Mathematical Modelling and Analysis
Volume 25, Issue 1, 71–87, 2020
https://doi.org/10.3846/mma.2020.10450

http://mma.vgtu.lt
ISSN: 1392-6292
eISSN: 1648-3510

Joint Discrete Approximation of a Pair of Analytic Functions by Periodic Zeta-Functions

Aidas Balčiūnas[a], **Virginija Garbaliauskienė**[b], **Julija Karaliūnaitė**[c], **Renata Macaitienė**[b], **Jurgita Petuškinaitė**[a] and **Audronė Rimkevičienė**[d]

[a] *Faculty of Mathematics and Informatics, Vilnius University*
Naugarduko str. 24, LT-03225 Vilnius, Lithuania
[b] *Institute of Regional Development, Šiauliai University*
P. Višinskio str. 25, LT-76351 Šiauliai, Lithuania
[c] *Faculty of Fundamental Sciences, Vilnius Gediminas Technical University*
Saulėtekio av. 11, LT-10223 Vilnius, Lithuania
[d] *Faculty of Business and Technologies, Šiauliai State College*
Aušros av. 40, LT-76241 Šiauliai, Lithuania
E-mail: aidas.balciunas@mif.vu.lt
E-mail(*corresp.*): virginija.garbaliauskiene@su.lt
E-mail: julija.karaliunaite@vgtu.lt
E-mail: renata.macaitiene@su.lt
E-mail: j.petuskinaite@gmail.com
E-mail: a.rimkeviciene@svako.lt

Received June 7, 2019; revised November 1, 2019; accepted November 1, 2019

Abstract. In the paper, the problem of simultaneous approximation of a pair of analytic functions by a pair of discrete shifts of the periodic and periodic Hurwitz zeta-function is considered. The above shifts are defined by using the sequence of imaginary parts of non-trivial zeros of the Riemann zeta-function. For the proof of approximation theorems, a weak form of the Montgomery pair correlation conjecture is applied.

Keywords: Hurwitz zeta-function, non-trivial zeros of the Riemann zeta-function, periodic zeta-function, periodic Hurwitz zeta-function, universality.

AMS Subject Classification: 11M06; 11M35; 11M41.

1 Introduction

In the paper, we consider the approximation of a pair of analytic functions by shifts of the periodic and periodic Hurwitz zeta-functions involving imaginary parts of non-trivial zeros of the Riemann zeta-function. We recall the definitions of the mentioned zeta-functions. Let $s = \sigma + it$ be a complex variable, and $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ and $\mathfrak{b} = \{b_m : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ be two periodic sequences of complex numbers with minimal periods $q_1 \in \mathbb{N}$ and $q_2 \in \mathbb{N}$, respectively. Then the periodic zeta-function $\zeta(s; \mathfrak{a})$ and the periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{b})$ with parameter α, $0 < \alpha \le 1$, are defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s} \text{ and } \zeta(s, \alpha; \mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m}{(m+\alpha)^s}.$$

If $a_m \equiv 1$, then $\zeta(s; \mathfrak{a})$ reduces to the Riemann zeta-function $\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$, $\sigma > 1$, and $\zeta(s, \alpha; \mathfrak{b})$, for $b_m \equiv 1$, becomes the classical Hurwitz zeta-function $\zeta(s; \alpha) = \sum_{m=0}^{\infty} 1/(m+\alpha)^s$. The periodicity of the sequences $\mathfrak{a}$ and $\mathfrak{b}$ implies the equalities

$$\zeta(s; \mathfrak{a}) = \frac{1}{q_1^s} \sum_{m=1}^{q_1} a_m \zeta\left(s, \frac{m}{q_1}\right), \tag{1.1}$$

$$\zeta(s, \alpha; \mathfrak{b}) = \frac{1}{q_2^s} \sum_{m=0}^{q_2-1} b_m \zeta\left(s, \frac{m+\alpha}{q_2}\right). \tag{1.2}$$

Thus, the well-known properties of the Hurwitz zeta-function show that the functions $\zeta(s; \mathfrak{a})$ and $\zeta(s, \alpha; \mathfrak{b})$ have analytic continuation to the whole complex plane, except for the point $s = 1$ that is a simple pole with residues

$$\frac{1}{q_1} \sum_{m=1}^{q_1} a_m \text{ and } \frac{1}{q_2} \sum_{m=0}^{q_2-1} b_m,$$

respectively. If the above quantities are zero, then the corresponding zeta-functions are entire. The approximation of analytic functions by the functions $\zeta(s; \mathfrak{a})$ and $\zeta(s, \alpha; \mathfrak{b})$ was studied in [8, 26, 28, 29] and [2, 7, 18, 22, 24, 25, 27], respectively.

The first joint results for a pair of functions $\big(\zeta(s; \mathfrak{a}), \zeta(s, \alpha; \mathfrak{b})\big)$ has been obtained in [9]. Assuming that the sequence $\mathfrak{a}$ is multiplicative, i. e., $a_1 = 1$ and $a_{mn} = a_m a_n$ for all coprimes m and n, and that the parameter is transcendental, a joint universality theorem on the approximation of a pair of analytic functions has been proved. Let $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$, $\mathcal{K}$ be the class of compact subsets of the strip D with connected complements, $H(K)$ with $K \in \mathcal{K}$ be the class of continuous functions on K that are analytic in the interior of K, and let $H_0(K)$ denote the subclass of $H(K)$ of non-vanishing functions. Then it was proved in [9] that if $K_1, K_2 \in \mathcal{K}$, $f_1(s) \in H_0(K_1)$ and

$f_2(s) \in H(K_2)$, then for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas}\Big\{ \tau \in [0,T] : \sup_{s\in K_1} |\zeta(s+i\tau;\mathfrak{a}) - f_1(s)| < \varepsilon, \sup_{s\in K_2} |\zeta(s+i\tau,\alpha;\mathfrak{b}) - f_2(s)| < \varepsilon \Big\} > 0,$$

where meas$A$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. A discrete version of the latter theorem has been presented in [15]. Let $\#A$ denote the cardinality of the set A, N run over non-negative integers, and $\mathbb{P}$ be the set of all prime numbers. For $h > 0$, define

$$L(\mathbb{P},\alpha,h,\pi) = \Big\{ (\log p : p \in \mathbb{P}), (\log(m+\alpha) : m \in \mathbb{N}_0), \frac{2\pi}{h} \Big\}.$$

If the set $L(\mathbb{P},\alpha,h,\pi)$ is linearly independent over the field of rational numbers $\mathbb{Q}$, and the sequence $\mathfrak{a}$ is multiplicative, then, for the same K_1, K_2 and $f_1(s), f_2(s)$ as above, it was proved in [15] that, for every $\varepsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{ 0 \le k \le N : \sup_{s\in K_1} |\zeta(s+ikh;\mathfrak{a}) - f_1(s)| < \varepsilon, \sup_{s\in K_2} |\zeta(s+ikh,\alpha;\mathfrak{b}) - f_2(s)| < \varepsilon \Big\} > 0. \quad (1.3)$$

Moreover, under hypothesis that the set

$$\big\{ (h_1 \log p : p \in \mathbb{P}), (h_2 \log(m+\alpha) : m \in \mathbb{N}_0, 2\pi \big\}$$

is linearly independent over $\mathbb{Q}$, it was obtained the following modification of inequality (1.3):

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{ 0 \le k \le N : \sup_{s\in K_1} |\zeta(s+ikh_1;\mathfrak{a}) - f_1(s)| < \varepsilon, \sup_{s\in K_2} |\zeta(s+ikh_2,\alpha;\mathfrak{b}) - f_2(s)| < \varepsilon \Big\} > 0.$$

Similar results also are given in [17] and [19]. Approximation results for more general collections consisting from periodic zeta functions were obtained in [3, 11, 12, 13, 14, 16, 20] and [23].

The aim of this paper is to replace in shifts $\zeta(s+ikh;\mathfrak{a})$ and $\zeta(s+ikh;\alpha;\mathfrak{b})$ the sequence $\{kh\}$ by more complicated one. Let $0 < \gamma_1 < \gamma_2 < ... \le \gamma_k \le ...$ be the sequence of imaginary parts of non-trivial zeros of the Riemann zeta-function. The behaviour of the sequence $\{\gamma_k : k \in \mathbb{N}\}$ is mysterious, therefore, we will use a certain hypothesis that is implied by the well-known Montgomery pair correlation conjecture [33]. Namely, we suppose that the estimate

$$\sum_{\substack{\gamma_k \le T \\ |\gamma_k - \gamma_l| < \frac{c}{\log T}}} \sum_{\gamma_l \le T} 1 \ll T \log T \quad (1.4)$$

holds for $c > 0$ as $T \to \infty$. The Montgomery conjecture gives the asymptotic formula for the left-hand side of (1.4). The condition (1.4) was applied in [6] for the approximation of analytic functions by shifts $\zeta(s + i\gamma_k h)$, in [30] for shifts $\zeta(s+i\gamma_n h, \alpha)$ and by shifts $(\zeta(s + i\gamma_k h), \zeta(s + i\gamma_k h, \alpha)))$ in [21]. In [4,5], in place of (1.4), the Riemann hypothesis was used. The paper [26] is devoted to joint approximation of analytic functions by shifts of Dirichlet L-functions $L(s + i\gamma_k h, \chi_1), ..., L(s + i\gamma_k h, \chi_r)$ also by using (1.4).

Now, we state the main theorems of the paper.

Theorem 1. *Suppose that the sequence $\mathfrak{a}$ is multiplicative, the parameter α is transcendental, and the bound (1.4) is true. Let $K_1, K_2 \in \mathcal{K}$, $f_1(s) \in H_0(K_1)$, $f_2(s) \in H(K_2)$ and $h > 0$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N} \# \left\{ 1 \le k \le N : \sup_{s\in K_1} |\zeta(s + i\gamma_k h; \mathfrak{a}) - f_1(s)| < \varepsilon, \right.$$
$$\left. \sup_{s\in K_2} |\zeta(s + i\gamma_k h, \alpha; \mathfrak{b}) - f_2(s)| < \varepsilon \right\} > 0.$$

The positivity of a lower density of the set of shifts approximating a given pair $(f_1(s), f_2(s))$ can be replaced by that of the density with some exception for $\varepsilon > 0$. More precisely, the following statement is true.

Theorem 2. *Under hypotheses of Theorem 1, the limit*

$$\lim_{N\to\infty} \frac{1}{N} \# \{ 1 \le k \le N : \sup_{s\in K_1} |\zeta(s + i\gamma_k h; \mathfrak{a}) - f_1(s)| < \varepsilon,$$
$$\sup_{s\in K_2} |\zeta(s + i\gamma_k h, \alpha; \mathfrak{b}) - f_2(s)| < \varepsilon \} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

For the proof of Theorems 1 and 2, the Fourier transform and weak convergence methods will be applied.

2 Uniform distribution modulo 1

In this section, we present some facts related to the uniform distribution modulo 1 of sequences of real numbers.

We recall that the sequence $\{x_k : k \in \mathbb{N}\} \subset \mathbb{R}$ is uniformly distributed modulo 1 if, for every interval $[a, b) \subset [0, 1)$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{[a,b)}\big(\{x_k\}\big) = b - a,$$

where $\chi_{[a,b)}$ is the indicator function of the interval $[a, b)$, and $\{x_k\}$ denotes the fractional part of x_k.

The next lemma is the well-known Weyl criterion.

Lemma 1. *A sequence $\{x_k : k \in \mathbb{N}\} \subset \mathbb{R}$ is uniformly distributed modulo 1 if and only if, for every $m \in \mathbb{Z} \setminus \{0\}$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} e^{2\pi i m x_k} = 0.$$

Proof of the lemma can be found, for example, in [10].

Lemma 2. *The sequence $\{\gamma_k a : k \in \mathbb{N}\}$ with every $a \in \mathbb{R} \setminus \{0\}$ is uniformly distributed modulo 1.*

Proof. The lemma was obtained in [34] and used in [6]. □

Lemmas 1 and 2 will be applied for weak convergence of probability measures on certain topological groups. Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, and

$$\Omega_1 = \prod_{p\in\mathbb{P}} \gamma_p \text{ and } \Omega_2 = \prod_{m\in\mathbb{N}_0} \gamma_m,$$

where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$ and $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. In view of the Tikhonov theorem, Ω_1 and Ω_2, with the product topology and pointwise multiplication, are compact topological Abelian groups. Define $\Omega = \Omega_1 \times \Omega_2$. Then again, Ω is a compact topological group, therefore, on $(\Omega, \mathcal{B}(\Omega))$ ($\mathcal{B}(\mathbb{X})$ is the Borel σ-field of the space $\mathbb{X}$) the probability Haar measure m_H exists, and we have the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega_1(p)$ the *pth* component of an element $\omega_1 \in \Omega_1$, $p \in \mathbb{P}$, and by $\omega_2(m)$ the *mth* component of an element $\omega_2 \in \Omega_2$. Elements of Ω are denoted by $\omega = (\omega_1, \omega_2)$, $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$.

For $A \in \mathcal{B}(\Omega)$, define

$$Q_{N,\alpha}(A)=\frac{1}{N}\#\Big\{1 \le k \le N : \Big(\big(p^{-i\gamma_k h} : p \in \mathbb{P}\big), (m+\alpha)^{-i\gamma_k h} : m \in \mathbb{N}_0\Big)\in A\Big\}.$$

The next lemma deals with weak convergence of $Q_{N,\alpha}$ as $N \to \infty$.

Lemma 3. *Suppose that α is a transcendental number. Then $Q_{N,\alpha}$ converges weakly to the Haar measure m_H as $N \to \infty$.*

Proof. We apply the Fourier transform method. Let $g_{N,\alpha}(\underline{k}, \underline{l})$, $\underline{k} = (k_p : k_p \in \mathbb{Z}, p \in \mathbb{P})$, $\underline{l} = (l_m : l_m \in \mathbb{Z}, m \in \mathbb{N}_0)$, be the Fourier transform of $Q_{N,\alpha}$. Then it is well known that

$$g_{N,\alpha}(\underline{k}, \underline{l}) = \int_{\Omega} \Big(\prod_{p\in\mathbb{P}}{}' \omega_1^{k_p}(p) \prod_{m\in\mathbb{N}_0}{}' \omega_2^{l_m}(m)\Big) \mathrm{d}Q_{N,\alpha},$$

where "′" means that only a finite number of integers k_p and l_m are distinct from zero. Thus, by the definition of $Q_{N,\alpha}$,

$$\begin{aligned} g_{N,\alpha}(\underline{k}, \underline{l}) &= \frac{1}{N} \sum_{k=1}^{N} \prod_{p\in\mathbb{P}}{}' p^{-ihk_p\gamma_k} \prod_{m\in\mathbb{N}_0}{}' (m+\alpha)^{-ihl_m\gamma_k} \qquad (2.1)\\ &= \frac{1}{N} \sum_{k=1}^{N} \exp\Big\{ -ih\gamma_k \Big(\sum_{p\in\mathbb{P}}{}' k_p \log p + \sum_{m\in\mathbb{N}_0}{}' l_m \log(m+\alpha)\Big)\Big\}. \end{aligned}$$

Clearly,

$$g_{N,\alpha}(\underline{0},\underline{0}) = 1. \tag{2.2}$$

Since α is transcendental, the set

$$\big\{(\log p : p \in \mathbb{P}), (\log(m+\alpha) : m \in \mathbb{N}_0)\big\}$$

is linearly independent over $\mathbb{Q}$ [9]. Therefore,

$$\sum_{p\in\mathbb{P}}{}' k_p \log p + \sum_{m\in\mathbb{N}_0}{}' l_m \log(m+\alpha) \neq 0$$

for $(\underline{k},\underline{l}) \neq (\underline{0},\underline{0})$. Hence, in view of Lemmas 2 and 1, we obtain by (2.1)

$$\lim_{N\to\infty} g_{N,\alpha}(\underline{k},\underline{l}) = 0$$

for $(\underline{k},\underline{l}) \neq (\underline{0},\underline{0})$. This together with (2.2) shows that

$$\lim_{N\to\infty} g_{N,\alpha}(\underline{k},\underline{l}) = \begin{cases} 1, & \text{if } (\underline{k},\underline{l}) = (\underline{0},\underline{0}), \\ 0, & \text{if } (\underline{k},\underline{l}) \neq (\underline{0},\underline{0}). \end{cases}$$

Since the right-hand side of the latter equality is the Fourier transform of the Haar measure m_H, a continuity theorem for probability measures on compact groups proves the lemma. □

Lemma 3 implies the weak convergence for probability measures defined by means of absolutely convergent Dirichlet series. We recall that $D = \big\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\big\}$. Denote by $H(D)$ the space of analytic functions on D endowed with the topology of uniform convergence on compacta, and $H^2(D) = H(D) \times H(D)$.

Let $\theta > \frac{1}{2}$ be a fixed number, and, for $m, n \in \mathbb{N}$,

$$v_n(m) = \exp\Big\{ - \Big(\frac{m}{n}\Big)^\theta \Big\},$$

and, for $m \in \mathbb{N}_0$, $n \in \mathbb{N}$,

$$v_n(m,\alpha) = \exp\Big\{ - \Big(\frac{m+\alpha}{n+\alpha}\Big)^\theta \Big\}.$$

Define the series

$$\zeta_n(s;\mathfrak{a}) = \sum_{m=1}^\infty \frac{a_m v_n(m)}{m^s}, \quad \zeta_n(s,\alpha;\mathfrak{b}) = \sum_{m=0}^\infty \frac{b_m v_n(m,\alpha)}{(m+\alpha)^s}.$$

The latter series are absolutely convergent for $\sigma > \frac{1}{2}$ [9]. Moreover, we set

$$\zeta_n(s,\omega_1;\mathfrak{a}) = \sum_{m=1}^\infty \frac{a_m \omega_1(m) v_n(m)}{m^s} \tag{2.3}$$

and

$$\zeta_n(s,\alpha,\omega_2;\mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m \omega_2(m) v_n(m,\alpha)}{(m+\alpha)^s}, \tag{2.4}$$

the series again being absolutely convergent for $\sigma > \frac{1}{2}$. For brevity, we put

$$\underline{\zeta}_n(s,\alpha;\mathfrak{a},\mathfrak{b}) = (\zeta_n(s;\mathfrak{a}),\zeta_n(s,\alpha;\mathfrak{b})),$$
$$\underline{\zeta}_n(s,\alpha,\omega;\mathfrak{a},\mathfrak{b}) = (\zeta_n(s,\omega_1;\mathfrak{a}),\zeta_n(s,\alpha,\omega_2;\mathfrak{b})).$$

Define the function $u_{n,\alpha} : \Omega \to H^2(D)$ by the formula

$$u_{n,\alpha}(\omega) = \underline{\zeta}_n(s,\alpha,\omega;\mathfrak{a},\mathfrak{b}).$$

Since the series (2.3) and (2.4) are absolutely convergent for $\sigma > \frac{1}{2}$, the function $u_{n,\alpha}$ is continuous, hence $(\mathcal{B}(\Omega),\mathcal{B}(H^2(D)))$ – measurable. Therefore, the measure m_H induces on $(H^2(D),\mathcal{B}(H^2(D)))$ the unique probability measure $m_H u_{n,\alpha}^{-1}$ defined, for $A \in \mathcal{B}(H^2(D))$ by

$$m_H u_{n,\alpha}^{-1}(A) = m_H(u_{n,\alpha}^{-1}A).$$

Let, for $A \in \mathcal{B}(H^2(D))$,

$$P_{N,n,\alpha}(A) = \frac{1}{N}\#\Big\{1 \le k \le N : \underline{\zeta}_n(s+i\gamma_k h,\alpha;\mathfrak{a},\mathfrak{b}) \in A\Big\}.$$

Then we have the following statement.

Lemma 4. *Suppose that α is a transcendental number. Then $P_{N,n,\alpha}$ converges weakly to $\hat{P}_{n,\alpha} \stackrel{def}{=} m_H u_{n,\alpha}^{-1}$ as $N \to \infty$.*

Proof. By the definition of $u_{n,\alpha}$,

$$u_{n,\alpha}\left((p^{-i\gamma_k h} : p \in \mathbb{P}),((m+\alpha)^{-i\gamma_k h} : m \in \mathbb{N}_0)\right) = \underline{\zeta}_n(s+i\gamma_k h,\alpha;\mathfrak{a},\mathfrak{b}).$$

Therefore, for every $A \in \mathcal{B}(H^2(D))$,

$$P_{N,n,\alpha}(A) = \frac{1}{N}\#\Big\{1 \le k \le N : \left((p^{-i\gamma_k h} : p \in \mathbb{P}),((m+\alpha)^{-i\gamma_k h} : m \in \mathbb{N}_0)\right) \in u_{n,\alpha}^{-1}A\Big\},$$

i. e., $P_{N,n,\alpha} = Q_{N,\alpha}u_{n,\alpha}^{-1}$, where $Q_{N,\alpha}$ is from Lemma 3. Thus, the assertion of the lemma is a consequence of Lemma 3, continuity of $u_{n,\alpha}$ and Theorem 5.1 of [1]. □

3 Mean square estimates

To pass from $\underline{\zeta}_n(s,\alpha;\mathfrak{a},\mathfrak{b})$ to $\underline{\zeta}(s,\alpha,\mathfrak{a},\mathfrak{b}) = (\zeta(s;\mathfrak{a}),\zeta(s,\alpha;\mathfrak{b}))$, we need a certain approximation result for $\underline{\zeta}(s,\alpha;\mathfrak{a},\mathfrak{b})$ by $\zeta_n(s,\alpha;\mathfrak{a},\mathfrak{b})$. For this aim, some mean square estimates are needed. In this step, the estimate (1.4) plays an important role. Equalities (1.1) and (1.2) imply for fixed σ, $\frac{1}{2} < \sigma < 1$, the estimates

$$\int_0^T |\zeta(\sigma + it; \mathfrak{a})|^2 \mathrm{d}t \ll_{\sigma,\mathfrak{a}} T \quad \text{and} \quad \int_0^T |\zeta(\sigma + it, \alpha; \mathfrak{b})|^2 \mathrm{d}t \ll_{\sigma,\alpha,\mathfrak{b}} T.$$

Hence, for $\tau \in \mathbb{R}$,

$$\int_0^T |\zeta(\sigma + it + i\tau; \mathfrak{a})|^2 \mathrm{d}t \ll_{\sigma,\mathfrak{a}} T(1 + |\tau|), \tag{3.1}$$

$$\int_0^T |\zeta(\sigma + it + i\tau, \alpha; \mathfrak{b})|^2 \mathrm{d}t \ll_{\sigma,\alpha,\mathfrak{b}} T(1 + |\tau|). \tag{3.2}$$

The above mean square estimates are of continuous type. The following Gallagher lemma connects discrete and continuous mean square estimates for certain functions.

Lemma 5. *Suppose that T_0, $T \geq \delta > 0$ are real numbers, and $\mathfrak{T} \neq \varnothing$ is a finite set in the interval $[T_0 + \frac{\delta}{2}, T_0 + T - \frac{\delta}{2}]$. Define*

$$N_\delta(x) = \sum_{t \in \mathfrak{T},\, |t - x| < \delta} 1.$$

Let $S(x)$ be a complex-valued continuous function on $[T_0, T_0 + T]$ having a continuous derivative on $(T_0, T_0 + T)$. Then

$$\sum_{t \in \mathfrak{T}} N_\delta^{-1}(t) |S(t)|^2 \leq \frac{1}{\delta} \int\limits_{T_0}^{T_0+T} |S(x)|^2 dx + \left(\int\limits_{T_0}^{T_0+T} |S(x)|^2 dx \int_{T_0}^{T_0+T} |S'(x)|^2 dx \right)^{\frac{1}{2}}.$$

Proof of the lemma is given in [32], Lemma 1.4.

The asymptotics of γ_k is given in

Lemma 6. *For $k \to \infty$, $\gamma_k \sim 2\pi k / \log k$.*

Proof of the lemma can be found in [35].

Now, we are in position to obtain discrete mean square estimates for the functions $\zeta(s, \mathfrak{a})$ and $\zeta(s, \alpha; \mathfrak{b})$.

Lemma 7. *Suppose that (1.4) is true. Then, for fixed σ, $\frac{1}{2} < \sigma < 1$, and $\tau \in \mathbb{R}$,*

$$\sum_{k=1}^N |\zeta(\sigma + i\gamma_k h + i\tau; \mathfrak{a}| \ll_{\sigma,\mathfrak{a},h} N(1 + |\tau|),$$

$$\sum_{k=1}^N |\zeta(\sigma + i\gamma_k h + i\tau, \alpha; \mathfrak{b}| \ll_{\sigma,\alpha,\mathfrak{b},h} N(1 + |\tau|).$$

Proof. In view of Lemma 6, $\gamma_k \le c_1 k/\log k$ with some $c_1 > 0$ for all $k \ge 2$. We apply Lemma 5 with $\delta = ch\left(\log \frac{\log N}{c_1 N}\right)^{-1}$, $T_0 = \gamma_1 h - \frac{\delta}{2}$, $T = \gamma_N h - T_0 + \frac{\delta}{2}$ and $\mathfrak{T} = \{\gamma_1 h, ..., \gamma_N h\}$. Then we have by (1.4)

$$\sum_{l=1}^{N} N_\delta(\gamma_l h) = \sum_{l=1}^{N} \sum_{\substack{\gamma_k \le \frac{c_1 N}{\log N} \\ |\gamma_l - \gamma_k| < \frac{\delta}{h}}} 1 = \sum\sum_{\substack{\gamma_l, \gamma_k \le \frac{c_1 N}{\log N} \\ |\gamma_l - \gamma_k| < \frac{\delta}{h}}} 1 \ll N. \tag{3.3}$$

By the Cauchy integral formula,

$$\zeta'(\sigma + it + i\tau; \mathfrak{a}) = \frac{1}{2\pi i} \int_L \frac{\zeta(z + it + i\tau; \mathfrak{a})}{(z - \sigma)^2} \mathrm{d}z,$$

where L is the circle with a center σ lying in D. Hence,

$$\left|\zeta'(\sigma + it + i\tau; \mathfrak{a})\right|^2 \ll \left| \int_L \frac{\zeta'(z + it + i\tau; \mathfrak{a})}{(z - \sigma)^2} \mathrm{d}z\right|^2 \ll \int_L \frac{|\mathrm{d}z|}{|z - \sigma|^4}$$
$$\times \int_L |\zeta(z + it + i\tau; \mathfrak{a})|^2 |\mathrm{d}z| \ll_\sigma \int_L |\zeta(z + it + i\tau; \mathfrak{a})|^2 |\mathrm{d}z|.$$

Therefore, in view of (3.2),

$$\int_0^T \left|\zeta'(\sigma + it + i\tau; \mathfrak{a})\right|^2 \mathrm{d}t \ll \int_L \left|\mathrm{d}z\right| \int_0^T \left|\zeta(\Re z + i\Im z + it + i\tau; \mathfrak{a})\right|^2 \mathrm{d}t$$
$$\ll_{\sigma,\mathfrak{a}} T(1 + |\tau|).$$

Now, this (3.1), (3.3) and Lemma 5 yield, for sufficiently large N,

$$\sum_{k=1}^{N} |\zeta(\sigma + i\gamma_k h + i\tau; \mathfrak{a})| = \sum_{k=1}^{N} \sqrt{N_\delta(\gamma_k h) N_\delta^{-1}(\gamma_k h)} |\zeta(\sigma + i\gamma_k h + i\tau; \mathfrak{a})|$$
$$\ll \left(\sum_{k=1}^{N} N_\delta(\gamma_k h) \sum_{k=1}^{N} N_\delta^{-1}(\gamma_k h) |\zeta(\sigma + i\gamma_k h + i\tau; \mathfrak{a})|^2 \right)^{\frac{1}{2}}$$
$$\ll_\sigma \sqrt{N} \Big(\frac{1}{\delta} \int_0^{2\gamma_N h} |\zeta(\sigma + it + i\tau; \mathfrak{a})|^2 \mathrm{d}t + \Big(\int_0^{2\gamma_N h} |\zeta(\sigma + it + i\tau; \mathfrak{a})|^2 \mathrm{d}t$$
$$\times \int_0^{2\gamma_N h} |\zeta'(\sigma + it + i\tau; \mathfrak{a})|^2 \mathrm{d}t \Big)^{\frac{1}{2}} \Big)^{\frac{1}{2}} \ll_{\sigma,\mathfrak{b},h} N(1 + |\tau|).$$

The bound for the function $\zeta(s, \alpha; \mathfrak{b})$ is obtained similarly. □

4 Approximation results

In this section, we will approximate $\underline{\zeta}(s + i\gamma_k h, \alpha; \mathfrak{a}, \mathfrak{b})$ by $\underline{\zeta}_n(s + i\gamma_k h, \alpha; \mathfrak{a}, \mathfrak{b})$ in the mean. For this, we recall the metric in the space $H^2(D)$. For $g_1, g_2 \in H(D)$, define

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|},$$

where $\{K_l : l \in \mathbb{N}\}$ is a sequence of compact subsets of the strip D such that $D = \bigcup_{l=1}^{\infty} K_l$, $K_l \subset K_{l+1}$ for $l \in \mathbb{N}$, and if $K \subset D$ is a compact set, then $K \subset K_l$ for some $l \in \mathbb{N}$. Then ρ is a metric in $H(D)$ inducing its topology of uniform convergence on compacta. For $\underline{g}_1 = (g_{11}, g_{12}), \underline{g}_2 = (g_{21}, g_{22}) \in H^2(D)$, we set

$$\underline{\rho}(\underline{g}_{11}, \underline{g}_{21}) = \max_{1 \le j \le 2} \rho(g_{1j}, g_{2j}).$$

Then $\underline{\rho}$ is a metric in $H^2(D)$ inducing the product topology.

Lemma 8. *Suppose that (1.4) is true. Then*

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \underline{\rho}\Big(\underline{\zeta}(s+i\gamma_k h, \alpha; \mathfrak{a}, \mathfrak{b}), \underline{\zeta}_n(s+i\gamma_k h, \alpha; \mathfrak{a}, \mathfrak{b})\Big) = 0.$$

Proof. By the definition of the metric $\underline{\rho}$, it suffices to prove that

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \rho\left(\zeta(s+i\gamma_k h; \mathfrak{a}), \zeta_n(s+i\gamma_k h; \mathfrak{a})\right) = 0, \tag{4.1}$$

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \rho\left(\zeta(s+i\gamma_k h, \alpha; \mathfrak{b}), \zeta_n(s+i\gamma_k h, \alpha; \mathfrak{b})\right) = 0. \tag{4.2}$$

Let

$$l_n(s) = \frac{s}{\theta} \Gamma\left(\frac{s}{\theta}\right) n^s,$$

where θ comes from the definition of $v_n(m)$, and $\Gamma(s)$ denotes the Euler gamma-function. Then it is known that

$$\zeta_n(s; \mathfrak{a}) = \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} \zeta(s+z; \mathfrak{a}) l_n(z) \frac{\mathrm{d}z}{z}. \tag{4.3}$$

Denote by a the residue of the function $\zeta(s; \mathfrak{a})$ at the point $s = 1$. Let $\hat{\theta} > 0$. Then, by (4.3),

$$\zeta_n(s; \mathfrak{a}) - \zeta(s; \mathfrak{a}) = \frac{1}{2\pi i} \int_{-\hat{\theta}-i\infty}^{-\hat{\theta}+i\infty} \zeta(s+z; \mathfrak{a}) l_n(z) \frac{\mathrm{d}z}{z} + \frac{a l_n(1-s)}{1-s}. \tag{4.4}$$

Suppose that K is a fixed compact set of the strip D, and take $\varepsilon > 0$ such that $\frac{1}{2} + 2\varepsilon \le \sigma \le 1 - \varepsilon$ for any point $s = \sigma + iv \in K$. Now, let

$$\hat{\theta} = \sigma - \varepsilon - \frac{1}{2} \text{ and } \theta = \frac{1}{2} + \varepsilon.$$

Then (4.4), implies, for $s \in K$, the inequality

$$\begin{aligned} |\zeta(s+i\gamma_k h; \mathfrak{a}) - \zeta_n(s+i\gamma_k h; \mathfrak{a})| \le & \frac{1}{2\pi} \int_{-\infty}^{\infty} |\zeta(s+i\gamma_k h - \hat{\theta} + it)| \frac{l_n(-\hat{\theta}+it)}{|-\hat{\theta}+it|} \mathrm{d}t \\ & + \frac{|a| l_n(1-s-i\gamma_k h)|}{|1-s-i\gamma_k h|}. \end{aligned}$$

In the latter integral, take t in place $t+v$. This gives

$$|\zeta(s+i\gamma_k h;\mathfrak{a})-\zeta_n(s+i\gamma_k h;\mathfrak{a})| \leq \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\zeta\Big(\frac{1}{2}+\varepsilon+i(t+\gamma_k h);\mathfrak{a}\Big)\right| \\ \times \frac{|l_n(\frac{1}{2}+\varepsilon-s+it)|}{|\frac{1}{2}+\varepsilon-s+it|}\mathrm{d}t+\frac{|a|l_n(1-s-i\gamma_k h)|}{|1-s-i\gamma_k h|}.$$

This leads to

$$\frac{1}{N}\sum_{k=1}^{N}\sup_{s\in K}|\zeta(s+i\gamma_k;\mathfrak{a})-\zeta_n(s+i\gamma_k h;\mathfrak{a})| \leq S_1+S_2, \tag{4.5}$$

where

$$S_1=\frac{1}{2\pi N}\int_{-\infty}^{\infty}\Big(\sum_{k=1}^{N}\left|\zeta\big(\frac{1}{2}+\varepsilon+i(t+\gamma_k h);\mathfrak{a}\big)\right|\sup_{s\in K}\frac{|l_n(\frac{1}{2}+\varepsilon-s+it)|}{|\frac{1}{2}+\varepsilon-s+it|}\Big)\mathrm{d}t,$$

$$S_2=\frac{|a|}{N}\sum_{k=1}^{N}\sup_{s\in K}\frac{|l_n(1-s-i\gamma_k h)|}{|1-s-i\gamma_k h|}.$$

For the function $\Gamma(\sigma+it)$, the estimate

$$\Gamma(\sigma+it)\ll\exp\{-c|t|\}\ ,\ c>0,$$

uniform in $\sigma_1\leq\sigma\leq\sigma_2$, is known. Therefore, the definition of the function $l_n(s)$ implies the bound, for $s\in K$,

$$\frac{l_n(\frac{1}{2}+\varepsilon-s+it)}{\frac{1}{2}+\varepsilon-s+it}\ll n^{-\varepsilon}\exp\Big\{-\frac{c|t-v|}{\theta}\Big\}\ll_K n^{-\varepsilon}\exp\{-c|t|\}. \tag{4.6}$$

By similar arguments, we find that

$$\frac{l_n(1-s-i\gamma_k h)}{1-s-i\gamma_k h}\ll_{K,h} n^{1-\sigma}\exp\{-c\gamma_k h\}. \tag{4.7}$$

From (4.6) and Lemma 7, it follows that

$$S_1\ll_{K,\mathfrak{a},h} n^{-\varepsilon}\int_{-\infty}^{\infty}(1+|t|)\exp\{-c|t|\}\mathrm{d}t\ll_{K,\mathfrak{a},h} n^{-\varepsilon},$$

while (4.7) shows that

$$S_2\ll_{K,\mathfrak{a},n} n^{\frac{1}{2}-2\varepsilon}\frac{\log N}{N}.$$

Therefore, in view of (4.5), we obtain that

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\sup_{s\in K}|\zeta(s+i\gamma_k h;\mathfrak{a})-\zeta_n(s+i\gamma_k h;\mathfrak{a})|=0,$$

and this and the definition of the metric ρ imply (4.1).

The equality (4.2) is proved similarly by using the representation

$$\zeta_n(s,\alpha;\mathfrak{b}) = \frac{1}{2\pi i} \int\limits_{-\theta+i\infty}^{\theta+i\infty} \zeta(s+z,\alpha;\mathfrak{b}) \frac{l_n(z,\alpha)}{z} dz,$$

where

$$l_n(s,\alpha) = \frac{s}{\theta} \Gamma\left(\frac{s}{\theta}\right)(n+\alpha)^s,$$

as well as the second bound of Lemma 7. □

5 A limit theorem

In this section, we will prove a limit theorem for $\underline{\zeta}(s,\alpha;\mathfrak{a},\mathfrak{b})$ in the space $H^2(D)$. For the statement of that theorem, a certain $H^2(D)$ - valued random element is used. On the probability space $(\Omega,\mathcal{B}(\Omega),m_H)$, define the $H^2(D)$ - valued random element

$$\underline{\zeta}(s,\omega,\alpha;\mathfrak{a},\mathfrak{b}) = \left(\sum_{m=1}^{\infty} \frac{a_m \omega_1(m)}{m^s}, \sum_{m=0}^{\infty} \frac{b_m \omega_2(m)}{(m+\alpha)^s}\right).$$

We observe that the latter series both are almost surely uniformly convergent on compact subsets of the strip D. Denote by $P_{\underline{\zeta},\alpha}$ the distribution of the random element $\underline{\zeta}(s,\omega,\alpha;\mathfrak{a},\mathfrak{b})$, i. e.,

$$P_{\underline{\zeta},\alpha}(A) = m_H\{\omega\in\Omega : \underline{\zeta}(s,\omega,\alpha;\mathfrak{a},\mathfrak{b}) \in A\}, \ A\in\mathcal{B}(H^2(D)).$$

Moreover, for $A\in\mathcal{B}(H^2(D))$,

$$P_{N,\alpha}(A) = \frac{1}{N}\#\{1\le k\le N : \underline{\zeta}(s+i\gamma_k h,\alpha;\mathfrak{a},\mathfrak{b})\in A\}.$$

Theorem 3. *Suppose that the sequence $\mathfrak{a}$ is multiplicative, the parameter α is transcendental, and the bound (1.4) is true. Then $P_{N,\alpha}$ converges weakly to $P_{\underline{\zeta},\alpha}$ as $N\to\infty$.*

Proof. We return to Lemma 4 and its limit measure $\hat{P}_{n,\alpha}$. Let θ_N be a random variable defined on a certain probability space with the measure μ and having the distribution

$$\mu\{\theta_N = \gamma_k h\} = \frac{1}{N}, k=1,...,N.$$

Define the $H^2(D)$ - valued random element

$$X_{N,n,\alpha} = X_{N,n,\alpha}(s) = \underline{\zeta}_n(s+i\theta_N,\alpha;\mathfrak{a},\mathfrak{b}).$$

Then, denoting by $\hat{X}_{n,\alpha}$ the $H^2(D)$-valued random element with the distribution $\hat{P}_{n,\alpha}$, we rewrite the assertion of Lemma 4 in the form

$$X_{N,n,\alpha} \xrightarrow[N\to\infty]{\mathcal{D}} \hat{X}_{n,\alpha}. \tag{5.1}$$

In [9], it is proved that the sequence of probability measures $\{\hat{P}_{n,\alpha} : n \in \mathbb{N}\}$ is tight, i.e., for every $\varepsilon > 0$, there exists a compact set $K = K(\varepsilon) \subset H^2(D)$ such that

$$\hat{P}_{n,\alpha}(K) > 1 - \varepsilon$$

for all $n \in \mathbb{N}$. By the Prokhorov theorem [1], Theorem 6.1, every tight family of probability measures is relatively compact. Thus, every subsequence of $\{\hat{P}_{n,\alpha}\}$ contains a subsequence $\{\hat{P}_{n_r,\alpha}\}$ such that $\hat{P}_{n_r,\alpha}$ converges weakly to a certain probability measure P_α on $(H^2(D), B(H^2(D)))$ as $r \to \infty$. This also can be writen in the form

$$\hat{X}_{n_r,\alpha} \xrightarrow[r\to\infty]{\mathcal{D}} P_\alpha. \tag{5.2}$$

Using the random variable θ_N, define one more $H^2(D)$-valued random element

$$X_{N,\alpha} = X_{N,\alpha}(s) = \underline{\zeta}(s + i\theta_N, \alpha; \mathfrak{a}, \mathfrak{b}).$$

Then Lemma 8 implies, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{N\to\infty} \mu\{\underline{\rho}(X_{N,\alpha}, X_{N,n,\alpha}) \geq \varepsilon\} \\
&\quad= \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N} \#\{1 \leq k \leq N : \\
&\qquad \underline{\rho}(\underline{\zeta}(s + i\gamma_k h, \alpha; \mathfrak{a}, \mathfrak{b}), \underline{\zeta}_n(s + i\gamma_k h, \alpha; \mathfrak{a}, \mathfrak{b})) \geq \varepsilon\} \\
&\quad\leq \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N\varepsilon} \sum_{k=1}^{N} \underline{\rho}\Big(\underline{\zeta}(s + i\gamma_k h, \alpha; \mathfrak{a}, \mathfrak{b}), \underline{\zeta}_n(s + i\gamma_k h, \alpha; \mathfrak{a}, \mathfrak{b})\Big) = 0.
\end{aligned}$$

Now, this equality, the relations (5.1) and (5.2), and Theorem 4.2 of [1] show that

$$X_{N,\alpha} \xrightarrow[N\to\infty]{\mathcal{D}} P_\alpha,$$

or $P_{N,\alpha}$ converges weakly to P_α as $N \to \infty$. Moreover, the latter relation shows that the measure P_α is independent of the subsequence $\{\hat{P}_{n_r}, \alpha\}$. This remark gives the relation

$$\hat{X}_{n,\alpha} \xrightarrow[n\to\infty]{\mathcal{D}} P_\alpha,$$

or $\hat{P}_{n,\alpha}$ converges weakly to P_α as $n \to \infty$. Thus, we obtained that $P_{N,\alpha}$ converges weakly to the limit measure P_α of $\hat{P}_{n,\alpha}$. In [15], it was shown that P_α coincides with $P_{\underline{\zeta},\alpha}$. □

6 Proof of universality theorems

Let

$$S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\} \times H(D).$$

Then it was proved in [9] that the support of the measure $P_{\zeta,\alpha}$ is the set S.

Proof of Theorem 1. By the Mergelyan theorem on the approximation analytic functions by polynomials [31], there exist polynomials $p_1(s)$ and $p_2(s)$ such that

$$\sup_{s\in K_1} \left| f_1(s) - e^{p_1(s)} \right| < \frac{\varepsilon}{2}, \tag{6.1}$$

$$\sup_{s\in K_2} |f_2(s) - p_2(s)| < \frac{\varepsilon}{2}. \tag{6.2}$$

Define

$$G_\varepsilon = \left\{ (g_1, g_2) \in H^2(D) : \sup_{s\in K_1} |g_1(s) - e^{p_1(s)}| < \frac{\varepsilon}{2}, \sup_{s\in K_2} |g_2(s) - p_2(s)| < \frac{\varepsilon}{2} \right\}.$$

Then G_ε is an open neighbourhood of the element $\left(e^{p_1(s)}, p_2(s)\right) \in S$. Therefore, by properties of the support,

$$P_{\underline{\zeta},\alpha}(G_\varepsilon) > 0. \tag{6.3}$$

Moreover, by Theorem 3 and the equivalent of weak convergence of probability measure in terms of open sets [1], Theorem 2.1, we have that

$$\liminf_{N\to\infty} P_{N,\alpha}(G_n) \geq P_{\underline{\zeta},\alpha}(G_\varepsilon) > 0.$$

This, the definitions of $P_{N,\alpha}$ and G_ε together with (6.1) and (6.2) prove the theorem.

Proof of Theorem 2. Define

$$\hat{G}_\varepsilon = \left\{ (g_1, g_2) \in H^2(D) : \sup_{1\leq j\leq 2} \sup_{s\in K_j} |g_j(s) - f_j(s)| < \varepsilon \right\}.$$

Then the boundary $\partial\hat{G}_\varepsilon$ of $\hat{G}_\varepsilon$ lies in the set

$$\left\{ (g_1, g_2) \in H^2(D) : \sup_{1\leq j\leq 2} \sup_{s\in K_j} |g_j(s) - f_j(s)| = \varepsilon \right\}.$$

Therefore, the boundaries $\partial\hat{G}_{\varepsilon_1}$ and $\partial\hat{G}_{\varepsilon_2}$ with different positive ε_1 and ε_2 do not intersect. Hence, the set $\hat{G}_\varepsilon$ is a continuity set $\left(P_{\underline{\zeta},\alpha}(\partial\hat{G}_\varepsilon) = 0\right)$ of the measure $P_{\underline{\zeta},\alpha}$ for all but at most countably many $\varepsilon > 0$. Therefore, by Theorem 3 and the equivalent of weak convergence of probability measures in terms of continuity sets [1], Theorem 2.1, we have that

$$\lim_{n\to\infty} P_{N,\alpha}(\hat{G}_\varepsilon) = P_{\underline{\zeta},\alpha}(\hat{G}_\varepsilon) \tag{6.4}$$

for all but at most countably many $\varepsilon > 0$. The definitions of G_ε and $\hat{G}_\varepsilon$, and (6.1) and (6.2) show that $G_\varepsilon \subset \hat{G}_\varepsilon$. Thus, in view of (6.3),

$$P_{\underline{\zeta},\alpha}(\hat{G}_\varepsilon) > 0.$$

This, the definitions of $P_{N,\alpha}$ and $\hat{G}_\varepsilon$ together with (6.4) give the assertion of the theorem.

References

[1] P. Billingsley. *Convergence of Probability Measures.* John Wiley and Sons, New York, 1968.

[2] V. Franckevič, A. Laurinčikas and D. Šiaučiūnas. On approximation of analytic functions by periodic Hurwitz zeta-functions. *Math. Modell. Analysis*, **24**(1):20–33, 2019. https://doi.org/10.3846/mma.2019.002.

[3] V. Garbaliauskienė, J. Karaliūnaitė and R. Macaitienė. On discrete value distribution of certain compositions. *Math. Modell. Analysis*, **24**(1):34–42, 2019. https://doi.org/10.3846/mma.2019.003.

[4] R. Garunkštis and A. Laurinčikas. Discrete mean square of the Riemann zeta-function over imaginary parts of its zeros. *Periodica Math. Hung.*, **76**(2):217–228, 2018. https://doi.org/10.1007/s10998-017-0228-6.

[5] R. Garunkštis and A. Laurinčikas. The Riemann hypothesis and universality of the Riemann zeta-function. *Math. Slovaca*, **68**(4):741–748, 2018. https://doi.org/10.1515/ms-2017-0141.

[6] R. Garunkštis, A. Laurinčikas and R. Macaitienė. Zeros of the Riemann zeta-function and its universality. *Acta Arith.*, **181**(2):127–142, 2017. https://doi.org/10.4064/aa8583-5-2017.

[7] A. Javtokas and A. Laurinčikas. Universality of the periodic Hurwitz zeta-function. *Integral Transf. Spec. Functions*, **17**:711–722, 2006. https://doi.org/10.1080/10652460600856484.

[8] J. Kaczorowski. Some remarks on the universality of periodic *L*-functions. In R. Steuding and J. Steuding(Eds.), *New directions in value-distribution theory of zeta and L-functions*, pp. 113–120. Shaker Verlag, Aachen, 2009.

[9] R. Kačinskaitė and A. Laurinčikas. The joint distribution of periodic zeta-functions. *Studia Sci. Math. Hungarica*, **48**(2):257–279, 2011. https://doi.org/10.1556/SScMath.48.2011.2.1162.

[10] L. Kuipers and H. Niederreiter. *Uniform Distribution of Sequences.* John Wiley and Sons, New York, 1974.

[11] A. Laurinčikas. Voronin-type theorem for periodic Hurwitz zeta-functions. *Sb. Math.*, **198**:231–262, 2007. https://doi.org/10.1070/SM2007v198n02ABEH003835.

[12] A. Laurinčikas. Joint universality for periodic Hurwitz zeta-functions. *Izv. Math.*, **72**:741–760, 2008. https://doi.org/10.1070/IM2008v072n04ABEH002421.

[13] A. Laurinčikas. Joint universality of zeta-functions with periodic coefficients. *Izv. Math.*, **74**(3):515–539, 2010. https://doi.org/10.1070/IM2010v074n03ABEH002497.

[14] A. Laurinčikas. Universality of composite functions of periodic zeta-functions. *Sb. Math.*, **203**:1631–1643, 2012. https://doi.org/10.1070/SM2012v203n11ABEH004279.

[15] A. Laurinčikas. The joint discrete universality of periodic zeta-functions. In J. Sander et al.(Ed.), *From Arithmetic to Zeta-Functions*, Number Theory in Memory of Wolfgang Schwarz, pp. 231–246. Springer, 2016. https://doi.org/10.1007/978-3-319-28203-9_15.

[16] A. Laurinčikas. Universality theorems for zeta-functions with periodic coefficients. *Sb. Math. J.*, **57**:330–339, 2016. https://doi.org/10.1134/S0037446616020154.

[17] A. Laurinčikas. A discrete version of the Mishou theorem. II. *Proc. Steklov Inst. Math.*, **296**(1):172–182, 2017. https://doi.org/10.1134/S008154381701014X.

[18] A. Laurinčikas. On discrete universality of the Hurwitz zeta-functions. *Results Math.*, **72**(1-2):907–917, 2017. https://doi.org/10.1007/s00025-017-0702-8.

[19] A. Laurinčikas. Joint value distribution theorems for the Riemann and Hurwitz zeta-functions. *Moscow Math. J.*, **18**(2):349–366, 2018. https://doi.org/10.17323/1609-4514-2018-18-2-349-366.

[20] A. Laurinčikas. Joint discrete universality for periodic zeta-functions. *Quaest. Math.*, **42**(5):687–699, 2019. https://doi.org/10.2989/16073606.2018.1481891.

[21] A. Laurinčikas. Non-trivial zeros of the Riemann zeta-function and joint universality theorems. *J. Math. Anal. Appl.*, **475**(1):395–402, 2019. https://doi.org/10.1016/j.jmaa.2019.02.047.

[22] A. Laurinčikas and R. Macaitienė. The discrete universality of the periodic Hurwitz zeta-function. *Integral Transf. Spec. Functions*, **20**:673–686, 2009. https://doi.org/10.1080/10652460902742788.

[23] A. Laurinčikas and R. Macaitienė. Joint approximation of analytic functions by shifts of the Riemann and periodic Hurwitz zeta-functions. *Appl. Anal. Discrete Math.*, **12**(2):508–527, 2018. https://doi.org/10.2298/AADM170713016L.

[24] A. Laurinčikas, R. Macaitienė, D. Mochov and D. Šiaučiūnas. Universality of the periodic Hurwitz zeta-function with rational parameter. *Siber. Math. J.*, **59**(5):894–900, 2018. https://doi.org/10.1134/S0037446618050130.

[25] A. Laurinčikas and D. Mochov. A discrete universality theorem for the periodic Hurwitz zeta-functions. *Chebysh. Sb.*, **17**:148–159, 2016.

[26] A. Laurinčikas and J. Petuškinaitė. Universality of Dirichlet L-functions and non-trivial zeros of the Riemann zeta-function. *Sb. Math.*, **210**, 2019. https://doi.org/10.1070/SM9194.

[27] A. Laurinčikas and D. Šiaučiūnas. Remarks on the universality of periodic zeta-functions (in Russian). *Math. Notes.*, **80**(3-4):532–538, 2006. https://doi.org/10.1007/s11006-006-0171-y.

[28] R. Macaitienė, M.Stoncelis and Šiaučiūnas. A weighted discrete universality theorem for periodic zeta-functions. II. *Math. Modell. Analysis*, **22**(6):750–762, 2017. https://doi.org/10.3846/13926292.2017.1365779.

[29] R. Macaitienė, M.Stoncelis and D. Šiaučiūnas. A weighted universality theorem for periodic zeta-functions. *Math. Modell. Analysis*, **22**(1):95–105, 2017. https://doi.org/10.3846/13926292.2017.1269373.

[30] R. Macaitienė and D. Šiaučiūnas. Joint universality of Hurwitz zeta-functions and nontrivial zeros of the Riemann zeta-function. *Lith. Math. J.*, **59**(1):81–95, 2019. https://doi.org/10.1007/s10986-019-09423-2.

[31] S.N. Mergelyan. Uniform approximations to functions of complex variable. *Usp. Mat. Nauk.*, **7**(2):31–122, 1952 (in Russian).

[32] H. L. Montgomery. *Topics in Multiplicative Number Theory.* Lect. Notes Math., Vol. 227, Springer, Berlin, Heidelberg, New York, 1971. https://doi.org/10.1007/BFb0060851.

[33] H. L. Montgomery. The pair correlation of zeros of the zeta function. In H.G. Diamond(Ed.), *Analytic Number Theory*, volume 24 of *Proc. Sympos. Pure Math.*, pp. 181–193. Amer. Math. Soc., Providence, RI, 1973. https://doi.org/10.1090/pspum/024/9944.

[34] J. Steuding. The roots of the equation $\zeta(s) = a$ are uniformly distributed modulo one. In A. Laurinčikas et al.(Ed.), *Analytic and Probabilistic Methods in Number Theory*, pp. 243–249, Vilnius, 2012. TEV.

[35] E. C. Titchmarsh. *The Theory of the Riemann zeta-function.* 2nd ed. revised by D.R. Heath-Brown, Clarendon Press, Oxford, 1986.