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Simultaneous Determination of a Source Term and Diffusion Concentration for a Multi-Term Space-Time Fractional Diffusion Equation

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Abstract. An inverse problem of determining a time dependent source term along with diffusion/temperature concentration from a non-local over-specified condition for a space-time fractional diffusion equation is considered. The space-time fractional diffusion equation involve Caputo fractional derivative in space and Hilfer fractional derivatives in time of different orders between 0 and 1. Under certain conditions on the given data we proved that the inverse problem is locally well-posed in the sense of Hadamard. Our method of proof based on eigenfunction expansion for which the eigenfunctions (which are Mittag-Leffler functions) of fractional order spectral problem and its adjoint problem are considered. Several properties of multinomial Mittag-Leffler functions are proved.

Keywords: inverse problem, fractional derivative, Bi-orthogonal system of functions, multinomial Mittag-Leffler function.

AMS Subject Classification: 26A33; 35R30; 35P10; 44A10; 33E12.

1 Introduction

In this article, we are concerned with the inverse problem of recovering the set of functions $\{u(x,t), a(t)\}$ for the following multi-term space-time fractional

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diffusion equation

$$D^{\alpha_0,\beta_0}_{0_+,t}u(x,t)+\sum_{i=1}^{m}\mu_i D^{\alpha_i,\beta_i}_{0_+,t}u(x,t)={}^{c}D^{\gamma}_{0_+,x}u(x,t)+a(t)f(x,t),\ (x,t)\in\Pi, \quad (1.1)$$

subject to the boundary conditions

$$u(0,t)=0=u(1,t), \quad t\in(0,T), \quad (1.2)$$

alongside non-homogenous initial conditions

$$\lim_{t\to 0} J^{(1-\alpha_i)(1-\beta_i)}_{0_+,t}u(x,t)={}_{i}\phi(x), \qquad i=0,1,...,m,\ m\in\mathbb{N}, \quad x\in(0,1), \quad (1.3)$$

where $D^{\alpha_i,\beta_i}_{0_+,t}$ stands for the Hilfer fractional derivatives in time variable of orders α_i, $0<\alpha_m<...<\alpha_1<\alpha_0<1$ and type $0\leq\beta_i\leq 1$, ${}^{c}D^{\gamma}_{0_+,x}$ represents Caputo fractional derivative in space variable of order $1<\gamma<2$, $\Pi:=(0,1)\times(0,T)$, μ_i, $i=1,2,...,m$ are positive real constants and $J^{(1-\alpha_i)(1-\beta_i)}_{0_+,t}$ are the Riemann-Liouville fractional integrals of orders $(1-\alpha_i)(1-\beta_i)$.

We need some additional data for unique solvability of the inverse problem, usually termed as over-determination condition and is given by

$$\int_0^1 u(x,t)dx=E(t), \quad t\in[0,T]. \quad (1.4)$$

The function $E(t)$ following consistency relation satisfies

$$\int_0^1 {}_{i}\phi(x)dx=J^{(1-\alpha_i)(1-\beta_i)}_{0_+,t}E(0), \quad i=0,1,...,m,\ m\in\mathbb{N}.$$

The structure of the source term arise in microwave heating process, in which the external energy is supplied to a target at a controlled level represented by $a(t)$ and $f(x,t)$ is the local conversion rate of the microwave energy.

The solution of the inverse problem $\{u(x,t),a(t)\}$ is said to be regular if

$$\begin{aligned}
&a\in C([0,T]),\ \ t^{\beta_0(1-\alpha_0)+\beta_i(1-\alpha_i)+1}u(x,t)\in C(\bar{\Pi}),\\
&t^{\beta_0(1-\alpha_0)+\beta_i(1-\alpha_i)+1}\Big(D^{\alpha_0,\beta_0}_{0_+,t}+\sum_{i=1}^{m}\mu_i D^{\alpha_i,\beta_i}_{0_+,t}\Big)u(x,t)\in C(\bar{\Pi}),\\
&t^{\beta_0(1-\alpha_0)+\beta_i(1-\alpha_i)+1}\ {}^{c}D^{\gamma}_{0_+,x}u(x,t)\in C(\bar{\Pi}).
\end{aligned}$$

We will present the existence, uniqueness and stability results for the solution of the inverse source problem.

Let us mention the importance of considering integrals and derivatives of arbitrary order which have been introduced in the mathematical modeling of many processes like anomalous diffusion [25], Continuous Time Random Walks (CTRW) (see [35] and references therein), predict earthquake [27], stochastic process [12], etc. In addition, in the case of heat conduction, the non-local generalization of Fourier's law and Fick's law in the case of diffusion leads to time-fractional Partial Differential Equations (PDEs) [28]. Time-fractional

PDEs are used to describe the physical phenomena that have the effect on memory and space-fractional PDEs deal with particle long-range interactions. A part from describing anomalous diffusion there are many applications of time or space or space-time PDEs, just to name a few in biology [11,13], finance [22], physics [10], viscoelasticity processes [23], forecasting of extreme events such as earthquake [8]. The order of fractional derivatives in some physical processes play a significant role and it doesn't remain the same, the fractional derivatives with variable order, the distributed order fractional derivatives are considered. In case of diffsion/transport, the order of fractional derivative used to explain the sub-diffusion processes. We used multi-term Hilfer fractional derivatives (as Hilfer fractional derivative has the important physical properties of both Riemann-Liouville and Caputo fractional derivatives) which could be used as a better model to explain anomalies by tuning the order of and type of fractional derivatives. For more detail about the applications of multi-term fractional PDEs, see [20, 24, 32] and references therein.

We provide a short survey of inverse problems for Fractional Differential Equations (FDEs). Inverse problems are considered in [6, 16, 34] to recover a space-dependent source term for Time Fractional Diffusion Equation (TFDE). Inverse problems of identifying a time-dependent source term for time-fractional telegraph equation are considered in [19], for time-fractional wave equation is considered in [33], and in [26], TFDEs are considered. For a fourth order parabolic equation with nonlocal boundary condition, two inverse source problems are considered by Sara and Malik [7]. An inverse problem involving generalized fractional derivative in diffusion and wave equations for time and space dependent source terms are discussed in [15]. In [4], for a Space-Time Fractional Diffusion Equation (STFDE) inverse problem of determining a temporal component in the source term from the total energy of the system is considered and the recovery of a space dependent source term from final data is discussed in [3]. Inverse problems of identifying the space and time dependent source terms for a STFDE are considered in [5]. Recovering order of fractional derivatives from multi-term TFDE with constant coefficients from boundary measurements is discussed in [18]. Inverse problem of simultaneously recovering the coefficient of diffusion and the source term for a multi-term TFDE is considered in [30]. Let us mention that in [1], an inverse problem of determining a space dependent source term for an equation involving only fractional derivatives in time and Bessel operator was discussed. Here we have n terms of Hilfer fractional derivative and we used Laplace transform technique for the solution of multi-term fractional order ordinary differential equation (see Section 1).

In Section 2, we presented preliminaries related to the Fractional Calculus (FC). In Section 3, we discussed the multinomial Mittag-Leffler function and related to its estimates. A bi-orthogonal system of functions from spectral problem and its adjoint problem are presented in Section 4. Our main results are discussed about the existence, uniqueness and stability of the inverse source problem in Section 5, some particular examples are also presented in the same section and in the last section we concluded our paper.

2 Preliminaries and spectral problem

In this section, we shall present some basic definitions from FC, properties and lemmas related to multinomial Mittag-Leffler function.

DEFINITION 1. [29], [14] Let $f \in L^1_{loc}([a,b]), -\infty < a < z < b < \infty$ be a locally integrable real-valued function. The left and right sided Riemann-Liouville fractional integral of order $\xi > 0$ are defined as

$$J^{\xi}_{a_+,z} f(z) := \frac{1}{\Gamma(\xi)} \int_a^z (z-\tau)^{\xi-1} f(\tau)\, d\tau,$$
$$J^{\xi}_{b_-,z} f(z) := \frac{1}{\Gamma(\xi)} \int_z^b (\tau-z)^{\xi-1} f(\tau)\, d\tau,$$

respectively.

DEFINITION 2. [10] Let $0 < \xi < 1, 0 \le \eta \le 1, f \in L^1([a,b]), -\infty < a < z < b < \infty$ and $J^{(1-\xi)(1-\eta)}_{a_+,z} f \in AC([a,b])$. The Hilfer fractional derivative of order ξ and type η is defined as

$$D^{\xi,\,\eta}_{a_+,z} f(z) := \left(J^{\eta(1-\xi)}_{a_+,z} \frac{d}{dz} J^{(1-\xi)(1-\eta)}_{a_+,z} f \right)(z).$$

The Hilfer fractional derivative interpolates both the Riemann-Liouville and the Caputo fractional derivative.

• For $\eta = 0$, the Hilfer fractional derivative becomes the Riemann-Liouville fractional derivative, i.e.,

$$D^{\xi,0}_{a_+,z} f(z) = \frac{d}{dz} J^{(1-\xi)}_{a_+,z} f(z) := {}^{RL}D^{\xi}_{a_+,z} f(z).$$

• For $\eta = 1$, the Hilfer fractional derivative becomes the Caputo fractional derivative, i.e.,

$$D^{\xi,1}_{a_+,z} f(z) = J^{1-\xi}_{a_+,z} \frac{d}{dz} f(z) := {}^{c}D^{\xi}_{a_+,z} f(z).$$

Notice that for $\beta_i = 1,\ i = 0,1,2,...,m, m \in \mathbb{N}$ the initial conditions (1.3) reduces to one condition, i.e., $u(x,0) = \phi(x),\ x \in (0,1)$.

Let $1 < \xi < 2$ and $f \in AC^2([a,b])$. The left-sided Caputo derivative of order ξ is defined as

$${}^{c}D^{\xi}_{a_+,z} f(z) := J^{2-\xi}_{a_+\tau} \frac{d^2}{dz^2} f(z) = \frac{1}{\Gamma(2-\xi)} \int_a^z (z-\tau)^{-\xi} f''(\tau)\, d\tau,$$

where for $n \in \mathbb{N}$, we have

$$AC^n([a,b]) = \left\{ f : [a,b] \to \mathbb{R} : \frac{d^{n-1}}{dx^{n-1}} f(x) \in AC([a,b]) \right\}.$$

Lemma 1. *[4] Assume that $0 < \xi < 1$, $h_1 \in AC([a,b])$ and $h_2 \in L^p([a,b])$, $1 \le p \le \infty$. Then, the following formulae of integration by parts hold*

$$\int_a^b h_1(z)\,{}^{RL}D^{\xi}_{a_+,z}h_2(z)dz = \int_a^b h_2(z)\,{}^{c}D^{\xi}_{b_-,z}h_1(z)dz + h_1(z)J^{1-\xi}_{a_+,z}h_2(z)\Big|_{z=a}^{z=b},$$
$$\int_a^b h_1(z)\,{}^{RL}D^{\xi}_{b_-,z}h_2(z)dz = \int_a^b h_2(z)\,{}^{c}D^{\xi}_{a_+,z}h_1(z)dz - h_1(z)J^{1-\xi}_{b_-,z}h_2(z)\Big|_{z=a}^{z=b}.$$

Lemma 2. *[29] Let g_i be a sequence of functions defined on $(a,b]$ for each $i \in \mathbb{N}$, such that*

(1) $D^{\xi,\eta}_{a_+,z}g_i(z)$ exists $\forall i \in \mathbb{N}$, $z \in (a,b]$,

(2) both series $\sum_{i=1}^{\infty} g_i(z)$ and $\sum_{i=1}^{\infty} D^{\xi,\eta}_{a_+,z}g_i(z)$ are uniformly convergent on the interval $[a+\epsilon, b]$ for any $\epsilon > 0$.
Then,

$$D^{\xi,\eta}_{a_+,z}\sum_{i=1}^{\infty} g_i(z) = \sum_{i=1}^{\infty} D^{\xi,\eta}_{a_+,z}g_i(z), \quad 0 < \xi \le \eta < 1,\ a < z < b.$$

Lemma 3. *[4] For $h_1(z), h_2(z) \in C^1([a,b])$, the following relation holds*

$$\frac{d}{dz}\big((h_1 * h_2)(z)\big) = h_1(z)h_2(a) + \big(h_1 * \frac{d}{dz}h_2\big)(z) = h_2(z)h_1(a) + \big(h_2 * \frac{d}{dz}h_1\big)(z).$$

3 Multinomial Mittag-Leffler functions

Definition 3. [21] For $\xi_i, \eta > 0$, $z_i \in \mathbb{C}, i = 1,2,...,m$, $m \in \mathbb{N}$, the multinomial Mittag-Leffler function is defined as

$$E_{(\xi_1,\xi_2,...,\xi_m),\eta}(z_1,z_2,...,z_m) := \sum_{k=0}^{\infty} \sum_{\substack{l_1+l_2+...+l_m=k \\ l_1\ge 0,...,l_m\ge 0}} (k; l_1,...,l_m)\frac{\Pi_{i=1}^m z_i^{l_i}}{\Gamma\Big(\eta + \sum_{i=1}^m \xi_i l_i\Big)},$$

where $(k; l_1,...,l_m) = \dfrac{k!}{l_1! \times ... \times l_m!}$.

Remark 1. Notice that the multinomial Mittag-Leffler function can be represented as

$$E_{(\xi_1,\xi_2,...,\xi_m),\eta}(z_1,z_2,...,z_m)$$
$$= \sum_{k=0}^{\infty}\sum_{l_1=0}^{k}\sum_{l_2=0}^{l_1}...\sum_{l_{m-1}=0}^{l_{m-2}} \frac{k!}{l_1!l_2!...(k-l_1-l_2-...-l_{m-1})!}$$
$$\times \frac{z_1^{k-l_1-l_2-...-l_{m-1}} z_2^{l_1}...z_m^{l_{m-1}}}{\Gamma(\eta + \xi_1(k-l_1-l_2-...-l_{m-1}) + \xi_2 l_1 + ... + \xi_m l_{m-1})}.$$

Remark 2. The parameter of multinomial Mittag-Leffler function commutes, i.e.,

$$E_{(\xi_1,\xi_2,...,\xi_m),\eta}(z_1, z_2, ..., z_m) = E_{(\xi_m,...,\xi_2,\xi_1),\eta}(z_m, ..., z_2, z_1). \tag{3.1}$$

Remark 3. For $z_1 \neq 0$ and $z_2 = z_3 = ... = z_m = 0,\ m \in \mathbb{N}$ the multinomial Mittag-Leffler function takes the following form

$$E_{\xi_1,\eta}(z_1) = \sum_{k=0}^{\infty} \frac{z_1^k}{\Gamma(\eta + \xi_1 k)}.$$

For convenience, we use the following notation

$$\mathcal{E}_{(\xi_1,\xi_2,...,\xi_m),\eta}(\tau; \sigma_1, ..., \sigma_m) := \tau^{\eta-1} E_{(\xi_1,\xi_2,...,\xi_m),\eta}(-\sigma_1 \tau^{\xi_1}, ..., -\sigma_m \tau^{\xi_m}), \tag{3.2}$$

where $\sigma_i > 0,\ i = 1, 2, ..., m$.

Lemma 4. *For $\xi_i, \eta, \tau, \sigma_i > 0,\ i = 1, 2, ..., m,\ m \in \mathbb{N}$ the Laplace transform of the multinomial Mittag-Leffler function is given by*

$$\mathcal{L}\{\mathcal{E}_{(\xi_1,\xi_2,...,\xi_m),\eta}(\tau; \sigma_1, ..., \sigma_m)\} = \frac{s^{-\eta}}{1 + \sum_{i=1}^{m} \sigma_i s^{-\xi_i}}, \quad \text{if } \left|\sum_{i=1}^{m} \sigma_i s^{-\xi_i}\right| < 1.$$

Proof. In view of notation (3.2) and Remark 1, we have

$$\begin{aligned}
&\mathcal{E}_{(\xi_1,\xi_2,...,\xi_m),\eta}(\tau; \sigma_1, \sigma_2, ..., \sigma_m) \\
&\quad = \tau^{\eta-1} \sum_{k=0}^{\infty} \sum_{i_1=0}^{k} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{m-1}=0}^{i_{m-2}} \frac{(-1)^k k!}{i_1! i_2! ... (k - i_1 - i_2 - ... - i_{m-1})!} \\
&\qquad \times \frac{(\sigma_1 \tau^{\xi_1})^{k-i_1-i_2-...-i_{m-1}} (\sigma_2 \tau^{\xi_2})^{i_1} ... (\sigma_m \tau^{\xi_m})^{i_{m-1}}}{\Gamma(\eta + \xi_1(k - i_1 - i_2 - ... - i_{m-1}) + \xi_2 i_1 + ... + \xi_m i_{m-1})}, \\
&\quad = \sum_{k=0}^{\infty} \sum_{i_1=0}^{k} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{m-1}=0}^{i_{m-2}} \frac{(-1)^k k! (\sigma_1)^{k-i_1-i_2-...-i_{m-1}} (\sigma_2)^{i_1} ... (\sigma_m)^{i_{m-1}}}{i_1! i_2! ... (k - i_1 - i_2 - ... - i_{m-1})!} \\
&\qquad \times \frac{\tau^{\eta-1+\xi_1(k-i_1-i_2-...-i_{m-1})+\xi_2 i_1+...+\xi_m i_{m-1}}}{\Gamma(\eta + \xi_1(k - i_1 - i_2 - ... - i_{m-1}) + \xi_2 i_1 + ... + \xi_m i_{m-1})}.
\end{aligned}$$

Taking Laplace transform, we obtain

$$\begin{aligned}
&\mathcal{L}\{\mathcal{E}_{(\xi_1,\xi_2,...,\xi_m),\eta}(\tau; \sigma_1, \sigma_2, ..., \sigma_m)\} \\
&= \sum_{k=0}^{\infty} \sum_{i_1=0}^{k} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{m-1}=0}^{i_{m-2}} \frac{(-1)^k k! (\sigma_1)^{k-i_1-i_2-...-i_{m-1}} (\sigma_2)^{i_1} ... (\sigma_m)^{i_{m-1}}}{i_1! i_2! ... (k - i_1 - i_2 - ... - i_{m-1})!} \\
&\quad \times \frac{1}{s^{\eta+\xi_1(k-i_1-i_2-...-i_{m-1})+\xi_2 i_1+...+\xi_m i_{m-1}}}.
\end{aligned} \tag{3.3}$$

Now, consider

$$(-1)^k(\sigma_1 s^{-\xi_1}+\sigma_2 s^{-\xi_2}+...+\sigma_m s^{-\xi_m})^k=\sum_{i_1=0}^{k}\sum_{i_2=0}^{i_1}...\sum_{i_{m-1}=0}^{i_{m-2}}(-1)^k k!$$
$$\times\frac{(\sigma_1 s^{-\xi_1})^{k-i_1-i_2-...-i_{m-1}}(\sigma_2 s^{-\xi_2})^{i_1}...(\sigma_m s^{-\xi_m})^{i_{m-1}}}{i_1!i_2!...(k-i_1-i_2-...-i_{m-1})!},$$
$$=\sum_{i_1=0}^{k}\sum_{i_2=0}^{i_1}...\sum_{i_{m-1}=0}^{i_{m-2}}\frac{(-1)^k k!}{i_1!i_2!...(k-i_1-i_2-...-i_{m-1})!}$$
$$\times\frac{(\sigma_1)^{k-i_1-i_2-...-i_{m-1}}(\sigma_2)^{i_i}...(\sigma_m)^{i_{m-1}}}{s^{\xi_1(k-i_1-i_2-...-i_{m-1})+\xi_2 i_1+...+\xi_m i_{m-1}}}.$$

From (3.3), we obtain

$$\mathcal{L}\{\mathcal{E}_{(\xi_1,\xi_2,...,\xi_m),\eta}(\tau;\sigma_1,...,\sigma_m)\}=s^{-\eta}/\Big(1+\sum_{i=1}^{m}\sigma_i s^{-\xi_i}\Big).$$

□

Remark 4. For $\sigma_1\neq 0$ and $\sigma_2=\sigma_3=...=\sigma_m=0$, $m\in\mathbb{N}$ the Laplace transform of the multinomial Mittag-Leffler function reduces to

$$\mathcal{L}\{\mathcal{E}_{\xi_1,\eta}(\tau;\sigma_1)\}=s^{\xi_1-\eta}/(s^{\xi_1}+\sigma_1),\quad |\sigma_1 s^{-\xi_1}|<1.$$

Lemma 5. *For $0<\eta<1$ and $0<\xi_m<...<\xi_1<1$ be given. Assume that $\xi_1\pi/2<\mu<\xi_1\pi$, $\mu\leq|arg(z_m)|\leq\pi$ and $z_i>0, i=1,2,...,m$. Then, there exists a constant depending only on $\mu,\xi_i, i=1,2,...,m$ such that*

$$|E_{(\xi_1-\xi_m,...,\xi_1-\xi_2,\xi_1),\eta}(z_m,...,z_2,z_1)|\leq C_1/(1+|z_m|).$$

Proof. From (3.1) and due to Lemma 3.2 of [17] required result can be proved. □

Lemma 6. *For $\xi_i,\tau,\sigma_i>0$, $i=1,2,...,m$, $m\in\mathbb{N}$, the multinomial Mittag-Leffler type function has the following form*

$$\frac{d}{d\tau}\Big(\mathcal{E}_{\boldsymbol{\xi},\xi_1+1}(\tau;\sigma_1,\sigma_2,...,\sigma_m)\Big)=\mathcal{E}_{\boldsymbol{\xi},\xi_1}(\tau;\sigma_1,\sigma_2,...,\sigma_m),$$

where $\boldsymbol{\xi}=(\xi_1,\xi_1-\xi_2,...,\xi_1-\xi_m)$.

Proof. Considering the notation (3.2), taking term by term differentiation and by arranging the terms, we get the required result. □

Lemma 7. *For $g\in C^1([a,b])$ and $\xi_i,\sigma_i>0$, for $i=1,2,...,m$, we have*

$$|g(\tau)*\mathcal{E}_{\boldsymbol{\xi},\xi_1}(\tau;\sigma_1,\sigma_2,...,\sigma_m)|\leq\frac{C_2}{\sigma_1}\|g\|_{C^1([0,T])},$$

where "$*$" represents the Laplace convolution and

$$\|g\|_{C^1([0,T])}=\sup_{t\in[0,T]}|g(t)|+\sup_{t\in[0,T]}|g'(t)|.$$

Proof. From Lemma 6, we have

$$|g(\tau) * \mathcal{E}_{\boldsymbol{\xi},\xi_1}(\tau;\sigma_1,\sigma_2,...,\sigma_m)| = \left|g(\tau) * \frac{d}{d\tau}\mathcal{E}_{\boldsymbol{\xi},\xi_1+1}(\tau;\sigma_1,\sigma_2,...,\sigma_m)\right|. \quad (3.4)$$

By using Lemma 3, we have the following relation

$$\begin{aligned}\left|g(\tau) * \frac{d}{d\tau}\mathcal{E}_{\boldsymbol{\xi},\xi_1+1}(\tau;\sigma_1,\sigma_2,...,\sigma_m)\right| =& \Big|\mathcal{E}_{\boldsymbol{\xi},\xi_1+1}(\tau;\sigma_1,\sigma_2,...,\sigma_m)g(0) \\ &+ \mathcal{E}_{\boldsymbol{\xi},\xi_1+1}(\tau;\sigma_1,\sigma_2,...,\sigma_m) * \frac{d}{d\tau}g(\tau)\Big|.\end{aligned}$$

By virtue of Lemma 5, we obtain

$$\begin{aligned}\left|g(\tau) * \frac{d}{d\tau}\mathcal{E}_{\boldsymbol{\xi},\xi_1+1}(\tau;\sigma_1,\sigma_2,...,\sigma_m)\right| &\leq \frac{C_1}{\sigma_1}|g(0)| + \frac{C_1}{\sigma_1}\int_0^t \left|\frac{d}{d\tau}g(\tau)\right|d\tau \\ &\leq \frac{C_1}{\sigma_1}(1+T)\|g\|_{C^1[0,T]}. \quad (3.5)\end{aligned}$$

Hence, due to (3.4) and (3.5), we have

$$|g(\tau) * \mathcal{E}_{\boldsymbol{\xi},\xi_1}(\tau;\sigma_1,\sigma_2,...,\sigma_m)| \leq \frac{C_2}{\sigma_1}\|g\|_{C^1[0,T]}.$$

□

Remark 5. For $\sigma_1 \neq 0, \sigma_2 = \sigma_3 = ... = \sigma_m = 0,\ m \in \mathbb{N}$ the above Lemma reduces to the following result proved in [3]

$$|g(\tau) * \tau^{\xi_1-1}E_{\xi_1,\xi_1}(-\sigma_1\tau^{\xi_1})| \leq \frac{C_2}{\sigma_1}\|g\|_{C^1([0,T])}.$$

Lemma 8. *For $\xi_i, \rho, \eta, \tau, \sigma > 0,\ i = 1,2,...,m\ m \in \mathbb{N}$, we have the following relation*

$$\begin{aligned}\tau^\rho * \mathcal{E}_{(\xi_1,\xi_2,...,\xi_m),\eta}(\tau;\sigma_1,\sigma_2,\ldots,\sigma_m) = \Gamma(\rho+1) \\ \times \mathcal{E}_{(\xi_1,\xi_2,...,\xi_m),\eta+\rho+1}(\tau;\sigma_1,\sigma_2,\ldots,\sigma_m).\end{aligned}$$

Proof. By taking the Laplace transform on the both sides, we obtained the result. □

Remark 6. For $\sigma_1 \neq 0, \sigma_2 = \sigma_3 = ... = \sigma_m = 0,\ m \in \mathbb{N}$ the above Lemma reduces the well known result

$$\tau^\rho * \mathcal{E}_{\xi_1,\eta}(\tau;\sigma_1) = \Gamma(\rho+1)\mathcal{E}_{\xi_1,\eta+\rho+1}(\tau;\sigma_1).$$

Proposition 1. *The following identities hold for Mittag-Leffler type function:*

- $\mathcal{E}_{\xi_1,3}(\tau;\sigma_1) = \tau^2/\Gamma(3) - \sigma_1\mathcal{E}_{\xi_1,3+\xi_1}(\tau;\sigma_1)$,
- $\mathcal{E}_{(\xi_1,\xi_1-\xi_2),3-\xi_2}(\tau;\sigma_1,\sigma_2) + \sigma_2\mathcal{E}_{(\xi_1,\xi_1-\xi_2),3+\xi_1-2\xi_2}(\tau;\sigma_1,\sigma_2)$
  $= \tau^{2-\xi_2}/\Gamma(3-\xi_2) - \sigma_1\mathcal{E}_{(\xi_1,\xi_1-\xi_2),3+\xi_1-\xi_2}(\tau;\sigma_1,\sigma_2)$.

Proof. Using the series expansion of $\mathcal{E}_{\xi_1,3}(\tau;\sigma_1)$ and $\mathcal{E}_{\xi_1,3+\xi_1}(\tau;\sigma_1)$, we have

$$\mathcal{E}_{\xi_1,3}(\tau;\sigma_1) = \tau^2 \sum_{k=0}^{\infty} \frac{(-\sigma_1\tau^{\xi_1})^k}{\Gamma(\xi_1 k+3)},\ \mathcal{E}_{\xi_1,3+\xi_1}(\tau;\sigma_1) = \tau^{2+\xi_1} \sum_{k=0}^{\infty} \frac{(-\sigma_1\tau^{\xi_1})^k}{\Gamma(\xi_1 k+\xi_1+3)}.$$

The first identity can be obtained from the above expression. Similarly, we can prove the second identity. □

4 Bi-orthogonal system

In this section, the spectral analysis corresponding to system (1.1)–(1.2) presented. We will construct the solution of the inverse source problem by using the Fourier's method, frequently known as separation of variables.

The spectral problem corresponding to (1.1)–(1.2) is

$$ {}^{c}D^{\gamma}_{0_+,x}X(x) = \lambda X(x), \qquad X(0) = 0 = X(1). \tag{4.1}$$

The spectral problem was considered in [2] and the eigenfunctions of the spectral problem are

$$\{X_n(x)\}_{n=1}^{\infty} = \{x^{\gamma-1}E_{\gamma,\gamma}(\lambda_n x^{\gamma})\}_{n=1}^{\infty},$$

corresponding to the eigenvalues λ_n which are the zeros of the function $E_{\gamma,\gamma}(\lambda_n)$ with $\mathrm{Im}(\lambda_n) > 0$.

The set $\{X_n(x)\}_{n=1}^{\infty}$ of eigenfunctions is complete but not orthogonal [2]. Due to fractional operator the spectral problem is non-self-adjoint. For the adjoint problem of the spectral problem (4.1), we have

$$\left\langle {}^{c}D^{\gamma}_{0_+,x}X(.), Y(.)\right\rangle = \left\langle J^{2-\gamma}_{0_+,x}\frac{d^2}{dx^2}X(.), Y(.)\right\rangle.$$

Due to Lemma 4.1 of [9], integration by parts, taking $Y(0) = 0 = Y(1)$ and then using Lemma 1, the adjoint problem of the spectral problem (4.1) is

$$ {}^{c}D^{\gamma}_{1_-,x}Y(x) = \lambda Y(x), \qquad Y(0) = 0 = Y(1).$$

For more details (see [5] page 7).

The adjoint problem has eigenfunctions $Y_n(x)$ corresponding to the same eigenvalues as that of spectral problem while eigenfunctions are

$$\{Y_n(x)\}_{n=1}^{\infty} = \{(1-x)^{\gamma-1}E_{\gamma,\gamma}\big(\lambda_n(1-x)^{\gamma}\big)\}_{n=1}^{\infty}.$$

The sets $\{X_n(x)\}_{n=1}^{\infty}$ and $\{Y_n(x)\}_{n=1}^{\infty}$ form a bi-orthogonal system of functions [2].

Now, we are going to describe some properties of the eigenvalues of the spectral problem.

Lemma 9. *[2] The eigenvalues λ_n, that are the zeros of the function $E_{\gamma,\gamma}(\lambda)$ with $Im(\lambda_n) > 0$, satisfy the following relations*

- $|\lambda_k| < |\lambda_{k+1}|$, *for* $k \geq 1$.
- *For* n *large enough and* $arg(\lambda_n) > \frac{\gamma\pi}{2}$, *we have* $|e^{\lambda_n t}| < 1$ *and* $|\lambda_n| \sim O(n^\gamma), \quad 1 < \gamma < 2$.

Lemma 10. *[5] For any* $h \in C^2([0,1])$ *such that* $h(0) = 0 = h(1)$, *we have the following relation*

$$h_n \leq \frac{C_1}{|\lambda_n|(1-\alpha)(2-\alpha)} \left(h'(0) + \int_0^1 h''(x)(1-x)^{2-\alpha} dx \right),$$

where C_1 *is a constant and* $h_n = \langle h(.), Y_n(.) \rangle$.

5 The main results

In this section, we will discuss the main results of this research article. Firstly, we are going to present the following theorem that states the conditions under which inverse problem has a classical solution.

Theorem 1. *Let the following conditions hold:*

(1) ${}_i\phi \in C^2([0,1])$ *be such that* ${}_i\phi(0) = 0 = {}_i\phi(1)$ *for* $i = 0, 1, ..., m$.

(2) $f(.,t) \in C^2([0,1])$ *be such that* $f(0,t) = 0 = f(1,t)$.

Furthermore,

$$\left(\int_0^1 f(x,t) dx \right)^{-1} \leq M_1$$

for some positive constant M_1,

(3) $E \in AC([0,T])$ *and satisfies the following consistency conditions*

$$\int_0^1 {}_i\phi(x) dx = \lim_{t \to 0} J_{0+,t}^{(1-\alpha_i)(1-\beta_i)} E(t), \qquad i = 0, 1, ..., m.$$

Then, the inverse problem (1.1)–(1.4), is locally well-posed in time.

Proof. Our proof consists of three steps, construction of the series solution by eigenfunction expansion method, unique existence of the solution and stability of the solution.

Construction of the solution: The solution of the inverse problem (1.1)–(1.4) can be written by using Fourier's method

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x),$$

where $T_n(t)$ satisfy the following fractional differential equation

$$D_{0+,t}^{\alpha_0,\beta_0} T_n(t) + \sum_{i=1}^{m} \mu_i D_{0+,t}^{\alpha_i,\beta_i} T_n(t) = \lambda_n T_n(t) + a(t) f_n(t)$$

and $f_n(t) = \langle f(.,t), Y_n(.)\rangle$.

By using the Laplace transform and the initial conditions (1.3), we get

$$\mathcal{L}\{T_n(t)\}=\frac{{}_0\phi_n s^{\beta_0(\alpha_0-1)}}{s^{\alpha_0}+\sum\limits_{i=1}^{m}\mu_i s^{\alpha_i}-\lambda_n}+\frac{\sum_{i=1}^{m}{}_i\phi_n\mu_i s^{\beta_i(\alpha_i-1)}}{s^{\alpha_0}+\sum\limits_{i=1}^{m}\mu_i s^{\alpha_i}-\lambda_n}+\frac{\mathcal{L}\{a(t)f_n(t)\}}{s^{\alpha_0}+\sum\limits_{i=1}^{m}\mu_i s^{\alpha_i}-\lambda_n},$$

where ${}_0\phi_n = \langle {}_0\phi(.), Y_n(.)\rangle$ and ${}_i\phi_n = \langle {}_i\phi(.), Y_n(.)\rangle$, for $i = 1, 2, ..., m$ are the coefficients of series expansion of ${}_i\phi(x)$.

By virtue of Lemma 4, we have

$$\begin{aligned} T_n(t) =& {}_0\phi_n\mathcal{E}_{\boldsymbol{\alpha},\alpha_0+\beta_0-\alpha_0\beta_0}(t;\lambda_n,\mu_1,\mu_2,...,\mu_m)\\ &+\sum_{i=1}^{m}{}_i\phi_n\mu_i\mathcal{E}_{\boldsymbol{\alpha},\alpha_0+\beta_i-\alpha_i\beta_i}(t;\lambda_n,\mu_1,\mu_2,...,\mu_m)\\ &+a(t)f_n(t)*\mathcal{E}_{\boldsymbol{\alpha},\alpha_0}(t;\lambda_n,\mu_1,\mu_2,...,\mu_m), \end{aligned}$$

where $\boldsymbol{\alpha}$ is defined as $\boldsymbol{\alpha} = (\alpha_0, \alpha_0 - \alpha_1, ..., \alpha_0 - \alpha_m)$.

Hence, the solution $u(x,t)$ takes the form

$$\begin{aligned} u(x,t) =& \sum_{n=1}^{\infty}\Big\{{}_0\phi_n\mathcal{E}_{\boldsymbol{\alpha},\alpha_0+\beta_0-\alpha_0\beta_0}(t;\lambda_n,\mu_1,\mu_2,...,\mu_m)\\ &+\sum_{i=1}^{m}{}_i\phi_n\mu_i\mathcal{E}_{\boldsymbol{\alpha},\alpha_0+\beta_i-\alpha_i\beta_i}(t;\lambda_n,\mu_1,\mu_2,...,\mu_m)\\ &+a(t)f_n(t)*\mathcal{E}_{\boldsymbol{\alpha},\alpha_0}(t;\lambda_n,\mu_1,\mu_2,...,\mu_m)\Big\}x^{\gamma-1}E_{\gamma,\gamma}(\lambda_n x^{\gamma}), \end{aligned} \tag{5.1}$$

where $a(t)$ is still to be determined.

By using over-determination condition (1.4), we get

$$\int_0^1\left(D_{0+,t}^{\alpha_0,\beta_0}u(x,t)+\sum_{i=1}^{m}\mu_i D_{0+,t}^{\alpha_i,\beta_i}u(x,t)\right)dx=\left(D_{0+,t}^{\alpha_0,\beta_0}+\sum_{i=1}^{m}\mu_i D_{0+,t}^{\alpha_i,\beta_i}\right)E(t).$$

From (1.1), we obtain

$$\int_0^1\left({}^cD_{0+,x}^{\gamma}u(x,t)+a(t)f(x,t)\right)dx=D_{0+,t}^{\alpha_0,\beta_0}E(t)+\sum_{i=1}^{m}\mu_i D_{0+,t}^{\alpha_i,\beta_i}E(t),$$

which leads to

$$a(t)=\left[\int_0^1 f(x,t)dx\right]^{-1}\Bigg[D_{0+,t}^{\alpha_0,\beta_0}E(t)+\sum_{i=1}^{m}\mu_i D_{0+,t}^{\alpha_i,\beta_i}E(t)$$
$$-\Bigg\{\sum_{n=1}^{\infty}\lambda_n\Big({}_0\phi_n\mathcal{E}_{\boldsymbol{\alpha},\alpha_0+\beta_0-\alpha_0\beta_0}(t;\lambda_n,\mu_1,\mu_2,...,\mu_m)$$
$$+\sum_{i=1}^{m}{}_i\phi_n\mu_i\mathcal{E}_{\boldsymbol{\alpha},\alpha_0+\beta_i-\alpha_i\beta_i}(t;\lambda_n,\mu_1,\mu_2,...,\mu_m)\Big)$$
$$+\int_0^t\sum_{n=1}^{\infty}a(\tau)f_n(\tau)(t-\tau)^{\alpha_0-1}E_{\boldsymbol{\alpha},\alpha_0}(t-\tau;\lambda_n,\mu_1,\mu_2,...,\mu_m)d\tau\Bigg\}$$
$$\Big(E_{\gamma,1}(\lambda_n)-1\Big)\Bigg]. \tag{5.2}$$

Setting

$$\digamma(t)=\sum_{n=1}^{\infty}\lambda_n\Big\{{}_0\phi_n\mathcal{E}_{\boldsymbol{\alpha},\alpha_0+\beta_0-\alpha_0\beta_0}(t;\lambda_n,\mu_1,\mu_2,\dots,\mu_m)$$
$$+\sum_{i=1}^{m}{}_i\phi_n\mu_i\mathcal{E}_{\boldsymbol{\alpha},\alpha_0+\beta_i-\alpha_i\beta_i}(t;\lambda_n,\mu_1,\mu_2,...,\mu_m)\Big\}\Bigg(E_{\gamma,1}(\lambda_n)-1\Bigg) \tag{5.3}$$

$$K(t,\tau)=\sum_{n=1}^{\infty}\lambda_n f_n(\tau)(t-\tau)^{\alpha_0-1}E_{\boldsymbol{\alpha},\alpha_0}(t-\tau;\lambda_n,\mu_1,\mu_2,\dots,\mu_m)\Big(E_{\gamma,1}(\lambda_n)-1\Big). \tag{5.4}$$

Thus, (5.2) can be written as

$$a(t)=\left(\int_0^1 f(x,t)dx\right)^{-1}\Bigg(D_{0+,t}^{\alpha_0,\beta_0}E(t)+\sum_{i=1}^{m}\mu_i D_{0+,t}^{\alpha_i,\beta_i}E(t)-\digamma(t)$$
$$-\int_0^t K(t,\tau)a(\tau)d\tau\Bigg). \tag{5.5}$$

Define the mapping $B:C([0,T])\to C([0,T])$ by

$$B(a(t)):=a(t), \tag{5.6}$$

where $a(t)$ is given by (5.5).

First, we will show that for $a\in C([0,T])$, i.e., the mapping $B(a(t))$ is well-defined then the mapping will be proved to be a contraction. Lemma 5 and Equation (5.3) yield the following relation

$$t^{\beta_0(1-\alpha_0)+\beta_i(1-\alpha_i)+1}|\digamma(t)|\le\sum_{n=1}^{\infty}\frac{C_1^2}{|\lambda_n|}\Big\{|{}_0\phi_n|t^{\beta_i(1-\alpha_i)+1}$$
$$+\sum_{i=1}^{m}\mu_i|{}_i\phi_n|t^{\beta_0(1-\alpha_0)+1}\Big\}. \tag{5.7}$$

The uniform convergence of the series involved in (5.7) is ensured by using Lemmas 9–10 and continuity of ${}_i\phi(x)$. Hence, by Weierstrass M-test, $t^{\beta_0(1-\alpha_0)+\beta_i(1-\alpha_i)+1}F(t)$ represents a continuous function.

Next, we will show that $K(t,\tau)$ represents a continuous function. Using Lemma 5 and Equation (5.4), we have the following inequality

$$(t-\tau)|K(t,\tau)| \le \sum_{n=1}^{\infty} \frac{C_1^2}{|\lambda_n|}|f_n(\tau)|. \tag{5.8}$$

By using Lemma 9, continuity of $f(x,t)$ and by Weierstrass M-test for the uniform convergence of the series involved in (5.8), we can deduce that $(t-\tau)K(t,\tau)$ represents a continuous function. Furthermore, we can have $M_2>0$ such that

$$\int_0^t |K(t,\tau)|d\tau \le TM_2.$$

Hence, the mapping defined by (5.6) is well-defined.

Now, we will show that the mapping $B(a(t)) := a(t)$ is a contraction. Consider

$$|B(a(t)) - B(b(t))| = \left(\int_0^1 f(x,t)dx\right)^{-1} \left(\int_0^t K(t,\tau)|a(\tau)-b(\tau)|d\tau\right).$$

By assumptions of Theorem 1, we obtain

$$\max_{0\le t\le T}|B(a(t)) - B(b(t))| \le M_1M_2T \max_{0\le t\le T}|a(\tau)-b(\tau)|,$$
$$\|B(a)-B(b)\|_{C([0,T])} \le M_1M_2T\|a-b\|_{C([0,T])}$$

for $T<\frac{1}{M_1M_2}$, where M_1 and M_2 are positive constants independent of n. Which shows that the mapping is a contraction. Hence, by Banach fixed point theorem unique existence of $a(t)$ is ensured.

Next, we will prove the regularity of the solution $u(x,t)$ given by (5.1), that is, $t^{\beta_0(1-\alpha_0)+\beta_i(1-\alpha_i)+1}u(x,t)$, $t^{\beta_0(1-\alpha_0)+\beta_i(1-\alpha_i)+1}\ {}^cD^{\gamma}_{0+,x}u(x,t)$ and $t^{\beta_0(1-\alpha_0)+\beta_i(1-\alpha_i)+1}\left(D^{\alpha_0,\beta_0}_{0+,t}u(x,t)+\sum_{i=1}^m \mu_i D^{\alpha_i,\beta_i}_{0+,t}u(x,t)\right)$ represent continuous functions.

Since, $a\in C([0,T])$ and for some $M>0$, we have $\|a\|_{C([0,T])}\le M$. From Equation (5.1) and due to Lemmas 5, 7, we obtain

$$\begin{aligned} t^{\beta_0(1-\alpha_0)+\beta_i(1-\alpha_i)+1}|u(x,t)| \le \sum_{n=1}^{\infty}\frac{C_1}{|\lambda_n|^2}\Big\{ & C_1|{}_0\phi_n|t^{\beta_i(1-\alpha_i)+1} \\ & + \sum_{i=1}^m C_1\mu_i|{}_i\phi_n|t^{\beta_0(1-\alpha_0)+1} + C_2M\|f_n\|_{C([0,T])}\Big\}. \end{aligned} \tag{5.9}$$

Consequently, continuity of ${}_i\phi(x), a(t)$, by virtue of Lemmas 9 and 10, Weierstrass M-test and inequality (5.9), the uniform convergence of series involved in $t^{\beta_0(1-\alpha_0)+\beta_i(1-\alpha_i)+1}\ u(x,t)$ is obtained. For the uniform convergence of $t^{\beta_0(1-\alpha_0)+\beta_i(1-\alpha_i)+1}\ {}^cD^{\gamma}_{0+,x}u(x,t)$, we consider

$$\begin{aligned}
&\sum_{n=1}^{\infty} {}^{c}D^{\gamma}_{0_+,x} T_n(t)X_n(x) = \sum_{n=1}^{\infty} \lambda_n T_n(t)X_n(x),\\
&\quad = \sum_{n=1}^{\infty} \lambda_n \Big\{ \phi_0 \mathcal{E}_{\boldsymbol{\alpha},\alpha_0+\beta_0-\alpha_0\beta_0}(t;\lambda_n,\mu_1,\mu_2,\dots,\mu_m)\\
&\quad + \sum_{i=1}^{m} {}_i\phi_n \mu_i \mathcal{E}_{\boldsymbol{\alpha},\alpha_0+\beta_i-\alpha_i\beta_i}(t;\lambda_n,\mu_1,\mu_2,\dots,\mu_m)\\
&\quad + a(t)f_n(t) * \mathcal{E}_{\boldsymbol{\alpha},\alpha_0}(t;\lambda_n,\mu_1,\mu_2,\dots,\mu_m)\Big\} x^{\gamma-1}E_{\gamma,\gamma}(\lambda_n x^{\gamma}).
\end{aligned}$$

By using Lemmas 5 and 7, we have the following inequality

$$t^{\beta_0(1-\alpha_0)+\beta_i(1-\alpha_i)+1}\left|\sum_{n=1}^{\infty} {}^{c}D^{\gamma}_{0_+,x} T_n(t)X_n(x)\right| \le \sum_{n=1}^{\infty} \frac{C_1}{|\lambda_n|}\Big[C_1 |{}_0\phi_n| t^{\beta_i(1-\alpha_i)+1}$$

$$+ \sum_{i=1}^{m} C_1 \mu_i |{}_i\phi_n| t^{\beta_0(1-\alpha_0)+1} + C_2 M \|f_n\|_{C([0,T])}\Big].$$

This implies $t^{\beta_0(1-\alpha_0)+\beta_i(1-\alpha_i)+1}\sum_{n=1}^{\infty} {}^{c}D^{\gamma}_{0_+,x}T_n(t)X_n(x)$ represents a continuous function. Hence, by using Lemma 2, $t^{\beta_0(1-\alpha_0)+\beta_i(1-\alpha_i)+1}\,{}^{c}D^{\gamma}_{0_+,x}u(x,t)$ is uniformly convergent. Similarly, the uniform convergence of $t^{\beta_0(1-\alpha_0)+\beta_i(1-\alpha_i)+1}\left(D^{\alpha_0,\beta_0}_{0_+,t} + \sum_{i=1}^{m}\mu_i D^{\alpha_i,\beta_i}_{0_+,t}\right)u(x,t)$ can be obtained.

Uniqueness of the Solution: Let $\{u(x,t),a_1(t)\}$ and $\{v(x,t),a_2(t)\}$ be the two regular solution sets of the inverse problem (1.1)–(1.3) and define $\tilde{u}(x,t) = u(x,t) - v(x,t)$, $\tilde{a}(t) = a_1(t) - a_2(t)$. Then, the function $\tilde{u}(x,t)$ satisfy the following equation

$$D^{\alpha_0,\beta_0}_{0_+,t}\tilde{u}(x,t) + \sum_{i=1}^{m}\mu_i D^{\alpha_i,\beta_i}_{0_+,t}\tilde{u}(x,t) = {}^{c}D^{\gamma}_{0_+,x}\tilde{u}(x,t), \quad (x,t)\in \mathit{\Pi}$$

with the boundary conditions

$$\tilde{u}(0,t) = 0 = \tilde{u}(1,t), \quad t\in[0,T]$$

and the initial conditions

$$\lim_{t\to 0} J^{(1-\alpha_i)(1-\beta_i)}_{0_+,t}\tilde{u}(x,t) = 0, \quad i = 0,1,...,m,\ m\in\mathbb{N}, \quad x\in[0,1]. \tag{5.10}$$

Consider the function

$$\tilde{T}_n(t) = \int_0^1 \tilde{u}(x,t)Y_n(x)dx.$$

Applying the multi-term Hilfer fractional derivatives, we have

$$D^{\alpha_0,\beta_0}_{0_+,t}\tilde{T}_n(t) + \sum_{i=1}^{m}\mu_i D^{\alpha_i,\beta_i}_{0_+,t}\tilde{T}_n(t) = \left(D^{\alpha_0,\beta_0}_{0_+,t} + \sum_{i=1}^{m}\mu_i D^{\alpha_i,\beta_i}_{0_+,t}\right)\int_0^1 \tilde{u}(x,t)Y_n(x)dx.$$

By virtue of (1.1), we have the following fractional differential equation

$$D_{0+,t}^{\alpha_0,\beta_0}\tilde{T}_n(t)+\sum_{i=1}^{m}\mu_i D_{0+,t}^{\alpha_i,\beta_i}\tilde{T}_n(t)=\lambda_n\tilde{T}_n(t)+\tilde{a}(t)f_n(t).$$

The solution of the above equation is

$$\begin{aligned}\tilde{T}_n(t)=&J_{0+,t}^{(1-\alpha_0)(1-\beta_0)}\tilde{T}_n(0)\mathcal{E}_{\boldsymbol{\alpha},\alpha_0+\beta_0-\alpha_0\beta_0}(t;\lambda_n,\mu_1,\mu_2,...,\mu_m)\\ &+\sum_{i=1}^{m}J_{0+,t}^{(1-\alpha_i)(1-\beta_i)}\tilde{T}_n(0)\mu_i\mathcal{E}_{\boldsymbol{\alpha},\alpha_0+\beta_i-\alpha_i\beta_i}(t;\lambda_n,\mu_1,\mu_2,...,\mu_m)\\ &+\tilde{a}(t)f_n(t)*\mathcal{E}_{\boldsymbol{\alpha},\alpha_0}(t;\lambda_n,\mu_1,\mu_2,...,\mu_m).\end{aligned}$$

By applying the initial conditions (5.10), uniqueness of $\tilde{a}(t)$ and completeness of $Y_n(x)$, we obtain

$$\tilde{T}_n(t)=0,\quad \forall\, n\in\mathbb{N}\cup\{0\}$$

and $u(x,t)=v(x,t)$, respectively.

Stability of the Solution: Let $\{u(x,t),a(t)\}$ and $\{\tilde{u}(x,t),\tilde{a}(t)\}$ be two solution sets of the inverse problem (1.1)–(1.4), corresponding to the data $\{{}_i\phi(x),E(t)\}$ and $\{{}_i\tilde{\phi}(x),\tilde{E}(t)\}$, respectively. Using (5.5), we have

$$\begin{aligned}|a(t)-\tilde{a}(t)|=&\left(\int_0^1 f(x,t)dx\right)^{-1}\Big\{\left(D_{0+,t}^{\alpha_0,\beta_0}E(t)-D_{0+,t}^{\alpha_0,\beta_0}\tilde{E}(t)\right)\\ &+\sum_{i=1}^{m}\mu_i\left(D_{0+,t}^{\alpha_i,\beta_i}E(t)-D_{0+,t}^{\alpha_i,\beta_i}\tilde{E}(t)\right)+\left(F(t)-\tilde{F}(t)\right)+\left(V(t)-\tilde{V}(t)\right)\Big\},\end{aligned}\tag{5.11}$$

where $F(t)$ and $\tilde{F}(t)$ are given by (5.3), corresponding to data $\{{}_i\phi(x),E(t)\}$ and $\{{}_i\tilde{\phi}(x),\tilde{E}(t)\}$, respectively, $V(t)$ and $\tilde{V}(t)$ are

$$V(t)=\sum_{n=1}^{\infty}\lambda_n a(t)f_n(t)*\mathcal{E}_{\boldsymbol{\alpha},\alpha_0}(t;\lambda_n,\mu_1,\mu_2,...,\mu_m)\Big(E_{\alpha,1}(\lambda_n)-1\Big),\tag{5.12}$$

$$\tilde{V}(t)=\sum_{n=1}^{\infty}\lambda_n\tilde{a}(t)f_n(t)*\mathcal{E}_{\boldsymbol{\alpha},\alpha_0}(t;\lambda_n,\mu_1,\mu_2...,\mu_m)\Big(E_{\alpha,1}(\lambda_n)-1\Big).\tag{5.13}$$

For the Equation (5.3) and due to Lemma 5, we obtain

$$\begin{aligned}t^{\beta_0(1-\alpha_0)+\beta_i(1-\alpha_i)+1}|F(t)-\tilde{F}(t)|\le&\sum_{n=1}^{\infty}\frac{C_1^2}{|\lambda_n|}\Big\{t^{\beta_i(1-\alpha_i)+1}|{}_0\phi_n-{}_0\tilde{\phi}_n|\\ &+\sum_{i=1}^{m}\mu_i t^{\beta_0(1-\alpha_0)+1}|{}_i\phi_n-{}_i\tilde{\phi}_n|\Big\}.\end{aligned}$$

By using Cauchy-Bunyakovsky-Schwarz inequality, we obtain

$$\begin{aligned}t^{\beta_0(1-\alpha_0)+\beta_i(1-\alpha_i)+1}\|F-\tilde{F}\|_{C([0,T])}\le&\sum_{n=1}^{\infty}\frac{C_1^2}{|\lambda_n|}\Big\{t^{\beta_i(1-\alpha_i)+1}\|{}_0\phi-{}_0\tilde{\phi}\|_{C^2([0,1])}\\ &+\sum_{i=1}^{m}\mu_i t^{\beta_0(1-\alpha_0)+1}\|{}_i\phi-{}_i\tilde{\phi}\|_{C^2([0,1])}\Big\}.\end{aligned}$$

Similarly, from the Equations (5.12) and (5.13), we obtain

$$\|V-\tilde{V}\|_{C([0,T])} \leq \sum_{n=1}^{\infty} \frac{A_1C_1C_2}{|\lambda_n|}\|a-\tilde{a}\|_{C([0,T])},$$

where $A_1 = \|f\|_{C(\Pi)}$.

Under the assumption $\|{}^{RL}D_{0+,t}^{\alpha_i+\beta_i-\alpha_i\beta_i}E(t)\|_{C([0,T])} \leq A_2\|E\|_{C([0,T])}$, where A_2 is constant, we can have (see [31])

$$\|D_{0+,t}^{\alpha_i,\beta_i}E - D_{0+,t}^{\alpha_i,\beta_i}\tilde{E}\|_{C([0,T])} \leq A_3\|E-\tilde{E}\|_{C([0,T])}, \quad i=0,1,...,m,\ m\in\mathbb{N},$$

for some constant A_3, and from (5.11), we obtain

$$\begin{aligned}\Big(1-\sum_{n=1}^{\infty}\frac{A_1C_1C_2}{|\lambda_n|}\Big)\|a-\tilde{a}\|_{C([0,T])} &\leq M_1\Big\{A_2\|E-\tilde{E}\|_{C([0,T])}\\ &+\sum_{i=1}^{m}\mu_iA_2\|E-\tilde{E}\|_{C([0,T])}+\sum_{n=1}^{\infty}\frac{C_1^2}{|\lambda_n|}\Big(t^{\beta_i(1-\alpha_i)+1}\|{}_0\phi-{}_0\tilde{\phi}\|_{C^2([0,1])}\\ &+\sum_{i=1}^{m}\mu_it^{\beta_0(1-\alpha_0)+1}\|{}_i\phi-{}_i\tilde{\phi}\|_{C^2([0,1])}\Big)\Big\}.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\|a-\tilde{a}\|_{C([0,T])} &\leq M_1\left(1-\sum_{n=1}^{\infty}\frac{A_1C_1C_2}{|\lambda_n|}\right)^{-1}\Big\{A_2\|E-\tilde{E}\|_{C([0,T])}\\ &+\sum_{i=1}^{m}\mu_iA_2\|E-\tilde{E}\|_{C([0,T])}+\sum_{n=1}^{\infty}\frac{C_1^2}{|\lambda_n|}\Big(t^{\beta_i(1-\alpha_i)+1}\|{}_0\phi-{}_0\tilde{\phi}\|_{C^2([0,1])}\\ &+\sum_{i=1}^{m}\mu_it^{\beta_0(1-\alpha_0)+1}\|{}_i\phi-{}_i\tilde{\phi}\|_{C^2([0,1])}\Big)\Big\},\end{aligned}$$

which shows the stability of time dependent term $a(t)$. Similarly, stability of $u(x,t)$ can be proved. □

Remark 7. For $\beta_i = 1, i = 0,1,2,...,m,\ m\in\mathbb{N}$, the solution $u(x,t)$ has the following form

$$\begin{aligned}u(x,t) = \sum_{n=1}^{\infty}\Big\{&\phi_n\mathcal{E}_{\boldsymbol{\alpha},1}(t;\lambda_n,\mu_1,\mu_2,...,\mu_m)+\sum_{i=1}^{m}\phi_n\mu_i\mathcal{E}_{\boldsymbol{\alpha},\alpha_0-\alpha_i+1}(t;\lambda_n,\mu_1,\\ &\mu_2,\dots,\mu_m)+a(t)f_n(t)*\mathcal{E}_{\boldsymbol{\alpha},\alpha_0}(t;\lambda_n,\mu_1,\mu_2,\dots,\mu_m)\Big\}x^{\gamma-1}E_{\gamma,\gamma}(\lambda_nx^{\gamma}).\end{aligned}$$

Remark 8. The inverse problem proved to be locally well-posed in Theorem 1, the result about global existence of the source term can be obtained by applying other fixed point arguments. The T is required to be small in proving existence of the solution of inverse problem, better estimates of multinomial Mittag-Leffler functions is one way for having global solution in time.

Example 1. In Equation (1.1), we take only one fractional derivative, that is, $\mu_i = 0,\ i = 1, 2, ..., m,\ f(x,t) = \Big(\frac{\Gamma(4)}{\Gamma(4-\alpha_0)} - \lambda_1 t^{\alpha_0}\Big) x^{\gamma-1} E_{\gamma,\gamma}(\lambda_1 x^{\alpha_0})$, in (1.3) the initial condition is taken to be zero and over-specified condition is

$$\int_0^1 u(x,t)dx = t^3\Big(E_{\gamma,1}(\lambda_1) - 1\Big).$$

Indeed, using (5.1) the solution of the system is given by

$$u(x,t) = \Big\{a(t)f_1(t) * \mathcal{E}_{\alpha_0,\alpha_0}(t;\lambda_1)\Big\} x^{\gamma-1} E_{\gamma,\gamma}(\lambda_1 x^{\gamma}),$$

where

$$f_1(t) = \Big(\Gamma(4)/\Gamma(4-\alpha_0) - \lambda_1 t^{\alpha_0}\Big).$$

By using Lemma 8 and Proposition 1, we obtain

$$u(x,t) = t^3 x^{\gamma-1} E_{\gamma,\gamma}(\lambda_1 x^{\gamma}).$$

The expression for $a(t)$ given by (5.5) takes the form

$$a(t) = \left(\int_0^1 f(x,t)dx\right)^{-1} \left(D_{0+,t}^{\alpha_0,\beta_0} E(t) - F(t) - \int_0^t K(t,\tau)a(\tau)d\tau\right),$$

where

$$\begin{aligned} \int_0^1 f(x,t)dx =& \left(\frac{\Gamma(4)}{\Gamma(4-\alpha_0)} - \lambda_1 t^{\alpha_0}\right)\Big(E_{\gamma,1}(\lambda_1) - 1\Big), \\ D_{0+,t}^{\alpha_0,\beta_0} E(t) =& \frac{\Gamma(4)}{\Gamma(4-\alpha_0)} t^{3-\alpha_0}\Big(E_{\gamma,1}(\lambda_1) - 1\Big), \quad F(t) = 0, \\ K(t,\tau) =& \left(\frac{\Gamma(4)}{\Gamma(4-\alpha_0)} - \lambda_1 \tau^{\alpha_0}\right)\mathcal{E}_{\alpha_0,\alpha_0}(t-\tau;\lambda_1)\Big(E_{\gamma,1}(\lambda_1) - 1\Big). \end{aligned}$$

In this case, we can find expression for $a(t)$ given by

$$a(t) = t^{3-\alpha_0}.$$

Example 2. In Equation (1.1), we take two fractional derivatives, that is, $\mu_i = 0,\quad i = 2, 3, ..., m,\quad f(x,t) = \Big(\frac{\Gamma(4-\alpha_1)}{\Gamma(4-\alpha_0-\alpha_1)} + \mu_1 \frac{\Gamma(4-\alpha_1)}{\Gamma(4-2\alpha_1)} t^{\alpha_0-\alpha_1} - \lambda_1 t^{\alpha_0}\Big)$ $x^{\gamma-1} E_{\gamma,\gamma}(\lambda_1 x^{\alpha_0})$, in (1.3) the initial condition is taken to be zero and over-specified condition is

$$\int_0^1 u(x,t)dx = t^{3-\alpha_1}\Big(E_{\gamma,1}(\lambda_1) - 1\Big).$$

The solution in this case involve multinomial Mittag-Leffler function and is given by

$$u(x,t) = \Big\{ a(t) f_1(t) * \mathcal{E}_{(\alpha_0,\alpha_0-\alpha_1),\alpha_0}(t;\lambda_1,\mu_1) \Big\} x^{\gamma-1} E_{\gamma,\gamma}(\lambda_1 x^\gamma),$$

where

$$f_1(t) = \left(\frac{\Gamma(4-\alpha_1)}{\Gamma(4-\alpha_0-\alpha_1)} + \mu_1 \frac{\Gamma(4-\alpha_1)}{\Gamma(4-2\alpha_1)} t^{\alpha_0-\alpha_1} - \lambda_1 t^{\alpha_0} \right).$$

Due to Lemma 8 and Proposition 1, we get

$$u(x,t) = t^{3-\alpha_1} x^{\gamma-1} E_{\gamma,\gamma}(\lambda_1 x^\gamma).$$

The expression for $a(t)$ given by (5.5) takes the form

$$a(t) = \left(\int_0^1 f(x,t)dx \right)^{-1} \left(D_{0+,t}^{\alpha_0,\beta_0} E(t) + \mu_1 D_{0+,t}^{\alpha_1,\beta_1} E(t) - F(t) - \int_0^t K(t,\tau) a(\tau) d\tau \right),$$

where

$$\int_0^1 f(x,t)dx = \left(\frac{\Gamma(4-\alpha_1)}{\Gamma(4-\alpha_0-\alpha_1)} + \mu_1 \frac{\Gamma(4-\alpha_1)}{\Gamma(4-2\alpha_1)} t^{\alpha_0-\alpha_1} - \lambda_1 t^{\alpha_0} \right) \Big(E_{\gamma,1}(\lambda_1) - 1 \Big),$$

$$D_{0+,t}^{\alpha_0,\beta_0} E(t) = \frac{\Gamma(4-\alpha_1)}{\Gamma(4-\alpha_0-\alpha_1)} t^{3-\alpha_0-\alpha_1} \Big(E_{\gamma,1}(\lambda_1) - 1 \Big),$$

$$D_{0+,t}^{\alpha_1,\beta_1} E(t) = \frac{\Gamma(4-\alpha_1)}{\Gamma(4-2\alpha_1)} t^{3-2\alpha_1} \Big(E_{\gamma,1}(\lambda_1) - 1 \Big), \quad F(t) = 0,$$

$$K(t,\tau) = \lambda_1 \left(\frac{\Gamma(4-\alpha_1)}{\Gamma(4-\alpha_0-\alpha_1)} + \mu_1 \frac{\Gamma(4-\alpha_1)}{\Gamma(4-2\alpha_1)} \tau^{\alpha_0-\alpha_1} - \lambda_1 \tau^{\alpha_0} \right) \times \mathcal{E}_{\alpha_0,\alpha_0}(t-\tau;\lambda_1,\mu_1)(E_{\gamma,1}(\lambda_1) - 1).$$

In this case, we can find expression for $a(t)$ given by $a(t) = t^{3-\alpha_0-\alpha_1}$.

6 Conclusions

An inverse problem of determining a time dependent source term and diffusion concentration for a space-time fractional differential equation (STFDE), with multi-term Hilfer fractional derivatives in time of orders α_i, $0 < \alpha_m < ... < \alpha_1 < \alpha_0 < 1$ and type $0 \leq \beta_i \leq 1$ and Caputo fractional derivative in space variable of order $0 < \gamma < 2$ with homogeneous boundary condition is investigated. A bi-orthogonal system of functions consisting of Mittag-Leffler functions is obtained from the spectral problem and its adjoint problem. The multi-term fractional order ordinary differential equations obtained by eigenfunction expansion method are solved by Laplace transformation method. A series representation involving multinomial Mittag-Leffler functions is obtained for $u(x,t)$ where as local existence and uniqueness of the source term has been proved by applying Banach fixed point theorem. The series solution $u(x,t)$

is proved to be classical solution, in order to do so, we have developed some estimates for multinomial Mittag-Leffler function. The solution of the inverse problem is proved to be locally well-posed. Let us mention that several other inverse problems for STFDE considered here yet to be considered for example, an important class of inverse problems is to determine the order of fractional derivatives, which explains the anomalies in the diffusion process specially the sub diffusion process. Let us mention that numerical investigation for the multinomial Mittag-Leffler function is the topic which has not been considered in the literature and hence the regularization techniques for the inverse problems for multi-term differential equations is also very rare in literature. Another interesting set of problems is related to the development of convergent numerical algorithms for solving direct and inverse problems related to STFDE.

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