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Investigation of Spectrum Curves for a Sturm–Liouville problem with Two-Point Nonlocal Boundary Conditions

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Abstract. The article investigates the Sturm–Liouville problem with one classical and another nonlocal two-point boundary condition. We analyze zeroes, poles and critical points of the characteristic function and how the properties of this function depend on parameters in nonlocal boundary condition. Properties of the Spectrum Curves are formulated and illustrated in figures for various values of parameter ξ.

Keywords: Sturm–Liouville problem, nonlocal two-point condition, eigenvalues, critical points, spectrum curves.

AMS Subject Classification: 34B24; 34B10.

1 Introduction

During the last two decades a lot of attention has been paid to problems of differential equations with different types of *Boundary Conditions* (BC). J.R. Cannon was the first investigator of the parabolic problems with integral boundary condition [3], now this condition is called *Nonlocal Boundary Conditions* (NBC). Later, Bitsadze and Samarskii formulated and investigated a *Boundary Value Problem* (BVP) for an elliptic equation with NBC [2]. One of the most important problem is to find eigenvalues of differential problem with

NBCs [24]. *Sturm–Liouville Problem* (SLP) for the second-order differential operator with *Nonlocal Condition* (NC) was formulated in [10]. The eigenvalue problem with NBCs is the part of the general nonselfadjoint operator theory [9]. In the article [18] Shkalikov described the result obtained on investigation of the properties of eigenfunctions for integral BC. *Finite Difference Schemes* (FDS) for parabolic problems with NBC were investigated in the papers [4,7,8], where the properties of the spectrum and stability of FDS were obtained.

M. Sapagovas with co-authors began to investigate the eigenvalue problem for one-dimensional differential operator with Bitsadze–Samarskii and integral type NBCs [5, 15, 16]. They showed that eigenvalues, which do not depend on some NBCs parameters can exist. Problem with NBC are not self-adjoint and spectrum for such problems may be not positive (or real, too). So, negative, multiple and complex eigenvalues for some values of NBC parameters can exist [12, 17, 20, 22]. M. Sapagovas scientific group results for problems with NBC's see in [24]. The structure of eigenvalues for some nonlocal boundary conditions presented in [1, 11].

In this paper we use *Characteristic Function* (CF) method [25] for investigation of the spectrum for differential SLP with two-point NBC. We describe zeroes, poles, critical points of CF, *Constant Eigenvalue* (CE) points. We investigate how Spectrum Curves depend on parameter ξ in NBC.

2 Sturm–Liouville problem with NBC

Let us analyze SLP with one classical BC and another two-point NBC

$$-u'' = \lambda u, \quad t \in (0,1), \tag{2.1}$$

$$u(0) = 0, \tag{2.2}$$

$$u(1) = \gamma u'(\xi), \tag{2.3}$$

where parameters $\gamma \in \mathbb{R}$ and $\xi \in [0,1]$. The eigenvalue $\lambda \in \mathbb{C}_\lambda := \mathbb{C}$ and eigenfunction $u(t)$ can be complex function.

If $\gamma = 0$, then we have the SLP with classical BCs. In this case eigenvalues and eigenfunction are known:

$$\lambda_k = (k\pi)^2, \qquad u_k(t) = \sin(k\pi t),\ k \in \mathbb{N} := \{1, 2, \ldots\}.$$

We also use notation $\mathbb{N}_o := \{2k-1,\, k \in \mathbb{N}\}$, $\mathbb{N}_e := \{2k,\, k \in \mathbb{N}\}$. In the case $\xi = 1$, we obtain the third type (classical) BC. The case $\gamma = \infty$ corresponds to (2.1) with clasical BCs $u(0) = 0$ and $u'(\xi) = 0$, $\xi \in [0,1]$, instead of condition (2.3) and eigenvalues and eigenfunction are:

$$\lambda_k = \bigl((k-1/2)\pi/\xi\bigr)^2, \qquad u_k(t) = \sin\bigl((k-1/2)\pi t/\xi\bigr),\ k \in \mathbb{N}.$$

If $\lambda = 0$, then a function $u(t) = Cu_0(t)$, where $u_0(t) := t$, satisfy (2.1) equation and BC (2.2). Substituting this function into the second NBC (2.3), we obtain that eigenvalue $\lambda = 0$ ($C \neq 0$) exists if and only if $\gamma = 1$.

If $\lambda \neq 0$ function $u(t) = Cu_q(t)$, $u_q(t) = \sin(\pi q t)/(\pi q)$, satisfies equation (2.1) and BC (2.2), where $\lambda = (\pi q)^2$. If we consider a map $\lambda : \mathbb{C}_\lambda \to \mathbb{C}$,

$\lambda(q) = (\pi q)^2$, the inverse map is multivalued and point $\lambda = 0$ is the first order *Branch Point* (BP) of the second order. This point is important for our investigation and we call $q = 0$ *Ramification Point* (RP).

In this article $q \in \mathbb{C}_q := \mathbb{R}_q \cup \mathbb{C}_q^+ \cup \mathbb{C}_q^-$, where $\mathbb{R}_q := \mathbb{R}_q^- \cup \mathbb{R}_q^+ \cup \mathbb{R}_q^0$, $\mathbb{R}_q^- := \{q = x + \iota y \in \mathbb{C} : x = 0, y > 0\}$, $\mathbb{R}_q^+ := \{q = x + \iota y \in \mathbb{C} : x > 0, y = 0\}$, $\mathbb{R}_q^0 := \{q = 0\}$, $\mathbb{C}_q^+ := \{q = x + \iota y \in \mathbb{C} : x > 0, y > 0\}$ and $\mathbb{C}_q^- := \{q = x + \iota y \in \mathbb{C} :\ x > 0,\ y < 0\}$. Then a map $\lambda = (\pi q)^2$ is the bijection between $\mathbb{C}_q$ and $\mathbb{C}_\lambda$ [25]. Here $q = 0$ corresponds to $\lambda = 0$. This bijection is a conformal map, except the point $q = 0$. For each eigenvalue λ for SLP corresponds *Eigenvalue Point* (EP) $q \in \mathbb{C}_q$. Real eigenvalues are described by EP $q \in \mathbb{R}_q$=$\{q \in \mathbb{C}_q \colon \lambda = (\pi q)^2 \in \mathbb{R}\}$. If $\lambda = 0$ is eigenvalue, then $q = 0$ we call as *Branch Eigenvalue Point* (BEP).

3 Constant Eigenvalues and Characteristic Function

We substitute $u_q(t)$ to BC (2.3) and get $u_q(1) = \gamma u'_q(\xi)$. So, a nontrivial solution of the problem (2.1)–(2.3) exists if $q \in \mathbb{C}_q$ is the root of a equation

$$\frac{\sin(\pi q)}{\pi q} = \gamma \cos(\xi \pi q). \tag{3.1}$$

For NBC (2.3) we introduce two entire functions:

$$Z(z) := \frac{\sin(\pi z)}{\pi z}; \quad P_\xi(z) := \cos(\xi \pi z), \quad z \in \mathbb{C}. \tag{3.2}$$

In this section we investigate relations between spectrum of SLP (2.1)–(2.3) and parameter γ for fixed ξ.

Zeroes points. Zeroes of these functions are important in analyzing and describing the spectrum. Moreover, zeroes z_k of the function $Z(q)$, $q \in \mathbb{C}_q$, coincide with EPs in the classical case $\gamma = 0$ (the graphs in 1a, 1d):

$$z_l = l \in \mathbb{N}.$$

We denote the corresponding set of points $\mathcal{Z}$. All zeroes are simple, real (positive integer numbers).

In the case $\xi \neq 0$ all zeroes of the function $P_\xi(q)$ (see (3.2)) in $\mathbb{C}_q$ are simple, real and positive:

$$\overline{\mathcal{Z}}_\xi := \{p_k = (k - 1/2)/\xi,\ \ k \in \mathbb{N}\}. \tag{3.3}$$

In the case $\xi = 0$ the set $\overline{\mathcal{Z}}_\xi = \varnothing$.

We rewrite the equation (3.1) in the form:

$$Z(q) = \gamma P_\xi(q), \quad q \in \mathbb{C}_q.$$

Constant Eigenvalues. We will define a *Constant Eigenvalue* (CE) as the eigenvalue which does not depend on the parameter γ [25]. Then for any CE $\lambda \in \mathbb{C}_\lambda$ there exists the *Constant Eigenvalue Point* (CEP) $q \in \mathbb{C}_q$. CEP are roots of the system:

$$Z(q) = 0, \qquad P_\xi(q) = 0, \tag{3.4}$$

i.e., CEP $c \in \mathbb{N}$ and $P_\xi(c) = 0$. We denote the set of all CEP as $\mathcal{C}_\xi = \overline{\mathcal{Z}}_\xi \cap \mathcal{Z}$. If $c \in \mathcal{C}_\xi$ and

$$Z(z) = (z-c)^\alpha \widetilde{Z}(z),\ \widetilde{Z}(c) \neq 0, \qquad P_\xi(z) = (z-c)^\beta \widetilde{P}(z),\ \widetilde{P}(c) \neq 0, \tag{3.5}$$

where $\alpha, \beta \in \mathbb{N}$, then number $\min\{\alpha, \beta\}$ is the order of CEP. For SLP (2.1)–(2.3) all CEPs are of the first order.

Remark 1. If the parameter $\xi = 0$, then from the formula (3.2) we obtain that $P_\xi \equiv 1$. So, $\overline{\mathcal{Z}}_\xi = \varnothing$ and CEPs do not exist. If $\xi = 1$, then there are no CEPs, because the functions $\sin(\pi q)$ and $\cos(\pi q)$ have no common zeroes (we have the third type BC).

Remark 2. If the parameter $\xi \notin \mathbb{Q}$, then CEPs do not exist, because the equation $\xi l = k - \frac{1}{2}$ have not roots for $l, k \in \mathbb{N}$.

Let us consider that $\xi \in \mathbb{Q}$, $\xi = m/n$, $m, n \in \mathbb{N}$, $0 < m \leqslant n$, and $\gcd(m, n) = 1$, where $\gcd(m,\, n)$ is the greatest common divisor of two (positive) integers m and n. In this case the system (3.4) is equivalent to equation $lm/n = k - 1/2$, where $k,\, l \in \mathbb{N}$ are unknowns. We can rewrite this equation in the following form

$$nk - lm = n/2. \tag{3.6}$$

Remark 3. If $n \in \mathbb{N}_e$, then the right hand side of this equation is integer number. If $n \in \mathbb{N}_o$, then equation (3.6) has no roots.

Let $a,\ b,\ c \in \mathbb{Z}$. Then the following theorem is valid.

Theorem 1. ([6], Gelfond 1978) *If* $\gcd(a, b) = 1$ *and* (α, β) *is any solution of equation:*

$$ax + by + c = 0, \tag{3.7}$$

then all solutions of this equation have a form;

$$x = \alpha - bt, \quad y = \beta + at, \quad t \in \mathbb{Z}.$$

Remark 4. Any solution (α, β) of (3.7), $\gcd(a, b) = 1$ can be found using Euclidean algorithm [6] and solving the classical equation

$$ax + by = \gcd(a, b).$$

Theorem 2. *For SLP* (2.1)–(2.3) *Constant eigenvalues exist only for rational parameter* $\xi = m/n \in (0,\, 1)$, $m \in \mathbb{N}_o$, $n \in \mathbb{N}_e$ *values and those eigenvalues are equal to* $\lambda_s = (\pi c_s)^2$, $c_s := (s - 1/2)n$, $s \in \mathbb{N}$.

Proof. The case $\xi \notin \mathbb{Q}$ was discussed in Remark 2. Let $\xi \in \mathbb{Q}$, $\xi = m/n$, and from Remark 3 we have that $m \in \mathbb{N}_o$, $n \in \mathbb{N}_e$. From equation (3.6) we see that $l = \tau n/2$, $\tau \in \mathbb{N}$. We rewrite equation (3.6) as $m\tau = 2k - 1$. Right side of this equation is odd number. So, τ must be odd number too, i.e., $\tau = 2t - 1$, $t \in \mathbb{N}$. For t and k we have equation $mt - k = (m-1)/2$. Because $t_0 = 0$ and $k_0 = -(m-1)/2$ satisfy this equation, then the solution of this equation is $t = s$ and $k = -(m-1)/2 + ms$, according to Theorem 1. If $s \in \mathbb{N}$, then $t, k \in \mathbb{N}$. Therefore, we obtain that $q = l = (s - 1/2)n$, $s \in \mathbb{N}$, is CEP. $\square$

Characteristic Functions. Let us consider the following meromorphic function:

$$\gamma_c(z) = \gamma_c(z;\xi) := \frac{Z(z)}{P_\xi(z)}, \quad z \in \mathbb{C}, \tag{3.8}$$

where Z and P_ξ are entire functions (3.2). If $\lim_{z\to p} \gamma_c = \infty$, then we have a *Pole Point* (PP) at $z = p$. We have pole of the β order in the case $Z(p) \neq 0$ and $P_\xi(z) = (z-p)^\beta \widetilde{P}(z)$, $\widetilde{P}((p) \neq 0$, $\beta \in \mathbb{N}$. If $p \in \mathcal{C}_\xi$ and condition (3.5) is valid with $\beta > \alpha$, then pole $z = c$ is of the order $\beta - \alpha$. In the case $0 \leqslant \beta < \alpha$ function γ_c has zero of order $\alpha - \beta$. If $\beta = \alpha$, then point $z = c$ is removable singularity isolated point and $0 < |\gamma_c(c)| < \infty$.

We call the expression of the meromorphic function γ_c on $\mathbb{C}_q$ *C*omplex Characteristic Function (Complex CF) [19,25]. γ-points of Complex CF define EPs (and Eigenvalues, too) which depend on parameter γ. We call such EPs *Nonconstant Eigenvalue's Points* (NEPs) and corresponding Eigenvalues as *Nonconstant Eigenvalues.*

As we noted, functions Z and P_ξ for SLP (2.1)–(2.3) have simple zeroes only. If $Z(z) = 0$ and $P_\xi(z) \neq 0$, then we have zero point of CF at the point z; if $Z(p) \neq 0$ and $P_\xi(p) = 0$, then we have PP of CF at the point p. A set of PPs for Complex CF is $\mathcal{P}_\xi := \overline{\mathcal{Z}}_\xi \setminus \mathcal{Z} = \overline{\mathcal{Z}}_\xi \setminus \mathcal{C}_\xi$. So, $p_k \in \overline{\mathcal{Z}}_\xi$ is PP if and only if $p_k \notin \mathbb{N}$. The set of zeroes for this Complex CF is $\mathcal{Z}_\xi := \mathcal{Z} \setminus \overline{\mathcal{Z}}_\xi = \mathcal{Z} \setminus \mathcal{C}_\xi$. If $c \in \mathcal{C}_\xi$, i.e. $Z(c) = 0$ and $P_\xi(c) = 0$, then we have removable singularity isolated point and we have (for $m \in \mathbb{N}_o$, $n \in \mathbb{N}_e$) sequence of such points

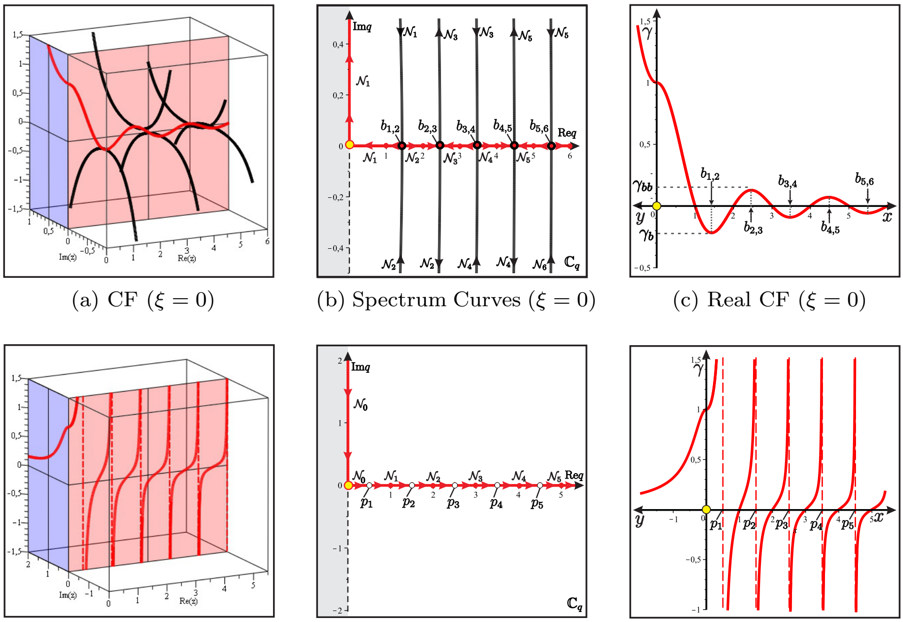
$$\begin{aligned} &c_s = p_{k_s} = z_{l_s} = n(s - 1/2),\ s \in \mathbb{N}, \\ &k_s = m(s - 1/2) + 1/2,\ l_s = n(s - 1/2). \end{aligned} \tag{3.9}$$

Remark 5. In the case $\xi = 0$ function $P_\xi \equiv 1$ and PPs do not exist. If $\xi > 0$, then a set of poles $\mathcal{P}_\xi$ is either empty or countable.

Remark 6. Case $\xi = 1/n$, $n \in \mathbb{N}$. From de Moivre formula:

$$\begin{aligned} \sin(2kz) &:= 2k\cos^{2k-1} z \sin z - \tbinom{2k}{3} \cos^{2k-3} z \sin^3 z + \ldots \\ &\quad + (-1)^{k-1} 2k \cos z \sin^{2k-1} z, \quad k \in \mathbb{N} \cup \{0\}, \\ \sin((2k+1)z) &:= (2k+1)\cos^{2k} z \sin z - \tbinom{2k+1}{3} \cos^{2k-2} z \sin^3 z + \ldots \\ &\quad + (-1)^k \sin^{2k} z, \quad k \in \mathbb{N} \cup \{0\}, \end{aligned}$$

we have $\mathcal{P}_\xi = \overline{\mathcal{Z}}_\xi$, $\mathcal{C}_\xi = \varnothing$ for $\xi = 1/(2k+1)$, $k \in \mathbb{N}$; $\mathcal{P}_\xi = \varnothing$, $\mathcal{C}_\xi = \overline{\mathcal{Z}}_\xi$ for $\xi = 1/(2k)$, $k \in \mathbb{N}$. So, for $n \in \mathbb{N}_o$ there are PPs, but CEPs do not exist, for $n \in \mathbb{N}_e$ there are no PPs and we have CEPs.



(a) CF ($\xi = 0$) (b) Spectrum Curves ($\xi = 0$) (c) Real CF ($\xi = 0$)

(d) CF ($\xi = 1$) (e) Spectrum Curves ($\xi = 1$) (f) Real CF ($\xi = 1$)

Figure 1. CF, Spectrum Domain, Real CF for $\xi = 0$, $\xi = 1$. ● – Zero Point, ○ – Pole Point, ○ – Ramification Point, ○ – Branch Eigenvalue Point, ● – Critical Point.

Remark 7. A point $q = \infty \notin \mathbb{C}_q$. This point is singular (isolated essential point if $\mathcal{P}_\xi = \varnothing$, otherwise we have cluster of poles) point.

Complex-Real Characteristic Function (CF) [25] $\gamma = \gamma(q)$ is the restriction of Complex CF γ_c on a set $\mathcal{D}_\xi := \{q \in \mathbb{C}_q : \mathrm{Im}\gamma_c(q) = 0\}$, i.e., $\gamma\colon \mathcal{D}_\xi \to \mathbb{R}$. CF $\gamma(q)$ describes the value of the parameter γ at the point $q \in \mathcal{D}_\xi$, such that there exist the Nonconstant Eigenvalue $\lambda = (\pi q)^2$. A set $\mathcal{E}_\xi(\gamma_0) := \gamma^{-1}(\gamma_0)$ is the set of all NEPs for $\gamma_0 \in \mathbb{R}$. So, $\mathcal{D}_\xi = \cup_{\gamma\in\mathbb{R}}\mathcal{E}_\xi(\gamma)$. The *Spectrum Domain* $\mathcal{N}_\xi$ is the set $\mathcal{D}_\xi \cup \mathcal{C}_\xi$ for fixed ξ [21]. We denote sets $\overline{\mathcal{D}}_\xi := \mathcal{D}_\xi \cup \mathcal{P}_\xi \cup \{\infty\}$, $\partial\mathcal{D}_\xi = \mathcal{P}_\xi \cup \{\infty\}$. We can see the Spectrum Domain in 6 for various ξ.

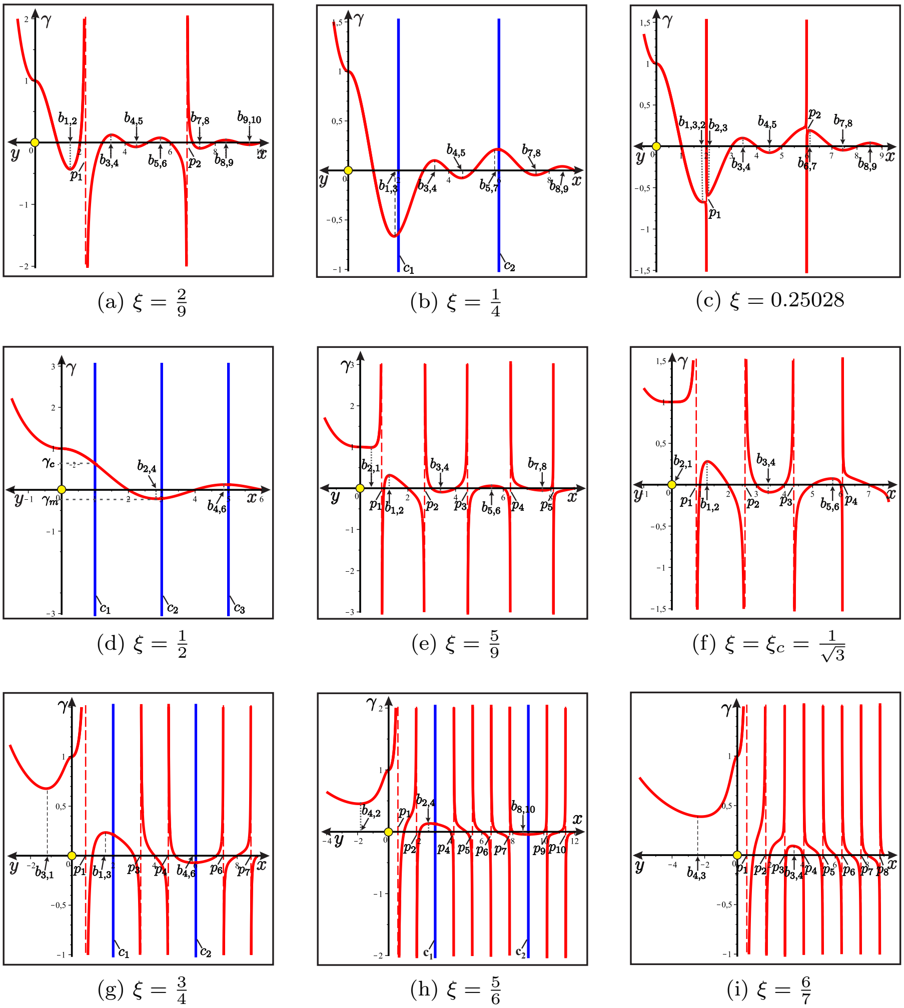
Remark 8. Examples of CF graphs are presented in 1a,1d for $\xi = 0$ and $\xi = 1$. We see Spectrum Domains for these cases in 1b,1e. In the case $\xi = 1$ Spectrum Domain $\mathcal{D}_1 \subset \mathbb{R}_q$. So, all eigenvalues are real. In the case $\xi = 0$ part of $\mathcal{D}_0$ belongs to $\mathbb{C}_q^+ \cup \mathbb{C}_q^-$ and complex eigenvalues exist for some values of γ. If the parameter $\xi = 0$ then from the equation (3.1) we obtain that CF have no PPs (see Figure 1 and Figures 1a–1c). If $\xi = 1$, then there are no CEPs, but we have family of poles p_k, $k \in \mathbb{N}$, defined by formula (3.3) (see Figures 1d–1f).

Real Eigenvalues. Real Characteristic Function (Real CF) describes only real nonconstant eigenvalues and it is restriction of the Complex CF $\gamma_c(q)$ on the set $\mathbb{R}_q$. We can use the argument $x \in \mathbb{R}$ for Real CF:

$$\gamma_r(x) = \gamma_r(x;\xi) := \begin{cases} \gamma(-\imath x;\xi) = \dfrac{\sinh(\pi x)}{\pi x \cosh(\xi\pi x)}, & x \leqslant 0; \\ \gamma(x;\xi) = \dfrac{\sin(\pi x)}{\pi x \cos(\xi \pi x)}, & x \geqslant 0. \end{cases}$$

This function is useful for investigation of real negative, zero and positive eigenvalues

$$\lambda = \lambda_r(x) = \lambda_r(x;\xi) := \begin{cases} -(\pi x)^2, & x \leqslant 0; \\ (\pi x)^2, & x \geqslant 0. \end{cases}$$



(a) $\xi = \frac{2}{9}$ (b) $\xi = \frac{1}{4}$ (c) $\xi = 0.25028$

(d) $\xi = \frac{1}{2}$ (e) $\xi = \frac{5}{9}$ (f) $\xi = \xi_c = \frac{1}{\sqrt{3}}$

(g) $\xi = \frac{3}{4}$ (h) $\xi = \frac{5}{6}$ (i) $\xi = \frac{6}{7}$

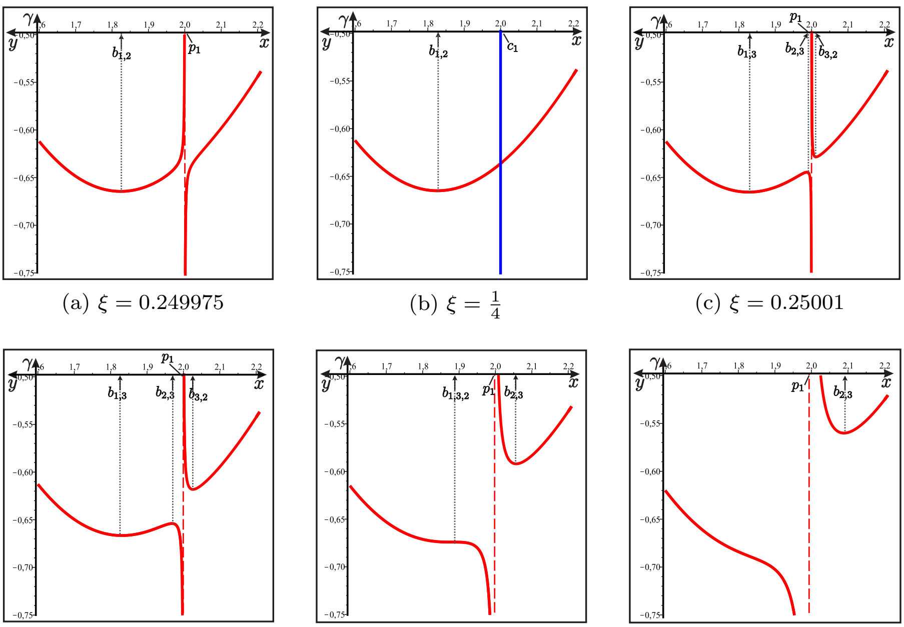
Figure 2. Real CF $\gamma_r(q;\xi)$ for various parameter ξ values.

For positive and zero eigenvalues EPs for CF γ and Real CF γ_r are the same. For negative eigenvalues EPs for CF γ and Real CF γ_r are related by formula

$q = -\imath x$, $x < 0$. The graphs of these Real CFs for some parameter ξ values are presented in Figures 1c, 1f, 2. In Figure 2 the vertical (blue) solid lines correspond to the CEP, vertical (red) dashed lines cross the x-axis at the PPs. Real CF for SLP (2.1)–(2.3) was investigated in [12]. In this paper values of Real CF and it's derivatives at CEP $c_s := (s-1/2)n$, $s \in \mathbb{N}$, $\xi = m/n \in (0, 1)$, $m \in \mathbb{N}_o$, $n \in \mathbb{N}_e$, were found:

$$\gamma_s(\xi) := \gamma(c_s;\xi) = (-1)^{s+(n-m+1)/2} c_s^{-1}\xi^{-1}\pi^{-1}, \tag{3.10}$$
$$\gamma_s'(\xi) := \gamma'(c_s;\xi) = -c_s^{-1}\gamma_s, \tag{3.11}$$
$$\gamma_s''(\xi) := \gamma''(c_s;\xi) = \big(2c_s^{-2} - \pi^2(1-\xi^2)/3\big)\gamma_s.$$

We see, that $\gamma_s \neq 0$, $\gamma_s' \neq 0$, $\gamma_s'' \neq 0$ for all ξ and s. Graphs of Real CF in the neighborhood CEP are presented in Figure 3b (see Figures 3a,3c, too).



(a) $\xi = 0.249975$ (b) $\xi = \frac{1}{4}$ (c) $\xi = 0.25001$

(d) $\xi = 0.25005$ (e) $\xi = \xi_{2b}$ (f) $\xi = 0.2508$

Figure 3. Real CF in the neighborhood CEP $x = 2$ for $\xi = 1/4$ and in the neighborhood of the second order critical point for $\xi = \xi_{2b} \approx 0.25028$.

Some results about the first two real eigenvalues are presented in Table 1. We note, that in the case $\xi = 0$ and $\gamma_r < \gamma_b$ real eigenvalues do not exist. The location of these two eigenvalues can be more accurate if we take smaller intervals of parameter ξ. In [14] statements about negative eigenvalues were formulated. For example, if $\xi = 1/\sqrt{3}$, then double negative eigenvalue exists. Some results about Real CF one can find in [12, 13].

Table 1. The first two real eigenvalues $\lambda_0 = \lambda_r(x_0)$ and $\lambda_1 = \lambda_r(x_1)$ $(\gamma \neq 0)$.

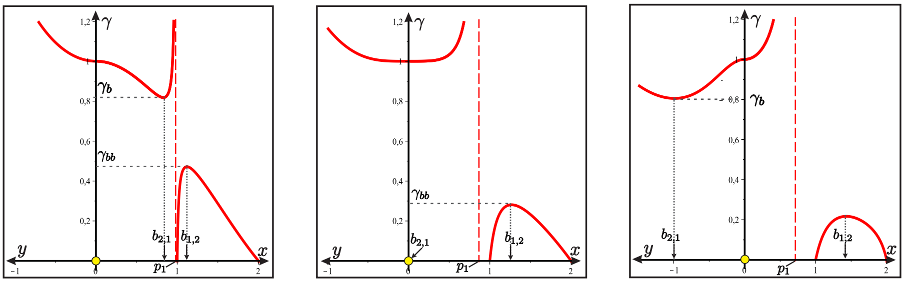
ξ	γ	x_0	x_1
$\xi = 0$	$\gamma_b \leqslant \gamma < 0$	$(1, x_b]$	$[x_b, 2)$
$\gamma_b = \min_{1<x<2} \gamma_r(x) = \gamma_r(x_b)$	$0 < \gamma \leqslant \gamma_{bb}$	$(0,1)$	$(2, x_{bb}]$
	$\gamma_{bb} < \gamma < 1$	$(0,1)$	$-$
$\gamma_{bb} = \max_{2<x<3} \gamma_r(x) = \gamma_r(x_{bb})$	$\gamma = 1$	$= 0$	$-$
	$\gamma > 1$	< 0	$-$
$0 < \xi < 1/2$	$\gamma < 0$	> 1	> 1
	$0 < \gamma < 1$	$(0,1)$	> 1
	$\gamma = 1$	$= 0$	> 1
	$\gamma > 1$	< 0	> 1
$\xi = 1/2$	$\gamma < \gamma_m$	$= 1$	$= 2$
	$\gamma_m \leqslant \gamma < 0$	$= 1$	$(2, x_m]$
	$0 < \gamma < \gamma_c$	$= 1$	$(1,2)$
$\gamma_c = 2/\pi$	$\gamma = \gamma_c$	$= 1$	$= 1$
$\gamma_m = \min_{2<x<3} \gamma_r(x) = \gamma_r(x_m)$	$\gamma_c < \gamma < 1$	$(0,1)$	$= 1$
	$\gamma = 1$	$= 0$	$= 1$
	$\gamma > 1$	< 0	$= 1$
$1/2 < \xi < 1/\sqrt{3}$	$\gamma \leqslant \gamma_{bb}$	$(p_1, x_{bb}]$	$[x_{bb}, 2)$
$\gamma_b = \min_{0<x<p_1} \gamma_r(x) = \gamma_r(x_b)$	$\gamma_{bb} < \gamma < \gamma_b$	$(p_2, 3)$	$(4, p_3)$
	$\gamma_b \leqslant \gamma < 1$	$(0, x_b]$	$[x_b, p_1)$
$\gamma_{bb} = \max_{1<x<2} \gamma_r(x) = \gamma_r(x_{bb})$	$\gamma = 1$	$= 0$	(x_b, p_1)
	$\gamma > 1$	< 0	(x_b, p_1)
$\xi = 1/\sqrt{3}$	$\gamma < 0$	$(p_1, 1)$	$(2, p_2)$
$p_1 = \sqrt{3}/2$, $p_2 = 3\sqrt{3}/2$,	$0 < \gamma \leqslant \gamma_{bb}$	$(1, x_{bb}]$	$[x_{bb}, 2)$
$p_3 = 5\sqrt{3}/2$	$\gamma_{bb} < \gamma < 1$	$(p_2, 3)$	$(4, p_3)$
$\gamma_{bb} = \max_{1<x<2} \gamma_r(x) = \gamma_r(x_{bb})$	$\gamma = 1$	$= 0$	$= 0$
	$\gamma > 1$	< 0	$(0, p_1)$
$1/\sqrt{3} < \xi < 1$	$\gamma < \gamma_b$	(p_1, p_3)	> 1
	$\gamma_b \leqslant \gamma < 1$	$\leqslant x_b$	$[x_b, 0)$
$\gamma_b = \min_{x<0} \gamma_r(x) = \gamma_r(x_b)$	$\gamma = 1$	$< x_b$	$= 0$
	$\gamma > 1$	$< x_b$	$(0, p_1)$
$\xi = 1$	$\gamma < 0$	$(p_1, 1)$	$(p_2, 2)$
$p_1 = 1/2$, $p_2 = 3/2$	$0 < \gamma < 1$	< 0	$(1, p_2)$
	$\gamma = 1$	$= 0$	$(1, p_2)$
	$\gamma > 1$	$(0, p_1)$	$(1, p_2)$

Ramification Point. The Taylor series for CF $\gamma(q)$ at RP $q = 0$ is:

$$\begin{aligned}\gamma(q;\xi) \;:=\; & 1 + \left(-\tfrac{1}{6} + \tfrac{1}{2}\xi^2\right)\pi^2 q^2 \\ & + \left(\tfrac{1}{120} - \tfrac{1}{24}\xi^4 - \left(\tfrac{1}{2}\left(\tfrac{1}{6} - \tfrac{1}{2}\xi^2\right)\right)\xi^2\right)\pi^4 q^4 + \mathcal{O}(q^6), \quad q \in \mathbb{C}_q. \qquad (3.12)\end{aligned}$$

Multiplier of q^2 is positive if, $\xi > \xi_c = 1/\sqrt{3}$, and negative if $\xi < \xi_c$. When $\xi = \xi_c$ the second term vanishes in (3.12), and the coefficient in the third term of this series is positive and equal $\pi^4/270 > 0$. Graphs of Real CF in the neighborhood RP $q = 0$ are presented in Figure 4.

Critical Points. For the SLP (2.1)–(2.3) with two-points NBCs we obtain three types of critical points: the first, the second and the third order. Let to consider function γ_c (3.8). If $\gamma_c'(b) = 0$, $b \in \mathbb{C}$, then b we call *Critical*



(a) $\xi = 0.51$ (b) $\xi = \xi_c = \frac{1}{\sqrt{3}} \approx 0,5773$ (c) $\xi = 0.7$

Figure 4. Real CF at RP $q = 0$.

Point (CP) of the function γ_c, and value $\gamma_c(b)$ is a *critical value* of the function γ_c [22]. CPs are saddle points of Complex CF. For Real CF it can be a half-saddle points, maximum or minimum points and also can be inflection points. CPs of the CF are important for investigation of multiple eigenvalues. The critical point b depend on the parameter ξ continuously. If the function γ_c at CP $b \in \mathbb{C}_q$ satisfies $\gamma_c'(b) = 0, \ldots, \gamma_c^{(k)}(b) = 0$, $\gamma_c^{(k+1)}(b) \neq 0$, then b is called the *k-order critical point* (kCP). The set of CPs we denote $\mathcal{K}_\xi$.

Remark 9. In the case $\xi = \xi_c = \frac{1}{\sqrt{3}}$ the point $q = 0$ is 3CP in the domain $\mathbb{C}_q$, but for $\lambda = 0$ it is only 1CP, because $q = 0$ is RP for map $\lambda = (\pi q)^2$. In the complex plane $\mathbb{C}_\lambda$ the Taylor series (3.12) have a form

$$\gamma(\lambda, \xi) := 1 + \left(-\tfrac{1}{6} + \tfrac{1}{2}\xi^2\right)\lambda + \left(\tfrac{1}{120} - \tfrac{1}{24}\xi^4 - \left(\tfrac{1}{2}\left(\tfrac{1}{6} - \tfrac{1}{2}\xi^2\right)\right)\xi^2\right)\lambda^2 + \mathcal{O}(\lambda^3).$$

If $\xi \neq \xi_c$, then point $q = 0$ and $\lambda = 0$ are not CPs.

The *first order real critical point* (1CP) $b \in \mathring{\mathbb{R}}_q = \mathbb{R}_q^- \cup \mathbb{R}_q^+$ can be found as root of an equation

$$\gamma'(b; \xi) = 0$$

for fixed ξ. For example, when $\xi = 0$ we can see 1CP ($b_{1,2} \approx 1.43$, $b_{2,3} \approx 2.46$) in Figure 1. If 1CP is between two zeroes of Real CF, then these zeroes we use to numerate CP in most cases. More precisely, the index of CP we explain in the next section.

The *second order critical points* (2CP) arise when two 1CP coincide in the same point b. 2CP $b \in \mathring{\mathbb{R}}_q$ and ξ value we can find by solving system:

$$\gamma'(b; \xi) = 0, \quad \gamma''(b; \xi) = 0.$$

For example, 2CP we obtain in the inflection point (see Figure 2c, when $\xi = \xi_{2b} \approx 0.25028$ at the point $b_{1,3,2} \approx 1.883$). Graphs of Real CF in the neighborhood 2CP are presented in Figure 3e (see Figures 3d, 3f, too).

The *third order real critical point* (3CP) $b \in \mathring{\mathbb{R}}_q$ satisfies the following system:

$$\gamma'(b; \xi) = 0, \quad \gamma''(b; \xi) = 0, \quad \gamma'''(b; \xi) = 0.$$

For problem (2.1)–(2.3) we have only one 3CP, $b_{2,1} = 0$, $\xi = \xi_c$. But as we note in Remark 6, this point is 1CP in the domain $\mathbb{C}_\lambda$ (see in Figure 2f and Figure 4b at the point $b_{2,1}$, $\xi = \xi_c$).

Lemma 1. *Zero point of CF can not be CP.*

Proof. First of all $q = 0$ is not a zero. For CF (3.8) (see (3.2), too) we have

$$\gamma' = \Big(\frac{\sin(\pi q)}{\pi q \cos(\xi\pi q)}\Big)' = \frac{-1}{q^2\pi}\cdot\frac{\sin(\pi q)}{\cos^2(\xi\pi q)} + \frac{\xi}{q}\cdot\frac{\sin(\pi q)}{\cos(\xi\pi q)}\sin(\xi\pi q) + \frac{1}{q}\cdot\frac{\cos(\pi q)}{\cos(\xi\pi q)}$$
$$= \big(\pi\xi\sin(\xi\pi q)/\cos(\xi\pi q) - 1/q\big)\gamma(q) + \cos(\pi q)/\cos(\xi\pi q)/q.$$

If $\gamma(q_z) = 0$, then $q_z \notin \mathcal{C}_\xi$ (see (3.10)), $\sin(\pi q_z) = 0$, $\cos(\pi q_z) \neq 0$, $\cos(\xi\pi q_z) \neq 0$. So, $\gamma'(q_z) \neq 0$. □

Remark 10. Pole Point of CF is not CP. Function γ^{-1} has CP at this point only if order of the pole is greater than the first.

4 Spectrum Curves

Spectrum Domain is a union $\mathcal{N}_\xi = \mathcal{D}_\xi \cup \mathcal{C}_\xi$ in the complex domain $\mathbb{C}_q$. In the classical case $\gamma = 0$ Spectrum is $\mathbb{N} = \mathcal{Z}_\xi + \mathcal{C}_\xi$, $\mathcal{Z}_\xi = \mathcal{E}_\xi(0)$.

First of all, we consider $q_0 \in \mathcal{D}_\xi = \cup_{\gamma\in\mathbb{R}}\mathcal{E}_\xi(\gamma)$ and $\gamma_c{}'(q_0) \neq 0$, i.e., q_0 is not CP. Then $\mathcal{E}_\xi(\gamma)$ is a smooth parametric curve $\mathcal{N} : (\gamma_0 - \delta_1, \gamma_0 + \delta_2) \subset \mathbb{R} \to \mathbb{C}_q$ in the neighborhood of the point q_0 and $\mathcal{N}(\gamma_0) = q_0$ (see Figures 5a,5e). We can add arrows to this curve.

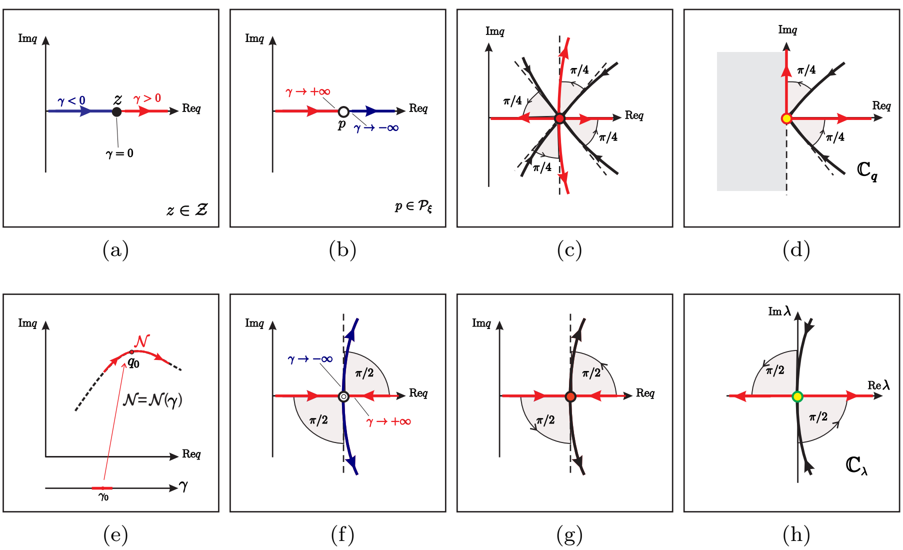
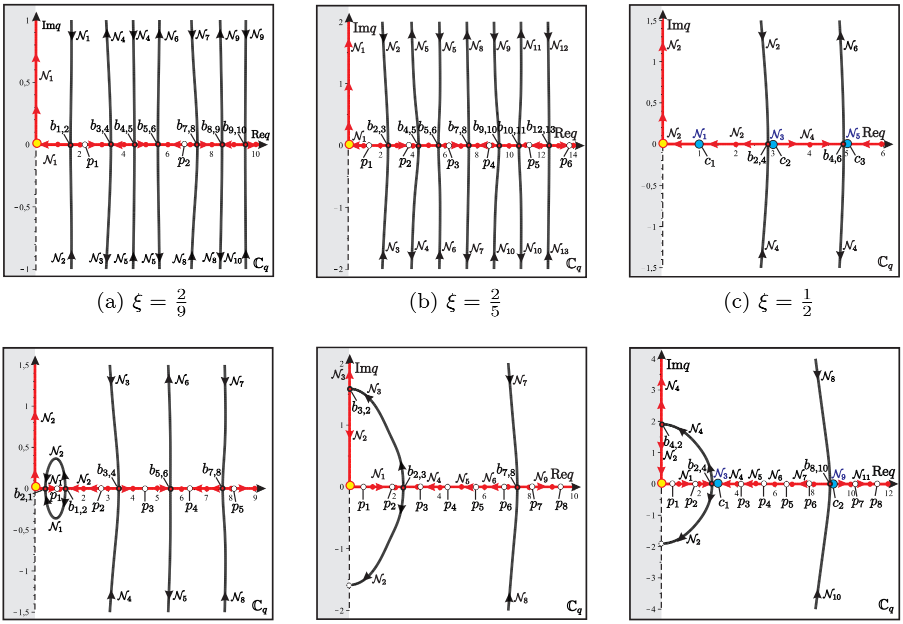


Figure 5. Spectrum Curves. ◎ – Pole Point of the second order, ○ – Branch Point

First of all, we consider $q_0 \in \mathcal{D}_\xi = \cup_{\gamma\in\mathbb{R}}\mathcal{E}_\xi(\gamma)$ and $\gamma_c{}'(q_0) \neq 0$, i.e., q_0 is not CP. Then $\mathcal{E}_\xi(\gamma)$ is a smooth parametric curve $\mathcal{N} : (\gamma_0 - \delta_1, \gamma_0 + \delta_2) \subset \mathbb{R} \to \mathbb{C}_q$ in the neighborhood of the point q_0 and $\mathcal{N}(\gamma_0) = q_0$ (see Figures 5a,5e). We can add arrows to this curve.



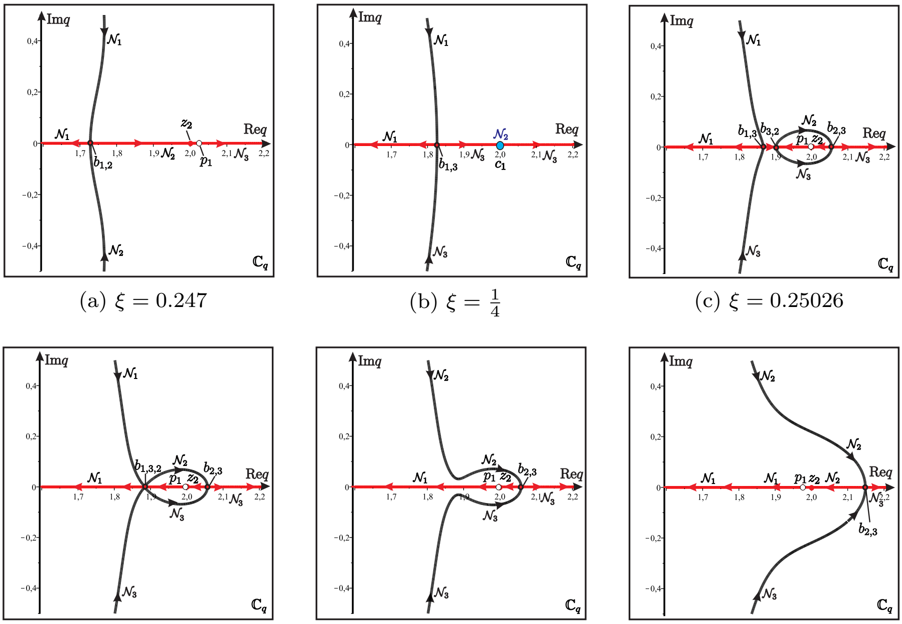
(a) $\xi = \frac{2}{9}$ (b) $\xi = \frac{2}{5}$ (c) $\xi = \frac{1}{2}$

(d) $\xi = \frac{5}{9}$ (e) $\xi = \frac{4}{5}$ (f) $\xi = \frac{5}{6}$

Figure 6. (a)–(f) Spectrum Domain (Spectrum Curves) for various parameter ξ values. ● – Constant Eigenvalue Point.

Arrows show the direction in which γ is increasing. So, EP from $\mathcal{E}_\xi(\gamma)$ is moving along this curve. We call this curve the *Spectrum Curve*. Zero Point is not CP (see Lemma 1) and in the neighborhood of Zero Point the Spectrum Curve belongs to $\mathbb{R}_q^+$ (see Figure 5a). When $\gamma \to \pm\infty$ the Spectrum Curve $\mathcal{N}(\gamma)$ is approaching to $\partial\mathcal{D}_\xi = \mathcal{P}_\xi \cup \{\infty\}$ (see Figure 6). PP is not CP, too, and all Poles are of the first order. So, in the neighborhood of PP, Spectrum Curves belong to $\mathbb{R}_q^+$ (see Figure 5b), and γ values in the limit correspond to $-\infty$ and $+\infty$. For other problems the structure of Spectrum Curves may be more complex (see Figure 5f for a pole of the second order [19, 21]).

If $q_0 \in \mathcal{K}_\xi$ and $q_0 \neq 0$, i.e. we have CP, then $0 < |\gamma(q_0)| < \infty$ (see Lemma 1). At this point eigenvalue is not simple. The view of Spectrum Curves near to CPs is shown in Figure 5c (3CP) and Figure 5g (1CP). For SLP (2.1)–(2.3) CPs of the first order and the second order exist (see Figure 6 and Figure 7). Since $0 < |\gamma(q_0)| < \infty$ at CP q_0, we can assume that a few Spectrum Curves are intersecting at this CP when parameter $\gamma = \gamma(q_0)$. At kCP Spectrum Curves change direction and the angle between the old and new direction is $\pi/(k + 1)$ (see Figure 5c and Figure 5g). We use the “right hand rule”. So, the Spectrum Curve turns to the right. Then the parameter $\gamma \in \mathbb{R}$ for all



(a) $\xi = 0.247$ (b) $\xi = \frac{1}{4}$ (c) $\xi = 0.25026$

(d) $\xi = \xi_{2b} \approx 0.25028$ (e) $\xi = 0.2503$ (f) $\xi = 0.253$

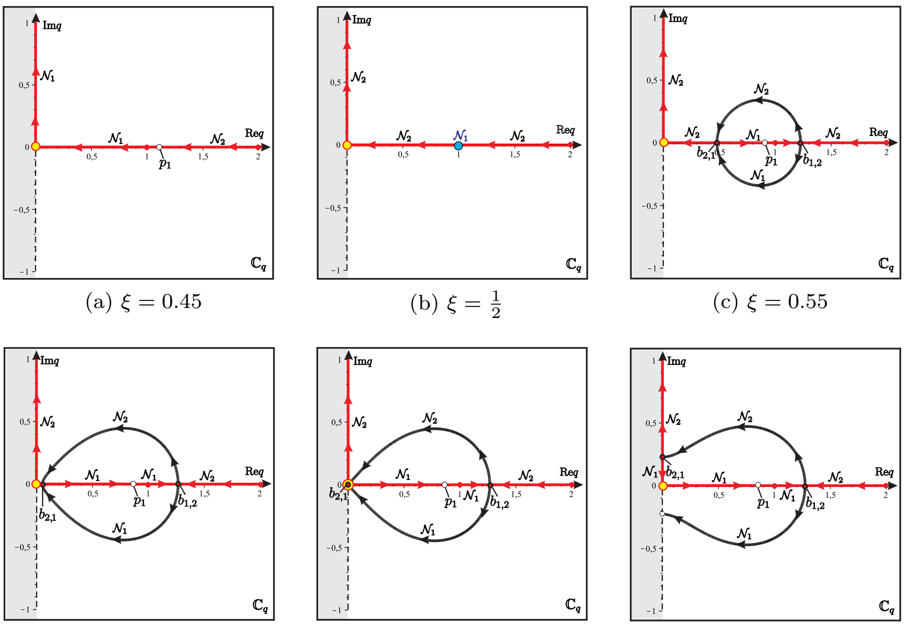
Figure 7. Spectrum Curves for various parameter ξ values, bifurcations.

Spectrum Curves, but we have exception in the case $\xi = 1$ when Spectrum Curve belongs to imaginary axis for $\gamma \in (0; 1]$ and real axis for $\gamma \in [1; +\infty)$ (see Figures 5d–5f). We enumerate Spectral Curves by classical case: if for $\gamma = 0$ the point $q = l \in \mathcal{Z}_\xi$ belongs to the Spectrum Curve, then we denote this *regular Spectrum Curve* $\mathcal{N}_l$. So, $\mathcal{N}_l = \{\mathcal{N}(\gamma), \gamma \in \mathbb{R}, \mathcal{N}(0) = l\}$, $l \in \mathcal{Z}_\xi$. In the case $\xi = 1$ we have additional *semi-regular Spectrum Curve* $\mathcal{N}_0 := \{\mathcal{N}(\gamma), \gamma \in (0; +\infty), \mathcal{N}(1) = 0\} = \mathbb{R}_q^- \cup [0; 1/2)$. Then we have $\mathcal{D}_\xi = \cup_{l \in \mathcal{Z}_\xi} \mathcal{N}_l$ for $\xi \neq 1$ and $\mathcal{D}_\xi = \cup_{l \in \{0\} \cup \mathcal{Z}_\xi} \mathcal{N}_l$ for $\xi = 1$.

Remark 11. In the case $\xi = 1$ and $\gamma \neq 0$ we can consider boundary condition $u'(1) = \tilde{\gamma} u(1)$, $\tilde{\gamma} \in \mathbb{R}$, where $\tilde{\gamma} = \gamma^{-1}$. Now CF is $\tilde{\gamma} = \pi q \cos(\pi q)/\sin(\pi q)$ and its zeroes are p_k, $k \in \mathbb{N}$, poles are z_k, $k \in \mathbb{N}$ (CEPs do not exist). For parameter $\tilde{\gamma} \in \mathbb{R}$ all Spectrum Curves will be regular. More generally, we can investigate SLP with parameter $\gamma \in \mathbb{R}P^1$ (projective line). In the case $\xi = 1$ we can consider one "super" Spectrum Curve for CF $\gamma \colon \mathbb{R}_q \to \mathbb{R}P^1$ (see [23], too).

The point $q = 0$ is RP. This point belongs to the regular Spectrum Curve in the case $\xi \neq 1$, $\xi \neq 1/\sqrt{3}$ and semi-regular Spectrum Curve in the case $\xi = 1$. RP $q = 0$ is CP for $\xi = \xi_c = 1/\sqrt{3}$ (see 9). The Spectrum Curve near to this RP $q = 0$ and BP $\lambda = 0$ has different properties. For example, the Spectrum Curve turns perpendicular to the right at this BEP for $\xi \in [0; 1/\sqrt{3})$ and the Spectrum Curve turns perpendicular to the left for $\xi \in (1/\sqrt{3}; 1]$. In $\mathbb{C}_\lambda$ the image of the Spectrum Curve lies in the real axis. For $\xi = \xi_c$ we have different angles (see Figures 5d, 5h) for $\mathbb{C}_q$ and $\mathbb{C}_\lambda$.

The index of CP is formed by the indices of the Spectrum Curves, intersecting in this CP. If the CP is real, then the left index coincides with the index of Spectral Curve, which is defined by the smaller real λ values, and the right index is defined by greater λ values (see Figures 6,7,8). We put the indices of other Spectrum Curves in the ascending order between left and right indices (see Figure 7d).



(a) $\xi = 0.45$ (b) $\xi = \frac{1}{2}$ (c) $\xi = 0.55$

(d) $\xi = 0.577$ (e) $\xi = \xi_c = \frac{1}{\sqrt{3}}$ (f) $\xi = 0.584$

Figure 8. Spectrum Curves for various parameter ξ values.
– Critical Point at Branch Eigenvalue Point.

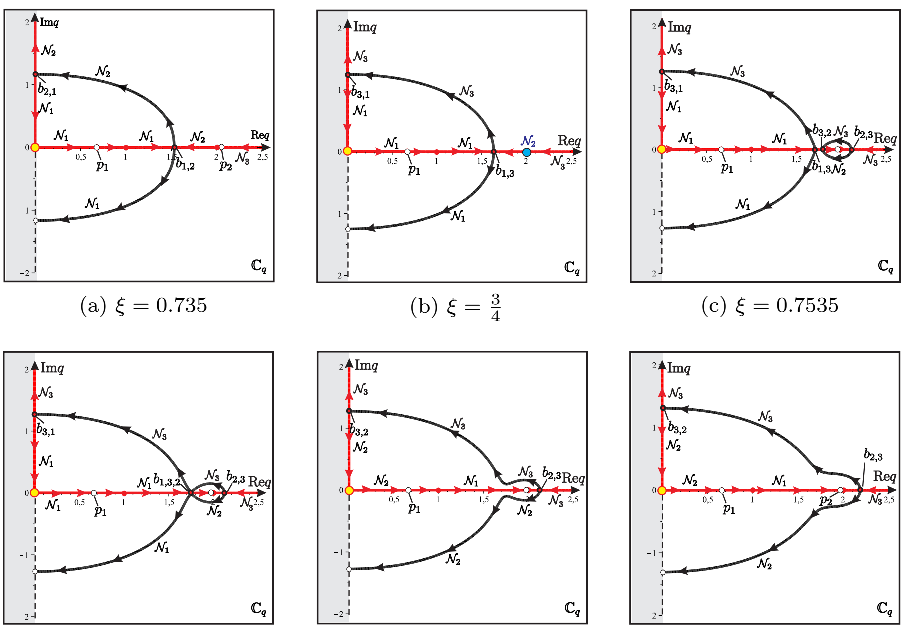
For every CEP $c_j = j$ we define *non-regular Spectrum Curve* (consisting of one point only) $\mathcal{N}_j = \{c_j\}$. We note, that non-regular Spectrum Curves can overlap with a point of a regular Spectrum Curve. In this point eigenvalue is not simple and generalized eigenvectors exist. Finally, we have that $\mathcal{N}_\xi$ is a countable union of Spectrum Curves $\mathcal{N}_l$, where $l \in \mathbb{N}$ for $\xi \neq 1$ and $l \in \{0\} \cup \mathbb{N}$ for $\xi = 1$.

4.1 Dynamics of Spectrum Curves

There are many papers, in which real eigenvalues of the Sturm–Liouville problem are analyzed. However, a complex spectrum of this problem is investigated that much and it is more complicated. By changing the value of the parameter ξ we get various type projection of Spectrum Curves in the complex domain

$\mathbb{C}_q$. In Figures 1b,1e and 6 we can see qualitative view of Spectrum Curves dependence on the parameter ξ.

If $\xi \in [0,1]$ is increasing from 0 to 1, then zeroes $p_k = (k-1/2)/\xi$, $k \in \mathbb{N}$, of the function $P_\xi(q)$ are moving to the left and zeroes $z_l = l$, $l \in \mathbb{N}$, of the function $Z(q)$ remain unchanged. In the limit case $\xi = 1$ PPs are $p_k = k - 1/2$, $k \in \mathbb{N}$, and we do not have PPs for $\xi = 0$. So, every p_k coincides with $z_l = l$, $l = k, k+1, \ldots$, for $\xi = (2k-1)/(2l)$ and we have CEP $c_s = p_{k_s} = z_{l_s} = n(s-1/2)$, $k_s = m(2s-1)/2 + 1/2$, $l_s = n(s-1/2)$, $s \in \mathbb{N}$ (see (3.9)). Formula (3.11) shows that we have the same situation for all CEPs. In Figures 3a–3c we can see how the Real CFs depend on the value of the parameter ξ in the neighbourhood of $\xi = \frac{1}{4}$. The Spectrum Curves near to $\xi = \frac{1}{4}$ are presented in Figures 7a–7c. When $\xi \lesssim \frac{1}{4}$ the PP p_1 is moving from right side to the zero point $z_2 = 2$. For $\xi = 1/4$ we have CEP at $c_1 = 2$. Next, if the value of $\xi \gtrsim \frac{1}{4}$ is increasing, the PP p_1 moves to the left from the zero point. A loop type curve appears, which consists of Spectrum Curves $\mathcal{N}_2$, $\mathcal{N}_3$ and two critical points $b_{3,2}$ and $b_{2,3}$ of the first order. While the value of ξ is increasing, the PP p_1 is moving to the left and the loop grows. We denote such *Zero and Pole bifurcation* type by $\beta_{ZP}\colon (z_{l_s}, p_{k_s}) \to c_s \to (b_{l_s+1,l_s}, p_{k_s}, z_{l_s}, b_{l_s,l_s+1})$. The CEPs exist for all rational $\xi = m/n$, $m \in \mathbb{N}_o$, $n \in \mathbb{N}_e$. We have such type bifurcation for $\xi = 1/2$ and $\xi = 3/4$ in the neighborhood of CEPs $c_1 = 1$ and $c_1 = 2$, respectively (see Figures 8a–8c and Figures 9a–9c).



(a) $\xi = 0.735$ (b) $\xi = \frac{3}{4}$ (c) $\xi = 0.7535$

(d) $\xi = \xi_{2b} \approx 0.7539...$ (e) $\xi = 0.7545$ (f) $\xi = 0.76$

Figure 9. Spectrum Curves for various parameter ξ values.

Every such ξ is a point of bifurcation near to all CEPs for this value of ξ.

Remark 12. For $\xi = 1/4$ we have $c_1 = 2$, $c_2 = 6$. β_{ZP} bifurcations at these two CEPs give similar view Spectrum Curves but the directions are opposite. The direction of Spectrum Curves depends on sign $\gamma(c_s;\xi)$ in formula (3.10).

CEP $c_1 = 1$ exists only for $\xi = 1/2$. If $\xi < 1/2$, then $\mathbb{R}_q^-$ and interval $[0,1)$ belong to regular Spectrum Curve $\mathcal{N}_1$. For $\xi = 1/2$ we have β_{ZP} type bifurcation and there are no CPs on $\mathcal{N}_2$ for $\gamma > 0$. If $\xi \gtrsim \frac{1}{2}$, then the smallest real eigenvalue is described by Spectrum Curve $\mathcal{N}_2$. We note that after bifurcation the CP $b_{2,1}$ exists (see Figure 8c) and this CP is moving to the left (Figure 8d). For $\xi = \xi_c = 1/\sqrt{3}$ this CP is at BEP $q = 0$. For $\xi > \xi_c$ this CP is moving along $\mathbb{R}_q^-$ (the imaginary axis) to ∞. So, for $\xi\colon \xi_c < \xi < 1$, we have loop type curve with one CP in $\mathbb{R}_q^-$ and another CP in $\mathbb{R}_q^+$ (see Figures 8e–8f).

Remark 13. During β_{ZP} bifurcation we get CEP c_s (see 2) and have a new configuration of Spectrum Curves:

1) regular Spectrum Curve $\mathcal{N}_{l_s}$ becomes non-regular;
2) on the left and on the rigth of CEP we get the same regular Spectrum Curve $\mathcal{N}_{l_s+1}$;
3) the CP b_{l_s-1,l_s} (if exists) becomes b_{l_s-1,l_s+1}, and the CP $b_{l_s,l_s-1} \in R_q^-$ (if exists) becomes b_{l_s+1,l_s-1}; at these CPs we have Spectrum Curves $\mathcal{N}_{l_s-1}$ and $\mathcal{N}_{l_s+1}$ instead $\mathcal{N}_{l_s-1}$ and $\mathcal{N}_{l_s}$.

The β_{ZP} bifurcation create configuration of the points $b_{l_s-1,l_s+1}, b_{l_s+1,l_s}$, p_{k_s}, z_{l_s}, b_{l_s,l_s+1} (see Figures 3c,3d, 7c, 9c) and loop type curve (formed by parts of the Spectrum Curves $\mathcal{N}_{l_s}$ and $\mathcal{N}_{l_s+1}$) in complex part of $\mathbb{C}_q$. While value of the parameter ξ is increasing, this loop type curve grows, the PP p_{k_s} is moving to the left and is pushing the CP b_{l_s+1,l_s} towards the CP b_{l_s-1,l_s+1}. These two 1CPs are between zero point z_{l_s-1} and PP p_{k_s}. For $\xi = \xi_{2b}$ these CPs merge into one 2CP b_{l_s-1,l_s+1,l_s} (see Figures 3e, 7d, 9d) and we have *the second order CP* bifurcation β_{2B}. We note, that ξ_{2b} depends on s and we can calculate the location of b_{l_s-1,l_s+1,l_s} only numerically ($z_{l_s-1} < b_{l_s-1,l_s+1,l_s} < p_{k_s}$). When $\xi \gtrsim \xi_{2b}$ loop type curve (around p_{k_s} and z_{l_s}) disappears and we have two Spectral Curves $\mathcal{N}_{l_s}$ and $\mathcal{N}_{l_s+1}$ which intersect in CP b_{l_s,l_s+1} (see Figures 3f,7e,7f, 9e,9f). So, $\beta_{2B}\colon (b_{l_s-1,l_s+1}, b_{l_s+1,l_s}) \to b_{l_s-1,l_s+1,l_s} \to \varnothing$.

Remark 14. During β_{2B} bifurcation we have a new configuration of Spectrum Curves:

1) the part of Spectrum Curve $\mathcal{N}_{l_s-1}$ becomes as part of Spectrum Curve $\mathcal{N}_{l_s}$;
2) after bifurcation Spectrum Curve $\mathcal{N}_{l_s-1}$ starts in PP p_{k_s} and not at infinity.

Finally, these two bifurcations β_{ZP} and β_{2B} interchange sequence of the points $(b_{l_s-1,l_s}, z_{l_s}, p_{k_s}) \to (p_{k_s}, z_{l_s}, b_{l_s,l_s+1})$ for $z_{l_s} \neq 1$. If $\xi = 1/2$, then the loop type curve around $q = 1$ exists for $1/2 < \xi < 1$. When value of ξ is increasing, then more and more zeroes and poles get inside the loop. This loop disappears only for $\xi = 1$. In Figures 1b, 6, 1e we see how global view of Spectrum Curves depends on parameter $\xi \in [0,1]$.

5 Conclusions

Investigation of the Spectrum Curves allows to get useful information about the spectrum for problems with two-point NBC. The properties of the spectrum for this problem depend on zeroes, poles, constant eigenvalue points and critical points of Characteristic Function. Critical points of Characteristic Function are important for analysis of complex eigenvalues and the Spectrum Curves in complex plane. In this paper we described how the Spectrum Curves depend on parameter ξ and find two types bifurcations. Some results are given as a graphs of Characteristic Function and Spectrum Curves in the complex domain $\mathbb{C}_q$.

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