(t)$ and $\ddot{k}_{\eta,\xi}(x,t) = \ddot{r}_\eta(x) \mathring{r}_\xi(t)$, respectively. See [11].

2.2 Reproducing kernel method

Consider $\mathcal{L} : \mathcal{W}_2^{(3,2)}(D_1) \longrightarrow \mathcal{W}_2^{(1,1)}(D_1)$ for the regular region in equation (1.1) where $\mathcal{L}$ is a invertible bounded linear differential operator, and $\mathcal{N}(y(x,t))$ is a continuous nonlinear differential operator, and $y(x,t)$ is an unknown function. The functions $\dot{\mathcal{K}}_{\eta,\xi}(x,t)$ and $\dot{k}_{\eta,\xi}(x,t)$ are reproducing kernels of $\mathcal{W}_2^{(3,2)}(D_1)$ and $\mathcal{W}_2^{(1,1)}(D_1)$ respectively. Now we choose a countable dense points $\{(x_i, t_j)\}_{i,j=1}^{\infty}$ on the region D_1 and define,

$$\phi_{ij}(x,t) = \dot{k}_{\eta,\xi}(x,t)|_{(\eta,\xi)=(x_i,t_j)}, \quad \psi_{ij}(x,t) = \mathcal{L}^* \phi_{ij}(x,t),$$

where $\mathcal{L}^*$ is adjoint operator of $\mathcal{L}$. It is clear that,

$$\psi_{ij}(x,t) = \mathcal{L}_{\eta,\xi}\dot{\mathcal{K}}_{\eta,\xi}(x,t)|_{(\eta,\xi)=(x_i,t_j)}, \quad \varphi_{ij}(x,t) = \dot{\mathcal{K}}_{\eta,\xi}(x,t)|_{(\eta,\xi)=(x_i,t_j)}$$

are complete functional systems for reproducing kernel space $\mathcal{W}_2^{(3,2)}(D_1)$ where $i, j = 1, 2, \ldots,$ [32, 33, 34].

Theorem 1. *Suppose that the inverse of the linear operator $\mathcal{L}$ exists. The exact solution of the equation (1.1) can be represented as,*

$$y(x,t) = \sum_{\ell=1}^{\infty} \varrho_\ell \varphi_\ell(x,t), \tag{2.1}$$

where $\{(x_i, t_j)\}_{i,j=1}^{\infty}$ are countable dense sequence of points on the D_1 and ϱ_ℓ are the unknown coefficients that must be determined and $\ell \equiv i, j$.

Proof. See [34]. □

Therefore by truncating the series (2.1), we can provide an approximate solution for the SPCDPP (1.1)

$$\dot{y}_{\dot{n},n}(x,t) = \sum_{\ell=1}^{\dot{n}} \varrho_{\ell,\dot{n},n} \varphi_\ell(x,t), \qquad n = 1, 2, \ldots, \tag{2.2}$$

where n is the number of iteration for nonlinear term $\mathcal{N}(y(x,t))$, and n must be sufficiently large. $\dot{n}$ is the number of collocation points to apply the present RKM in the regular region D_1 and $\dot{n}_1 \times \dot{n}_2 = \dot{n}$. $\ell \equiv i, j$, where $i = 1, 2, \ldots, \dot{n}_1$ and $j = 1, 2, \ldots, \dot{n}_2$. Based on the general technique for nonlinear problem, it is clear that the nonlinear problem (1.1) turns to a sequence of iterations to solve a linear problem in the form, $\mathcal{L}(\dot{y}_{\dot{n},n}(x,t)) = \mathcal{N}(\dot{y}_{\dot{n},n-1}(x,t)) + f(x,t)$. For $n = 1$ we choose the initial function $\dot{y}_{\dot{n},0}(x,t)$ such that satisfies in the regular region conditions and for each iteration $n = 2, 3, \ldots$ we obtain $\mathcal{N}(\dot{y}_{\dot{n},n-1}(x,t))$. Now we determine the unknown coefficients $\varrho_{\ell,\dot{n},n}$ by using the fundamental concepts of the Galerkin method. Consider the following equation

$$R_{\dot{n}}(x,t) = \mathcal{L}(\dot{y}_{\dot{n},n}(x,t)) - \mathcal{N}(\dot{y}_{\dot{n},n-1}(x,t)) - f(x,t), \tag{2.3}$$

such that $\langle R_{\dot{n}}(x,t), \varphi_\ell(x,t) \rangle_{\mathcal{W}_2^{(3,2)}(D_1)} = 0$ for $\ell \equiv i, j$ where $i = 1, 2, \ldots, \dot{n}_1$ and $j = 1, 2, \ldots, \dot{n}_2$ and $\dot{n}_1 \times \dot{n}_2 = \dot{n}$. Using equations (2.2) and (2.3) it is

easy to see that we have the following system of algebraic equations to obtain coefficients $\varrho_{\ell,\dot{n},n}$:

$$\sum_{\ell=1}^{\dot{n}} \varrho_{\ell,\dot{n},n} \mathcal{L}\varphi_\ell(x,t)|_{(x,t)=(x_i,t_j)} = \mathcal{N}(\dot{y}_{\dot{n},n-1}(x,t))|_{(x,t)=(x_i,t_j)} + f(x_i,t_j),$$ (2.4)

$$n = 1, 2, \ldots, \quad i = 1, 2, \ldots, \dot{n}_1 \quad j = 1, 2, \ldots, \dot{n}_2 \quad \dot{n}_1 \times \dot{n}_2 = \dot{n}.$$

2.3   Implementing RKM for SPCDPP

Consider the regular region $(x,t) \in D_1 \equiv [0, 1-\mu] \times [0,1]$

$$\begin{cases} -\varepsilon \partial_x^2 y(x,t) + p(x,t)\partial_x y(x,t) + q(x,t)y(x,t) + \partial_t y(x,t) \\ \quad = \mathcal{N}(y(x,t)) + f(x,t), \quad (x,t) \in D_1, \\ y(0,t) = 0, \quad t \in [0,1], \\ y(x,0) = y_0(x), \quad x \in [0, 1-\mu]. \end{cases}$$ (2.5)

Now homogenize the equation (2.5) and solve it by using the RKM in the space $\mathcal{W}_2^{(3,2)}(D_1)$. In the right layer region $(x,t) \in D_2 \equiv [1-\mu, 1] \times [0,1]$ we have

$$\begin{cases} -\varepsilon \partial_x^2 y(x,t) + p(x,t)\partial_x y(x,t) + q(x,t)y(x,t) + \partial_t y(x,t) \\ \quad = \mathcal{N}(y(x,t)) + f(x,t), \quad (x,t) \in D_2, \\ y(1,t) = 0, \quad y(1-\mu,t) \text{ is known}, \quad t \in [0,1], \\ y(x,0) = y_0(x), \quad x \in [1-\mu, 1]. \end{cases}$$ (2.6)

Suppose that $x = \mu z + 1$ and $y(x,t) = u(z,t)$ such that $\partial_z u(z,t) = \mu \partial_x y(x,t)$ and $\partial_z^2 u(z,t) = \mu^2 \partial_x^2 y(x,t)$ therefore the equation (2.6) turns into,

$$\begin{cases} \dfrac{-\varepsilon}{\mu^2} \partial_z^2 u(z,t) + \dfrac{1}{\mu} p(\mu z+1,t)\partial_z u(z,t) + q(\mu z+1,t)u(z,t) + \partial_t u(z,t) \\ \quad = \mathcal{N}(u(z,t)) + f(\mu z+1,t), \quad (z,t) \in D_3 \equiv [-1,0] \times [0,1], \\ u(0,t) = 0, \quad u(-1,t) \text{ is known}, \quad t \in [0,1], \\ u(z,0) = u_0(z), \quad z \in [-1,0]. \end{cases}$$ (2.7)

Now again homogenize the equation (2.7) and solve it by using the RKM in the space $\mathcal{W}_2^{(3,2)}(D_3)$.

2.4   Convergence analysis

Lemma 1. *Suppose* $\mathcal{Q} = \{\dot{y}_{\dot{n},n}(x,t)| \ \|\dot{y}_{\dot{n},n}(x,t)\|_{\mathcal{W}_2^{(3,2)}(D_1)} \leq \mathring{c}\}$, *then* $\mathcal{Q}$ *is compact set in space* $C(D_1)$, *where* $\mathring{c}$ *is a constant and* $n = 1, 2, \ldots$.

Proof.   See [21, 25, 32].   □

Theorem 2. *Suppose* $\dot{y}_{\dot{n},n}(x,t), y(x,t)$ *are the approximate solution in space* $\mathcal{W}_2^{(3,2)}(D_1)$ *and the exact solution for the problem (1.1) respectively, and* $\{(x_i,t_j)\}_{i,j=1}^\infty$ *are dense points on the region* D_1 *then* $\dot{y}_{\dot{n},n}(x,t) \longrightarrow y(x,t)$ *when* $\dot{n} \longrightarrow \infty$.

Proof. First, we need to show that $\|\dot{y}_{\dot{n},n}(x,t) - y(x,t)\|_{\mathcal{W}_2^{(3,2)}} \longrightarrow 0$ when $\dot{n} \longrightarrow \infty$ then we prove that $\dot{y}_{\dot{n},n}(x,t)$ uniformly convergent to $y(x,t)$ using the reproducing properties. See [25]. $\square$

Corollary 1. Similar to Theorem 2 we have $\partial_t \dot{y}_{\dot{n},n}(x,t) \longrightarrow \partial_t y(x,t)$ when, $\dot{n} \to \infty$.

3 Error analysis

3.1 Preliminaries and notations for the boundary layer region

Suppose $y(x,t)$ is exact solution for the problem (1.1) with a boundary layer behavior and $\mathcal{Y}_n(x,t)$ is approximate solution as following form,

$$\mathcal{Y}_{N,n}(x,t) = \begin{cases} \dot{y}_{\dot{n},n}(x,t), & x \in D_1 \equiv [0, 1-\mu] \times [0,1], \\ \ddot{y}_{\ddot{n},n}(x,t), & x \in D_2 \equiv [1-\mu, 1] \times [0,1], \end{cases}$$

where $\dot{y}_{\dot{n}}(x,t)$ and $\ddot{y}_{\ddot{n}}(x,t)$ are approximate solutions that obtained from the present method in the regions D_1 and D_2, respectively. Number of collocation points throughout the region D is N and $\dot{n}, \ddot{n}$ are number of collocation points on D_1 and D_2, respectively and $N = \dot{n} + \ddot{n}$.

Theorem 3. *Suppose the problem (1.1) as linear form which has homogeneous initial-boundary conditions and $f(x,t)$ is sufficiently smooth function such that the following conditions are satisfied*

$$f(1,0) = 0, \quad \left| \frac{\partial^{k+m} f(x,t)}{\partial x^k \partial t^m} \right|_{(x,t)=(0,0)} = 0,$$

where $k+2m \leq 3$, then we have the following bound for solution of the problem (1.1),

$$\left| \frac{\partial^{k+m} y(x,t)}{\partial x^k \partial t^m} \right| \leq C(1 + \varepsilon^{-k} e^{\frac{-\beta(1-x)}{\varepsilon}}), \quad (x,t) \in D, \ \ k=0,1, \ \ k+m \leq 2.$$

Proof. See [10,31]. $\square$

Corollary 2. According to Theorem 3 upper bound for derivative of $y(x,t)$ relative to x is infinite when $\varepsilon \to 0$ and $x \to 1$,

$$\lim_{x \to 1, \ \varepsilon \to 0} \partial_x y(x,t) \to \infty.$$

Corollary 3. According to Theorem 3 upper bound for derivative of $y(x,t)$ relative to t is independent of the negative powers of the ε and therefore it is finite.

We know that the solution of problem (1.1) and its derivative relative to x ($\partial_x y(x,t)$) has the boundary layer behavior. Moreover, in the boundary layer region, the derivative of the solution relative to x is of great value, see Figure 7

(Appendix). Therefore, in order to solve problem (1.1) region D is split into two regions, so that one of them would contain a boundary layer behavior and uses a proper variable change in this region. It should be remarked that the present method is employed only to approach the solution of problem (1.1) and is not applicable to the approximate derivative of the solution, since on the boundary layer region, the value of the solution derivative $(\partial_x y(x,t))$ is very high.

Remark 3. In summary, since the derivative of solution (1.1) relative to x has the boundary layer behavior and its value is large, the present method is not appropriate to approach the derivation of the solution according to Corollary 2.

Remark 4. According to Remark 3, error analysis theorems are valid if, in the process of proving theorems, we do not use the derivative of the approximate solution $(\partial_x \ddot{y}_{\ddot{n},n}(x,t))$. Accordingly, in the present work the proof process for the error analysis theorems are provided without using the derivative of the approximate solution $(\partial_x \dot{y}_{\dot{n},n}(x,t))$, but using the $\partial_t \dot{y}_{\dot{n},n}(x,t)$, instead. (Theorems 6, 7).

Remark 5. Since in the process of proving error analysis theorems, uniform convergence to the exact solution is required, the approximate solution is illustrated, and its derivation relative to t $(\partial_t y(x,t))$ is uniformly convergent in Theorem 2 and Corollary 1.

3.2 Error estimation

Theorem 4. *Suppose $\dot{y}_{\dot{n},n}(x,t)$ is the approximate solution of the problem (2.5) in space $\mathcal{W}_2^{(3,2)}(D_1)$ and $y(x,t)$ is the exact solution, if $y(x,t) \in \mathcal{W}_2^{(3,2)}(D_1)$ then,*

$$\|y(x,t) - \dot{y}_{\dot{n},n}(x,t)\|_\infty \leq C_1 h_x + C_2 h_t,$$

where $(x,t) \in D_1$ and $\|\dot{y}_{\dot{n},n}(x,t) - y(x,t)\|_\infty = \max_{(x,t)\in D_1} |\dot{y}_{\dot{n}}(x,t) - y(x,t)|$ and C_1, C_2 are positive constants, $h_x = max_{1\leq i\leq \dot{n}_1}|x_{i+1} - x_i|$, and $h_t = max_{1\leq j\leq \dot{n}_2}|t_{j+1} - t_j|$. $\dot{n} = \dot{n}_1 \times \dot{n}_2$ are number of collocation points in region D_1.

Proof. From [7, 8, 13, 19], and in each $[x_i, x_{i+1}] \times [t_j, t_{j+1}] \subset D_1$ we have,

$$\begin{aligned}
\|\mathcal{L}^{-1} R_{\dot{n}}(x,t)\|_\infty &= \|y(x,t) - \dot{y}_{\dot{n},n}(x,t)\|_\infty \\
&= \|y(x,t) - y(x_i,t_j) + \dot{y}_{\dot{n},n}(x_i,t_j) - \dot{y}_{\dot{n},n}(x,t) + y(x_i,t_j) - \dot{y}_{\dot{n},n}(x_i,t_j)\|_\infty.
\end{aligned} \tag{3.1}$$

By expansion of $y(x,t)$ at the point (x_i, t_j) we have,

$$y(x,t) \simeq y(x_i,t_j) + \big[(x-x_i)\partial_x y(x_i,t_j) + (t-t_j)\partial_t y(x_i,t_j)\big],$$
$$|y(x,t) - y(x_i,t_j)| \leq \big|(x-x_i)\partial_x y(x_i,t_j)\big| + \big|(t-t_j)\partial_t y(x_i,t_j)\big|,$$

since $y(x,t) \in \mathcal{W}_2^{(3,2)}(D_1)$, constants $\tilde{C}_1, \tilde{C}_2$ exist such that, $\forall (x,t) \in D_1$ $|\partial_x y(x,t)| \leq \tilde{C}_1$ and $|\partial_t y(x,t)| \leq \tilde{C}_2$ and we have,

$$\|y(x,t) - y(x_i,t_j)\|_\infty \leq \tilde{C}_1 h_x + \tilde{C}_2 h_t, \tag{3.2}$$

moreover, we can write

$$\dot{y}_{\dot{n},n}(x_i,t_j) - \dot{y}_{\dot{n},n}(x,t) = -\int_{x_i}^{x} \partial_s \dot{y}_{\dot{n},n}(s,t_j)ds - \int_{t_j}^{t} \partial_w \dot{y}_{\dot{n},n}(x,w)dw,$$

$$|\dot{y}_{\dot{n},n}(x_i,t_j) - \dot{y}_{\dot{n},n}(x,t)| \leq \int_{x_i}^{x} |\partial_s \dot{y}_{\dot{n},n}(s,t_j)|ds + \int_{t_j}^{t} |\partial_w \dot{y}_{\dot{n},n}(x,w)|dw,$$

since $\dot{y}_{\dot{n},n}(x,t) \in \mathcal{W}_2^{(3,2)}(D_1)$, constants $\hat{C}_1, \hat{C}_2$ exist such that, $|\partial_x \dot{y}_{\dot{n},n}(x,t)| \leq \hat{C}_1$ and $|\partial_t \dot{y}_{\dot{n},n}(x,t)| \leq \hat{C}_2$ and therefore we have,

$$\|\dot{y}_{\dot{n},n}(x_i,t_j) - \dot{y}_{\dot{n},n}(x,t)\|_\infty \leq \hat{C}_1 h_x + \hat{C}_2 h_t. \tag{3.3}$$

From Theorem 2 we have $\dot{y}_{\dot{n},n}(x,t) \longrightarrow y(x,t)$ when $\dot{n} \to \infty$, therefore for sufficiently large $\dot{n}$, $|\dot{y}_{\dot{n},n}(x_i,t_j) - y(x_i,t_j)| < \epsilon_1$ and by combining the above inequalities equations,

$$\|R_{\dot{n}}(x,t)\|_\infty \leq \mathring{C}_1 h_x + \mathring{C}_2 h_t, \quad \|y(x,t) - \dot{y}_{\dot{n},n}(x,t)\|_\infty \leq C_1 h_x + C_2 h_t.$$

$\square$

Theorem 5. *Consider assumptions of the Theorem 4 then,*

$$\|\partial_t y(x,t) - \partial_t \dot{y}_{\dot{n},n}(x,t)\|_\infty \leq K_1 h_x + K_2 h_t,$$

where $(x,t) \in D_1$ and $\|\partial_t \dot{y}_{\dot{n},n}(x,t) - \partial_t y(x,t)\|_\infty = \max_{(x,t)\in D_1} |\partial_t \dot{y}_{\dot{n},n}(x,t) - \partial_t y(x,t)|$ and K_1, K_2 are positive constants.

Proof. Similar to Theorem 4 and in each $[x_i,x_{i+1}] \times [t_j,t_{j+1}] \subset D_1$ we have,

$$\|\mathcal{L}^{-1}\partial_t R_{\dot{n}}(x,t)\|_\infty = \|\partial_t y(x,t) - \partial_t \dot{y}_{\dot{n},n}(x,t)\|_\infty = \|\partial_t y(x,t) - \partial_t y(x_i,t_j)$$
$$+ \partial_t \dot{y}_{\dot{n},n}(x_i,t_j) - \partial_t \dot{y}_{\dot{n},n}(x,t) + \partial_t y(x_i,t_j) - \partial_t \dot{y}_{\dot{n},n}(x_i,t_j)\|. \tag{3.4}$$

By expansion of $\partial_t y(x,t)$ at the point (x_i,t_j) we have,

$$\partial_t y(x,t) = \partial_t y(x_i,t_j) + \left[(x-x_i)\frac{\partial^2 y(x_i,t_j)}{\partial x \partial t} + (t-t_j)\frac{\partial^2 y(x_i,t_j)}{\partial t \partial x}\right],$$

since $y(x,t) \in \mathcal{W}_2^{(3,2)}(D_1)$ there exists constant $\tilde{K}_1$ such that, $|\frac{\partial^2 y(x,t)}{\partial x \partial t}| \leq \tilde{K}_1$,

$$\|\partial_t y(x,t) - \partial_t y(x_i,t_j)\|_\infty \leq \tilde{K}_1 h_x + \tilde{K}_1 h_t, \tag{3.5}$$

moreover, we can write

$$\partial_t \dot{y}_{\dot{n},n}(x_i,t_j) - \partial_t \dot{y}_{\dot{n},n}(x,t) = -\int_{x_i}^{x} \partial_t \partial_s \dot{y}_{\dot{n},n}(s,t_j)ds - \int_{t_j}^{t} \partial_w^2 \dot{y}_{\dot{n},n}(x,w)dw,$$

and since $\dot{y}_{\dot{n},n}(x,t) \in \mathcal{W}_2^{(3,2)}(D_1)$ there exist constants $\hat{K}_1, \hat{K}_2$ where $|\partial_t \partial_x \dot{y}_{\dot{n},n}(x,t)| \leq \hat{K}_1$, $|\partial_t^2 \dot{y}_{\dot{n},n}(x,t)| \leq \hat{K}_2$ and therefore we have,

$$\|\partial_t \dot{y}_{\dot{n},n}(x_i,t_j) - \partial_t \dot{y}_{\dot{n},n}(x,t)\|_\infty \leq \hat{K}_1 h_x + \hat{K}_2 h_t. \tag{3.6}$$

From Theorem 2 and properties of reproducing kernel we have $\partial_t \dot{y}_{\dot{n},n}(x,t) \longrightarrow$ $\partial_t y(x,t)$ when $\dot{n} \to \infty$, therefore for sufficiently large $\dot{n}$, $|\partial_t \dot{y}_{\dot{n},n}(x_i,t_j) - \partial_t y(x_i,t_j)| < \epsilon_2$ and by combining above equations,

$$\|\partial_t R_{\dot{n}}(x,t)\|_\infty \le \mathring{K}_1 h_x + \mathring{K}_2 h_t, \quad \|\partial_t y(x,t) - \partial_t \dot{y}_{\dot{n},n}(x,t)\|_\infty \le K_1 h_x + K_2 h_t.$$

$\square$

Now, in order to obtain error estimation between $\ddot{y}_{\ddot{n},n}(x,t)$ and $y(x,t)$ on the region D_2, first provide some essential preliminaries, so consider equation (2.6) as follows [15],

$$\begin{cases} -\varepsilon \partial_x^2 y(x,t) + p(x,t)\partial_x y(x,t) + q(x,t)y(x,t) + \partial_t y(x,t) \\ = \mathcal{N}(y(x,t)) + f(x,t), \quad (x,t) \in D_2, \\ y(1,t) = 0, \quad y(1-\mu,t) = \dot{y}_{\dot{n},n}(1-\mu,t), \quad t \in [0,1], \\ y(x,0) = y_0(x), \quad x \in [1-\mu,1]. \end{cases} \tag{3.7}$$

Consider following assumptions and change region D_2 into D_3, $x = \mu z + 1$ and $y(x,t) = u(z,t)$ such that $\partial_z u(z,t) = \mu \partial_x y(x,t)$ and $\partial_z^2 u(z,t) = \mu^2 \partial_x^2 y(x,t)$ therefore the equation (3.7) turns into,

$$\begin{cases} \frac{-\varepsilon}{\mu^2} \partial_z^2 u(z,t) + \frac{1}{\mu} p(\mu z+1,t)\partial_z u(z,t) + q(\mu z+1,t)u(z,t) + \partial_t u(z,t) \\ = \mathcal{N}(u(z,t)) + f(\mu z+1,t), \quad (z,t) \in D_3 \equiv [-1,0] \times [0,1], \\ u(0,t) = 0, \quad u(-1,t) = \dot{y}_{\dot{n},n}(1-\mu,t), \quad t \in [0,1], \\ u(z,0) = u_0(z), \quad z \in [-1,0]. \end{cases} \tag{3.8}$$

Now by homogenizing the initial and boundary conditions of problem (3.8) with function $u(z,t) = \bar{u}(z,t) + H_1(z,t)$ such that $H_1(z,0) = u_0(z) = y_0(\mu z+1)$ and $H_1(-1,t) = \dot{y}_{\dot{n},n}(1-\mu,t)$, therefore we have following equation,

$$\begin{cases} \frac{-\varepsilon}{\mu^2} \partial_z^2 \bar{u}(z,t) + \frac{1}{\mu} p(\mu z+1,t)\partial_z \bar{u}(z,t) + q(\mu z+1,t)\bar{u}(z,t) + \partial_t \bar{u}(z,t) \\ = \mathcal{N}(\bar{u}(z,t)) + \bar{f}(\mu z+1,t), \quad (z,t) \in D_3 \equiv [-1,0] \times [0,1], \\ \bar{u}(0,t) = 0, \quad \bar{u}(-1,t) = 0, \quad t \in [0,1], \\ \bar{u}(z,0) = 0, \quad z \in [-1,0]. \end{cases} \tag{3.9}$$

Suppose $\underline{u}(z,t)$ is the approximate solution of the problem (3.9) that has been obtained using RKM in space $W_2^{(3,2)}(D_3)$, therefore

$$\ddot{y}_{\ddot{n},n}(x,t) = \underline{u}\big((x-1)/\mu, t\big) + H_1\big((x-1)/\mu, t\big).$$

Consider the following equation,

$$\begin{cases} -\varepsilon \partial_x^2 v(x,t) + p(x,t)\partial_x v(x,t) + q(x,t)v(x,t) + \partial_t v(x,t) \\ = \mathcal{N}(v(x,t)) + g(x,t), \quad (x,t) \in D_2, \\ v(1,t) = 0, \quad v(1-\mu,t) = y(1-\mu,t), \quad t \in [0,1], \\ v(x,0) = v_0(x), \quad x \in [1-\mu,1]. \end{cases} \tag{3.10}$$

It is clear that the solution of the problem (3.10) is $y(x,t)$ where $(x,t) \in D_2$. By using the following variable change, $x = \mu z + 1$ and $v(x,t) = \nu(z,t)$ such

that $\partial_z \nu(z,t) = \mu \partial_x v(x,t)$ and $\partial_z^2 \nu(z,t) = \mu^2 \partial_x^2 v(x,t)$, we have shifted problem (3.8) to the problem (3.10),

$$
\begin{cases}
\frac{-\varepsilon}{\mu^2} \partial_z^2 \nu(z,t) + \frac{1}{\mu} p(\mu z + 1, t) \partial_z \nu(z,t) + q(\mu z + 1, t)\nu(z,t) + \partial_t \nu(z,t) \\
\quad = \mathcal{N}(\nu(z,t)) + g(\mu z + 1, t), \quad (z,t) \in D_3 \equiv [-1,0] \times [0,1], \\
\nu(0,t) = 0, \quad \nu(-1,t) = y(1 - \mu, t), \quad t \in [0,1], \\
\nu(z,0) = \nu_0(z), \quad z \in [-1,0].
\end{cases}
$$
(3.11)

Therefore we homogenize obtained problem (3.11), with $\nu(z,t) = \overline{\nu}(z,t) + H_2(z,t)$ such that $H_2(z,0) = \nu_0(z) = v_0(\mu z + 1)$ and $H_2(-1,t) = y(1 - \mu, t)$, therefore problem (3.11) turns to,

$$
\begin{cases}
\frac{-\varepsilon}{\mu^2} \partial_z^2 \overline{\nu}(z,t) + \frac{1}{\mu} p(\mu z + 1, t) \partial_z \overline{\nu}(z,t) + q(\mu z + 1, t)\overline{\nu}(z,t) + \partial_t \overline{\nu}(z,t) \\
\quad = \mathcal{N}(\overline{\nu}(z,t)) + \overline{g}(\mu z + 1, t), \quad (z,t) \in D_3 \equiv [-1,0] \times [0,1], \\
\overline{\nu}(0,t) = 0, \quad \overline{\nu}(-1,t) = 0, \quad t \in [0,1], \\
\overline{\nu}(z,0) = 0, \quad z \in [-1,0],
\end{cases}
$$
(3.12)

and suppose $\underline{\nu}(z,t)$ is the approximate solution of the problem (3.12) that has been obtained using RKM in space $\mathcal{W}_2^{(3,2)}(D_3)$.

Theorem 6. *Suppose $\underline{u}(z,t)$ is the approximate solution of the equation (3.9) and $\overline{\nu}(z,t)$ is the solution of the equation (3.12) and $p(\mu z + 1, t), q(\mu z + 1, t) \in \mathcal{C}^2(D_3)$ then,*
$$\|\underline{u}(z,t) - \overline{\nu}(z,t)\|_\infty \le M_1 h_x + M_2 h_t,$$
where $(x_i, t_j) \in D_3$ and $\|\underline{u}(z,t) - \overline{\nu}(z,t)\|_\infty = \max_{(x,t) \in D_3} = |\underline{u}(z,t) - \overline{\nu}(z,t)|$ and M_1, M_2 are constants, $h_x = \max_{1 \le i \le \ddot{n}_1} |x_{i+1} - x_i|$, and $h_t = \max_{1 \le j \le \ddot{n}_2} |t_{j+1} - t_j|$. Number of collocation points in region D_3 are $\ddot{n} = \ddot{n}_1 \times \ddot{n}_2$.

Proof. Since $\underline{u}(z,t), \overline{\nu}(z,t)$ are approximate solutions of the equations (3.9) and (3.12), respectively, therefore we have

$$
\|R_{\ddot{n}}(z,t)\|_\infty = \|\mathcal{L}\underline{u}(z,t) - \mathcal{L}\overline{\nu}(z,t)\|_\infty = \|\mathcal{L}\underline{u}(z,t) - \overline{f}(\mu z + 1, t) + \overline{f}(\mu z + 1, t)
$$
$$
- \mathcal{L}\overline{\nu}(z,t)\|_\infty \le \|\mathcal{L}\underline{u}(z,t) - \overline{f}(\mu z + 1, t)\|_\infty + \|\overline{f}(\mu z + 1, t) - \overline{g}(\mu z + 1, t)\|_\infty. \quad (3.13)
$$

By using Theorems 4 it is clear that $\|\mathcal{L}\underline{u}(z,t) - \overline{f}(\mu z + 1, t)\|_\infty \max_{(x,t) \in D_3} = |\mathcal{L}\underline{u}(z,t) - \overline{f}(\mu z + 1, t)| \le \hat{M}_1 h_x + \hat{M}_2 h_t$. From problem (3.8) and $u(z,t) = \underline{u}(z,t) + H_1(z,t)$ we have,

$$
\overline{f}(\mu z + 1, t) = \frac{-\varepsilon}{\mu^2} \partial_z^2 H_1(z,t) + \frac{1}{\mu} p(\mu z + 1, t) \partial_z H_1(z,t) + q(\mu z + 1, t) H_1(z,t)
$$
$$
+ \partial_t H_1(z,t).
$$

Using the problem (3.11) and $\nu(z,t) = \overline{\nu}(z,t) + H_2(z,t)$ we have,

$$
\overline{g}(\mu z + 1, t) = \frac{-\varepsilon}{\mu^2} \partial_z^2 H_2(z,t) + \frac{1}{\mu} p(\mu z + 1, t) \partial_z H_2(z,t) + q(\mu z + 1, t) H_2(z,t)
$$
$$
+ \partial_t H_2(z,t),
$$

it is easy to see that homogenization functions $H_1(z,t), H_2(z,t)$ can be defined in the from, $H_1(z,t) = a(z,t)u_0(z) + b(z,t)\dot{y}_{\dot{n},n}(1-\mu,t)$ and $H_2(z,t) = a(z,t)\nu_0(z) + b(z,t)y(1-\mu,t)$ such that $a(z,0) = 1, b(z,0) = 0$ and $a(-1,t) = 0, b(-1,t) = 1$ therefore we have,

$$\partial_z^2 H_1(z,t) = a_1(z,t) + \dot{y}_{\dot{n},n}(1-\mu,t)\partial_z^2 b(z,t),$$
$$\partial_z H_1(z,t) = a_2(z,t) + \dot{y}_{\dot{n},n}(1-\mu,t)\partial_z b(z,t),$$
$$\partial_t H_1(z,t) = a_3(z,t) + \dot{y}_{\dot{n},n}(1-\mu,t)\partial_t b(z,t) + b(z,t)\partial_t \dot{y}_{\dot{n},n}(1-\mu,t),$$
$$\partial_z^2 H_2(z,t) = a_1(z,t) + y(1-\mu,t)\partial_z^2 b(z,t),$$
$$\partial_z H_2(z,t) = a_2(z,t) + y(1-\mu,t)\partial_z b(z,t),$$
$$\partial_t H_2(z,t) = a_3(z,t) + y(1-\mu,t)\partial_t b(z,t) + b(z,t)\partial_t y(1-\mu,t),$$

also $|p(\mu z + 1, t)| \leq m_1$, $|q(\mu z + 1, t)| \leq m_2$ and we can write,

$$\|\overline{f}(\mu z + 1, t) - \overline{g}(\mu z+1,t)\|_\infty \leq \frac{-\varepsilon}{\mu^2}\|\partial_z^2 H_1(z,t) - \partial_z^2 H_2(z,t)\|_\infty + \frac{m_1}{\mu}\|\partial_z H_1(z,t)$$
$$- \partial_z H_2(z,t)\|_\infty + m_2\|H_1(z,t) - H_2(z,t)\|_\infty + \|\partial_t H_1(z,t) - \partial_t H_2(z,t)\|_\infty,$$

where,

$$\|H_1(z,t) - H_2(z,t)\|_\infty = \|a(z,t)u_0(z) - a(z,t)\nu_0(z) + b(z,t)\dot{y}_{\dot{n},n}(1-\mu,t)$$
$$- b(z,t)y(1-\mu,t)\|_\infty \leq \|a(z,t)u_0(z) - a(z,t)\nu_0(z)\|_\infty$$
$$+ \|b(z,t)\dot{y}_{\dot{n},n}(1-\mu,t) - b(z,t)y(1-\mu,t)\|_\infty \leq \tilde{M}_1 h_x + \tilde{M}_2 h_t. \qquad (3.14)$$

Similar to equation (3.14) and using Theorems 4, 5 we have, $\|\overline{f}(\mu z+1,t) - \overline{g}(\mu z+1,t)\|_\infty \leq \check{M}_1 h_x + \check{M}_2 h_t$, and therefore we have

$$\|R_{\ddot{n}}(z,t)\|_\infty = \|\mathcal{L}\underline{u}(z,t) - \mathcal{L}\overline{\nu}(z,t)\|_\infty \leq \mathring{M}_1 h_x + \mathring{M}_2 h_t$$

and from Theorem 4 it is clear $\|\underline{u}(z,t) - \overline{\nu}(z,t)\|_\infty \leq M_1 h_x + M_2 h_t$. □

Theorem 7. *Suppose $\ddot{y}_{\ddot{n},n}(x,t)$ is the approximate solution of the problem (2.6) in space $\mathcal{W}_2^{(3,2)}(D_3)$ and $y(x,t)$ is the exact solution, if $y(x,t) \in \mathcal{W}_2^{(3,2)}(D)$ then,*

$$\|y(x,t) - \ddot{y}_{\ddot{n},n}(x,t)\|_\infty \leq C_3 h_x + C_4 h_t,$$

where $(x_i, t_j) \in D_3$ and $\|y(x,t) - \ddot{y}_{\ddot{n},n}(x,t)\|_\infty = \max_{(x,t)\in D_3} = |y(x,t) - \ddot{y}_{\ddot{n},n}(x,t)|$ and C_3, C_4 are constants, $h_x = \max_{1\leq i \leq \ddot{n}_1}|x_{i+1} - x_i|$, and $h_t = \max_{1\leq j \leq \ddot{n}_2}|t_{j+1} - t_j|$. Number of collocation points in region D_3 are $\ddot{n} = \ddot{n}_1 \times \ddot{n}_2$.

Proof. From mentioned preliminaries, $\underline{u}(z,t)$ is approximate solution of equation (3.9) and $y(x,t)$ is exact solution of equation (3.10) and $\ddot{y}_{\ddot{n},n}(x,t) = \underline{u}(z,t) + H_1(z,t)$, $y(x,t) = \overline{\nu}(z,t) + H_2(z,t)$ where $z = \frac{x-1}{\mu}$ therefore we have,

$$\|y(x,t) - \ddot{y}_{\ddot{n},n}(x,t)\|_\infty = \|\underline{u}(z,t) + H_1(z,t) - \overline{\nu}(z,t) - H_2(z,t)\|_\infty$$
$$\leq \|\underline{u}(z,t) - \overline{\nu}(z,t)\|_\infty + \|H_1(z,t) - H_2(z,t)\|_\infty, \tag{3.15}$$

using Theorem 6 the proof is completed. □

Theorem 8. *Suppose* $\mathcal{Y}_{N,n}(x,t)$ *is the approximate solution of the problem* (1.1) *that has been obtained using RKM in space* $\mathcal{W}_2^{(3,2)}$ *and* $y(x,t)$ *is the exact solution, if* $p(x,t), q(x,t) \in \mathcal{C}^2(D)$ *and* $D \equiv [0,1] \times [0,1]$ *then,*

$$\|\mathcal{Y}_{N,n}(x,t) - y(x,t)\|_\infty = \max_{(x,t)\in D} |\mathcal{Y}_{N,n}(x,t) - y(x,t)| \le C_x h_x + C_t h_t,$$

where $(x_i, t_j) \in D$ *and* C_x, C_t *are constants,* $h_x = max_{1 \le i \le N}|x_{i+1} - x_i|$, *and* $h_t = max_{1 \le j \le N}|t_{j+1} - t_j|$. *Number of collocation points in throughout region* D *is* N, *and* $N = \dot{n} + \ddot{n}$.

Proof. By using Theorems 4 and 7 and definition of $\mathcal{Y}_{N,n}(x,t)$ on the region D,

$$\mathcal{Y}_{N,n}(x,t) = \begin{cases} \dot{y}_{\dot{n},n}(x,t) & (x,t) \in D_1, \\ \ddot{y}_{\ddot{n},n}(x,t) & (x,t) \in D_2, \end{cases}$$

and the proof is completed. $\square$

4 Numerical results

We consider the following numerical examples [17],

Example 1. $-\varepsilon\partial_x^2 y(x,t) + (1 + x - x^2)\partial_x y(x,t) + \partial_t y(x,t) = f_1(x,t)$.

Example 2. $-\varepsilon\partial_x^2 y(x,t) + (-x^2 + x + 1)\partial_x y(x,t) + \partial_t y(x,t) + e^{y(x,t)} = f_2(x,t)$.

Example 3. $-\varepsilon\partial_x^2 y(x,t) + (-x^2 + x + 1)\partial_x y(x,t)y(x,t) + \partial_t y(x,t) + e^{y(x,t)} = f_3(x,t)$.

Example 4. $-\varepsilon\partial_x^2 y(x,t) + (2 - x^2)\partial_x y(x,t) + xy(x,t) + \partial_t y(x,t) = 10t^2 e^{-t}x(1-x)$.

For Examples 1, 2 and 3, the exact solution is, $y(x,t) = e^{-t}\left(x\left(1 - e^{-1/\epsilon}\right) - e^{\frac{x-1}{\epsilon}} + e^{-1/\epsilon}\right)$ where $y_0(x) = x\left(1 - e^{-1/\epsilon}\right) - e^{\frac{x-1}{\epsilon}} + e^{-1/\epsilon}$ and for Example 4 the exact solution is not known and $y_0(x) = 0$. See Figure 1 for Example 1, Figure 2 for Example 2, Figure 3 for Example 3, and Figure 4 for Example 4 which they show the maximum absolute error for various ϵ. If perturbation parameter is not too small ($\varepsilon = 1, 10^{-2}$), then nonuniform collocation points are $\left(1 - (\frac{i}{n_1 + \frac{1}{3}})^2, \frac{j}{n_2 + \frac{1}{2}}\right)$, in throughout region D where $i = 1, 2, ..., n_1$, $j = 1, 2, ..., n_2$. Otherwise ($\varepsilon = 10^{-4, -6, -8, \cdots}$) nonuniform collocation points for boundary layer region, are $\left(-(\frac{i}{\ddot{n}_1 + \frac{1}{3}})^2, \frac{j}{\ddot{n}_2 + \frac{1}{2}}\right)$ throughout D_3 where $i = 1, 2, ..., \ddot{n}_1$, $j = 1, 2, ..., \ddot{n}_2$, and collocation points for the regular region are $\left(1 - (\frac{i}{\dot{n}_1 + \frac{1}{7}})^2, (\frac{j}{\dot{n}_2 + \frac{1}{6}})^2\right)$ throughout D_1, where $i = 1, 2, ..., \dot{n}_1$, $j = 1, 2, ..., \dot{n}_2$. Examples 1, 2 and 3 have the same exact solution, see Figure 5 for different values of ϵ, and Figure 6 for Example 4. All present numerical results are obtained using Wolfram Mathematica 10 software.

Remark 6. For solving problem (1.1), first we divided D to regular region $D_1 \equiv [0, 1 - \mu] \times [0,1]$ and boundary layer region $D_2 \equiv [1 - \mu, 1] \times [0,1]$

and we shifted region D_2 into $D_3 \equiv [-1,0] \times [0,1]$ and obtain the reproducing kernels $\dot{\mathcal{K}}_{\eta,\xi}(x,t)$ and $\ddot{\mathcal{K}}_{\eta,\xi}(x,t)$ for regular and shifted boundary layer regions, respectively. Then we construct complete functional systems $\varphi_{ij}(x,t) = \dot{\mathcal{K}}_{\eta,\xi}(x,t)|_{(\eta,\xi)=(x_i,t_j)}$ and $\varphi_{ij}(x,t) = \ddot{\mathcal{K}}_{\eta,\xi}(x,t)|_{(\eta,\xi)=(x_i,t_j)}$ for reproducing kernel spaces $\mathcal{W}_2^{(3,2)}(D_1)$ and $\mathcal{W}_2^{(3,2)}(D_3)$ respectively. For regular region D_1, $\dot{n} = \dot{n}_1 \times \dot{n}_2$ where $i = 1, 2, \ldots, \dot{n}_1$ and $j = 1, 2, \ldots, \dot{n}_2$. For shifted boundary layer region D_3, $\ddot{n} = \ddot{n}_1 \times \ddot{n}_2$ where $i = 1, 2, \ldots, \ddot{n}_1$ and $j = 1, 2, \ldots, \ddot{n}_2$ and $\ell \equiv i, j$. Therefor we solve problem (1.1) in the regular and shifted boundary layer region such that approximate solutions are as following form,

$$\dot{y}_{\dot{n},n}(x,t) = \sum_{\ell=1}^{\dot{n}} \varrho_{\ell,\dot{n},n} \varphi_\ell(x,t), \quad \ddot{y}_{\ddot{n},n}(x,t) = \sum_{\ell=1}^{\ddot{n}} \varrho_{\ell,\ddot{n},n} \varphi_\ell(x,t),$$

where we obtain coefficients $\varrho_{\ell,\dot{n},n}$ and $\varrho_{\ell,\ddot{n},n}$ by solving following system of algebraic equations

$$\sum_{\ell=1}^{\dot{n}} \varrho_{\ell,\dot{n},n} \mathcal{L}\varphi_\ell(x,t)|_{(x,t)=(x_i,t_j)} = \mathcal{N}(\dot{y}_{\dot{n},n-1}(x,t))|_{(x,t)=(x_i,t_j)} + f(x_i,t_j),$$

$$\sum_{\ell=1}^{\ddot{n}} \varrho_{\ell,\ddot{n},n} \mathcal{L}\varphi_\ell(x,t)|_{(x,t)=(x_i,t_j)} = \mathcal{N}(\ddot{y}_{\ddot{n},n-1}(x,t))|_{(x,t)=(x_i,t_j)} + f(x_i,t_j),$$

where $n = 2, 3, \ldots$ is number of iterative for nonlinear term $\mathcal{N}(\dot{y}_{\dot{n},n-1}(x,t))$ and $\mathcal{N}(\ddot{y}_{\ddot{n},n-1}(x,t))$ with initial functions $\dot{y}_{\dot{n},0}(x,t)$ and $\ddot{y}_{\ddot{n},0}(x,t)$ for $n = 1$ respectively.

Remark 7. According to Corollary 3 and Remark 3 we compared maximum absolute errors for derivative of the approximate solutions relative to t ($\mathbb{E}^{\partial_t \mathcal{Y}_{N,n}(x,t)} = Max\, |\partial_t \mathcal{Y}_{N,n}(x,t) - \partial_t y(x,t)|$) and x ($\mathbb{E}^{\partial_x \mathcal{Y}_{N,n}(x,t)} = Max\, |\partial_x \mathcal{Y}_{N,n}(x,t) - \partial_x y(x,t)|$) throughout D. We presented results in Table 3.

Remark 8. According to error analysis Theorems errors ratio for approximate solution $\mathcal{Y}_{N,n}(x,t)$ throughout D must be about 0.5. We showed errors ratio in Tables 1 and 2 for all numerical examples.

5 Conclusions

Hereby, a technique is introduced based on RKM to solve SPCDPPs. One of the advantages of this technique is that it could be used to solve SPCDPPs, so that SPCDPPs would not be easily solved employing common numerical methods or numerical commands in mathematical softwares that are available for free. Numerical examples demonstrated that the present method has higher precision compared to other methods. That is in account of the fact that the Gram-Schmidt process is removed.

Acknowledgements

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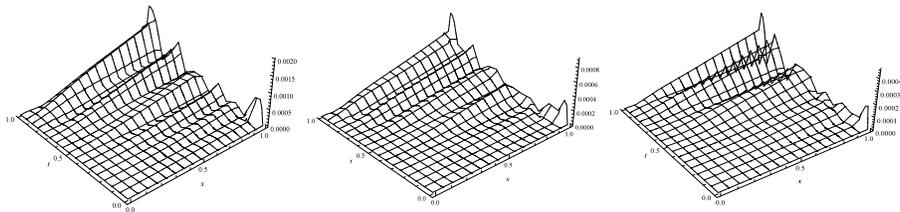
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Appendix: tables and figures

Table 1. Max point-wise error ($\mathbb{E}^{\mathcal{Y}_{N,n}(x,t)} = Max\,|\mathcal{Y}_{2N,n}(x,t) - \mathcal{Y}_{N,n}(x,t)|,\ (x,t) \in D$) and error ratio $\mathbb{E}^{\mathcal{Y}_{2N,n}}/\mathbb{E}^{\mathcal{Y}_{N,n}}$ for Example 4.

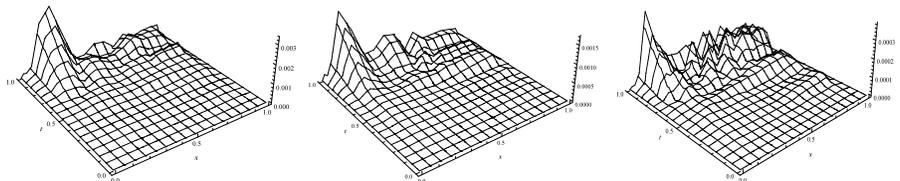|	PM			[17]	
N	200	400	800	512×160	1024×320
ε					
10^0	7.00×10^{-5}	1.80×10^{-5}	8.50×10^{-6}	6.21×10^{-5}	3.13×10^{-5}
	0.257143	0.472222			
10^{-2}	2.80×10^{-3}	7.00×10^{-4}	3.00×10^{-4}	6.07×10^{-4}	3.10×10^{-4}
	0.250	0.428571			
10^{-4}	3.20×10^{-3}	1.60×10^{-3}	3.40×10^{-4}	8.70×10^{-4}	4.39×10^{-4}
	0.50	0.2125			
10^{-6}	3.20×10^{-3}	1.70×10^{-3}	3.60×10^{-4}	8.93×10^{-4}	4.48×10^{-4}
	0.53125	0.211765			
10^{-8}	3.40×10^{-3}	1.70×10^{-3}	3.60×10^{-4}	8.94×10^{-4}	4.49×10^{-4}
	0.50	0.211765			

Figure 1. Max absolute error with $\varepsilon = 10^{-8}$ (Left: $N = 200$; Middle: $N = 400$; Right: $N = 800$) for Example 1.
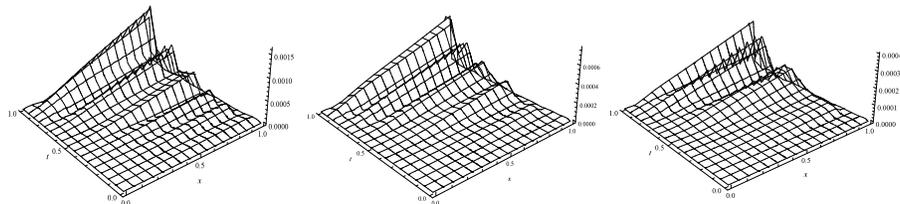
Figure 2. Max absolute error with $\varepsilon = 10^{-6}$ and $n = 10$ (Left: $N = 200$; Middle: $N = 400$; Right: $N = 800$) for Example 2.
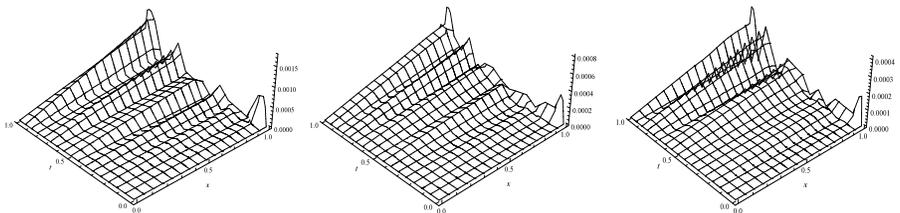
Figure 3. Max absolute error with $\varepsilon = 10^{-8}$ and $n = 10$ (Left: $N = 200$; Middle: $N = 400$; Right: $N = 800$) for Example 3.


Figure 4. Max error with $\varepsilon = 10^{-6}$ (Left: $N = 200$; Middle: $N = 400$; Right: $N = 800$) for Example 4.

Table 2. Max absolute error $(\mathbb{E}^{\mathcal{Y}_{N,n}(x,t)} = Max\,|\mathcal{Y}_{N,n}(x,t) - y(x,t)|,\ (x,t) \in D)$ with $n = 10$ and error ratio $\mathbb{E}^{\mathcal{Y}_{2N,n}}/\mathbb{E}^{\mathcal{Y}_{N,n}}.$

	PM			[17]	
N	200	400	800	512×160	1024×320
ε					
Ex.1					
10^0	1.50×10^{-4}	5.50×10^{-5}	2.30×10^{-5}	5.41×10^{-5}	2.67×10^{-5}
	0.366667	0.418182			
10^{-2}	3.20×10^{-3}	1.30×10^{-3}	5.50×10^{-4}	4.55×10^{-3}	2.52×10^{-3}
	0.40625	0.423077			
10^{-4}	1.60×10^{-3}	8.00×10^{-4}	4.00×10^{-4}	5.42×10^{-3}	2.74×10^{-3}
	0.50	0.50			
10^{-6}	1.60×10^{-3}	8.00×10^{-4}	4.00×10^{-4}	5.52×10^{-3}	2.76×10^{-3}
	0.50	0.50			
10^{-8}	2.00×10^{-3}	8.50×10^{-4}	4.40×10^{-4}	5.55×10^{-3}	2.77×10^{-3}
	0.425	0.517647			
Ex.2					
10^0	1.50×10^{-4}	5.00×10^{-5}	2.20×10^{-5}	4.53×10^{-5}	2.28×10^{-5}
	0.333333	0.440			
10^{-2}	2.60×10^{-3}	1.10×10^{-3}	4.60×10^{-4}	2.83×10^{-3}	1.42×10^{-3}
	0.423077	0.418182			
10^{-4}	1.60×10^{-3}	7.50×10^{-4}	4.00×10^{-4}	3.84×10^{-3}	1.93×10^{-3}
	0.46875	0.533333			
10^{-6}	1.60×10^{-3}	7.50×10^{-4}	4.00×10^{-4}	3.92×10^{-3}	1.96×10^{-3}
	0.46875	0.533333			
10^{-8}	1.80×10^{-3}	8.50×10^{-4}	4.40×10^{-4}	3.95×10^{-3}	1.96×10^{-3}
	0.472222	0.517647			
				$N=256 \times 80$	$N=512 \times 160$
Ex.3					
10^0	1.50×10^{-4}	5.50×10^{-5}	2.20×10^{-5}	2.83×10^{-5}	1.42×10^{-5}
	0.366667	0.40			
10^{-2}	2.60×10^{-3}	1.10×10^{-3}	4.60×10^{-4}	5.21×10^{-3}	2.70×10^{-3}
	0.423077	0.418182			
10^{-4}	1.60×10^{-3}	7.50×10^{-4}	4.00×10^{-4}	8.28×10^{-3}	4.17×10^{-3}
	0.46875	0.533333			
10^{-6}	1.60×10^{-3}	7.50×10^{-4}	3.80×10^{-4}	8.85×10^{-3}	4.28×10^{-3}
	0.46875	0.506667			
10^{-8}	1.70×10^{-3}	8.00×10^{-4}	4.20×10^{-4}	8.65×10^{-3}	4.30×10^{-3}
	0.470588	0.525			

Table 3. Max absolute errors for derivative of the approximate solutions in Example 3.

	Present Method		
	$N = 200$	$N = 400$	$N = 800$
$\varepsilon = 10^{-6}$			
$\mathbb{E}^{\partial_t \mathcal{Y}_{N,n}(x,t)}$	1.90×10^{-2}	1.20×10^{-2}	8.50×10^{-3}
$\mathbb{E}^{\partial_x \mathcal{Y}_{N,n}(x,t)}$	2.00×10^{13}	2.00×10^{14}	6.00×10^{13}
$\varepsilon = 10^{-8}$			
$\mathbb{E}^{\partial_t \mathcal{Y}_{N,n}(x,t)}$	3.40×10^{-2}	2.30×10^{-2}	1.60×10^{-2}
$\mathbb{E}^{\partial_x \mathcal{Y}_{N,n}(x,t)}$	8.00×10^{22}	8.50×10^{22}	2.00×10^{23}
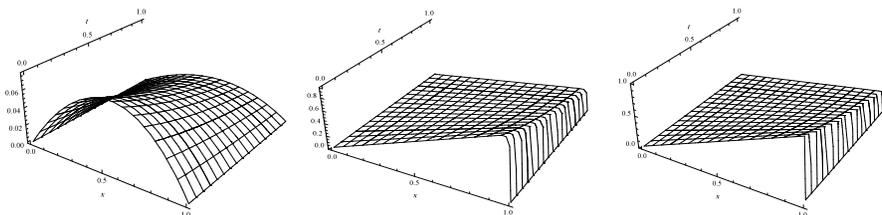
Figure 5. Approximate solution with $N = 200$ and $n = 10$ (Left: $\varepsilon = 1$; Middle: $\varepsilon = 10^{-2}$; Right: $\varepsilon = 10^{-6}$) for Examples 1, 2, 3.
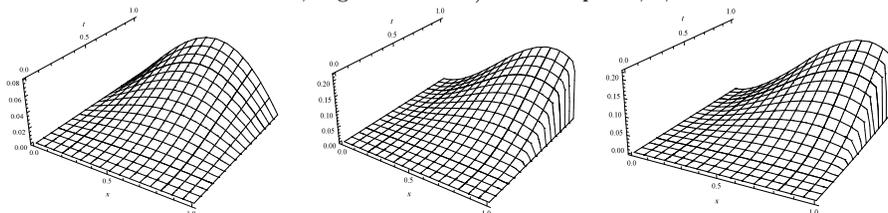
Figure 6. Approximate solution with $N = 200$ (Left: $\varepsilon = 1$; Middle: $\varepsilon = 10^{-2}$; Right: $\varepsilon = 10^{-6}$) for Example 4.
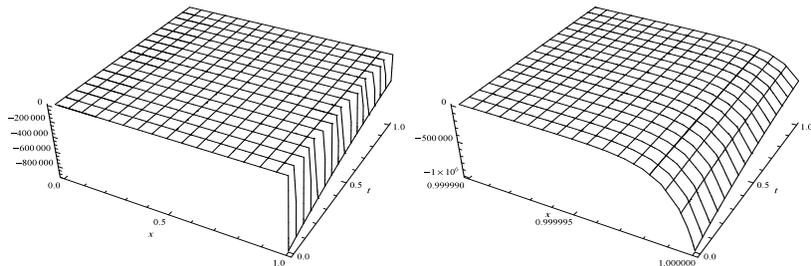
Figure 7. Derivative of the solution relative to x $(\partial_x y(x,t))$ throughout region D for Examples 1, 2, 3, with $\varepsilon = 10^{-6}$ (Left: derivative of the solution; Right: boundary layer behavior).