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On the Modulus of the Selberg Zeta-Functions in the Critical Strip

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Abstract. We investigate the behavior of the real part of the logarithmic derivatives of the Selberg zeta-functions $Z_{\mathrm{PSL}(2,\mathbb{Z})}(s)$ and $Z_C(s)$ in the critical strip $0 < \sigma < 1$. The functions $Z_{\mathrm{PSL}(2,\mathbb{Z})}(s)$ and $Z_C(s)$ are defined on the modular group and on the compact Riemann surface, respectively.

Keywords: Selberg zeta-function, modular group, compact Riemann surface, Riemann zeta-function, critical strip.

AMS Subject Classification: 11M36.

1 Introduction

Let $s = \sigma + it$ denote a complex variable. We start with the definition and some properties of the Riemann zeta-function. For $\sigma > 1$, the Riemann zeta-function is given by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and can be analytically continued to the whole complex plane, except for a simple pole at $s = 1$ with residue 1. Trivial zeros of $\zeta(s)$ are located at the negative even integers. The remaining, the so-called non-trivial zeros, lie on the critical strip $0 < \sigma < 1$. The Riemann zeta-function satisfies the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \zeta(1-s) \Gamma(1-s) \sin \frac{\pi s}{2},$$

or $\xi(s) = \xi(1-s)$, where $\xi(s) = \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$, and $\Gamma(s)$ denotes the Euler gamma-function. The function $\xi(s)$ is an entire function whose zeros are the non-trivial zeros of $\zeta(s)$, see [19, §II].

In the paper [11], it was proved the following relation between functions $\zeta(s)$ and $\xi(s)$.

Theorem 1. *The functions $\zeta(s)$ and $\xi(s)$ satisfy, for $|t| \geq 8$ and $\sigma < 1/2$, the inequality*

$$\mathrm{Re}\frac{\zeta'(s)}{\zeta(s)} < \mathrm{Re}\frac{\xi'(s)}{\xi(s)}.$$

Sondow and Dumitrescu proved in [17] the following theorem for the function $\xi(s)$.

Theorem 2. *The function $\xi(s)$ is increasing in modulus along every horizontal half-line lying in any open right half-plane that contains no its zeros. Similarly, the modulus decreases on each horizontal half-line in any zero-free, open left half-plane.*

In the same paper, the following reformulation for the Riemann hypothesis that all non-trivial zeros of $\zeta(s)$ lie on the line $\sigma = 1/2$ was given.

Theorem 3. *The following statements are equivalent:*
I. *If t is any fixed real number, then $|\xi(\sigma+it)|$ is increasing for $1/2 < \sigma < \infty$.*
II. *If t is any fixed real number, then $|\xi(\sigma+it)|$ is decreasing for $-\infty < \sigma < 1/2$.*
III. *The Riemann hypothesis is true.*

Later, Theorem 3 was reproved in [11] in a slightly different way.

Related properties of the functions $\zeta(s)$ and $\xi(s)$ in the critical strip were also investigated in [15].

In this paper, we ask whether Selberg zeta-functions have similar properties as the Riemann-zeta function has in Theorems 1 - 3. Note that, for Selberg zeta-functions, the analogue of the Riemann hypothesis is usually valid. We consider Selberg zeta-functions for cocompact and modular subgroups.

Let $\mathbb{H}$ be the upper half-plane, and Γ be a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. Let $\Gamma\backslash\mathbb{H}$ be a hyperbolic Riemann surface of finite area. The Selberg zeta-function $Z(s)$ is defined [5], for $\sigma > 1$, by

$$Z(s) = \prod_{\{P\}} \prod_{k=0}^{\infty} (1 - N(P)^{-s-k}),$$

where $\{P\}$ runs trough all primitive hyperbolic conjugacy classes of Γ, and $N(P) = \alpha^2$ if the eigenvalues of P are α and α^{-1}, $|\alpha| > 1$. The Selberg zeta-function has a meromorphic continuation to the whole complex plane [5].

If $\Gamma\backslash\mathbb{H}$ is a compact Riemann surface of genus $g \geq 2$, we use the notation $Z(s) = Z_C(s)$. If $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$, then we denote $Z(s) = Z_{\mathrm{PSL}(2,\mathbb{Z})}(s)$. Similarly, as the Riemann zeta-function, the Selberg zeta-function $Z_{\mathrm{PSL}(2,\mathbb{Z})}(s)$ has a meromorfic continuation to the whole complex plane, and satisfies the symmetric functional equation [8]

$$\Xi(s) = \Xi(1-s),$$

where

$$\Xi(s) = Z_{\mathrm{PSL}(2,\mathbb{Z})}(s) Z_{id}(s) Z_{ell}(s) Z_{par}(s),$$

and

$$Z_{id}(s) = \left(\frac{(2\pi)^s}{\Gamma(s)}\right)^{1/6} (\Gamma_2(s))^{1/3}, \quad Z_{par}(s) = \frac{\pi^s}{\Gamma(s)\zeta(2s)\Gamma(s+1/2)2^s},$$
$$Z_{ell}(s) = \Gamma\left(\frac{s}{2}\right)^{-1/2} \Gamma\left(\frac{s+1}{2}\right)^{1/2} \Gamma\left(\frac{s}{3}\right)^{-2/3} \Gamma\left(\frac{s+2}{3}\right)^{2/3}. \tag{1.1}$$

The function $\Gamma_2(s)$ is called the double Barnes gamma-function, and is defined by the canonic product

$$\frac{1}{\Gamma_2(s+1)} = (2\pi)^{\frac{s}{2}} \exp\left\{-\frac{s}{2} - \frac{(\gamma_0+1)s^2}{2}\right\} \prod_{k=1}^{\infty} \left\{\left(1+\frac{s}{k}\right)^k \exp\left(-s+\frac{s^2}{2k}\right)\right\},$$

where γ_0 denotes the Euler constant. The function $\Gamma_2(s)$ satisfies the relations

$$\Gamma_2(1) = 1, \quad \Gamma_2(s+1) = \frac{\Gamma_2(s)}{\Gamma(s)}, \quad \Gamma_2(n+1) = \frac{1^2 \cdot 2^2 \cdots n^2}{(n!)^n},$$

see, for example [1], [16] or [20].

The function $\Xi(s)$ is an entire function of order 2, and has zeros at the points $s = 1/2 + ir_n$, $n \geq 0$, where $r_n = \sqrt{\lambda_n - \frac{1}{4}}$, and $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$ are the eigenvalues of the Laplace operator [3], [7]. The function $Z_{\mathrm{PSL}(2,\mathbb{Z})}(s)$ has poles and zeros at the following points [6]:

Poles of $Z_{\mathbf{PSL}(2,\mathbb{Z})}(s)$:
(1) $s = 0$; order 1,
(2) $s = 1/2 - k$, $k \geq 0$; order 1.

Zeros of $Z_{\mathbf{PSL}(2,\mathbb{Z})}(s)$:
(1) $s = 1$; order 1,
(2) $s = -6k - j$, $k \geq 0, j = 1, 2, 3, 4, 6$; order $2k + 1$,
(3) $s = -6k - 5$, $k \geq 0$; order $2k + 3$,
(4) $s = \rho/2$, where ρ are non-trivial zeros of $\zeta(s)$),
(5) $s = 1/2 \pm ir_n$, $n \geq 0$.

We prove the following theorem.

Theorem 4. *There exists a positive number C such that, for $t > C$ and $0 < \sigma < 1/2$,*

$$\mathrm{Re}\frac{\Xi'(s)}{\Xi(s)} < 0.$$

Furthermore, if we assume the Riemann hypothesis for $\zeta(s)$, then there exists a positive number C_1 such that

$$\mathrm{Re}\frac{Z'_{\mathrm{PSL}(2,\mathbb{Z})}(s)}{Z_{\mathrm{PSL}(2,\mathbb{Z})}(s)} < \mathrm{Re}\frac{\Xi'(s)}{\Xi(s)}$$

for $t > C_1$ *and* $0 < \sigma < 1/4$. *Conversely, if*

$$\mathrm{Re}\frac{Z'_{\mathrm{PSL}(2,\mathbb{Z})}(s)}{Z_{\mathrm{PSL}(2,\mathbb{Z})}(s)} < 0$$

for $t > C_1$ *and* $0 < \sigma < 1/4$, *then the function* $\zeta(s)$, *for* $t > C_1$, *has zeros only for* $\sigma = 1/2$.

Theorem 4 is proved in the next section. Below, we formulate a couple of corollaries of Theorem 4. We also want to mention that a part of assertions of Theorem 4 can be obtained following the proof of Theorem 6.1 in [12].

Corollary 1. For a fixed sufficiently large t, the function $|\Xi(\sigma+it)|$ is decreasing for $0 < \sigma < 1/2$, and is increasing for $1/2 < \sigma < 1$ with respect to σ.

Corollary 2. If the Riemann hypotesis is true for $\zeta(s)$, then, for a sufficiently large fixed t, the function $|Z_{\mathrm{PSL}(2,\mathbb{Z})}(\sigma+it)|$ is decreasing for $0 < \sigma < 1/4$ with respect to σ.

Proofs of Corollaries 1 and 2 follow from Lemma 1, functional equation $\Xi(s) = \Xi(1-s)$ and equality $\Xi(\overline{s}) = \overline{\Xi(s)}$.

We return to Selberg zeta-functions attached to compact Riemann surfaces. The function $Z_C(s)$ is an entire function of order 2 [4, §2.4, Theorem 2.4.25] and satisfies the functional equation [4, §2.4, Theorem 2.4.12]

$$Z_C(s) = f(s)Z_C(1-s),$$

where

$$f(s) = \exp\Big(4\pi(g-1)\int_0^{s-1/2} v\tan(\pi v)\,dv\Big),$$

and $g \geq 2$ is the genus of a Riemann surface. The above functional equation is equivalent to $M(s) = M(1-s)$, where

$$M(s) = Z_C(s)\exp\Big(2\pi(g-1)\int_0^{1/2-s} v\tan\pi v\,dv\Big).$$

The Selberg zeta-function $Z_C(s)$ has trivial zeros at $s = 1, 0, -1, -2, \ldots$, non-trivial zeros on the critical line $\sigma = 1/2$ and also, possibly, on the interval $(0,1)$ of the real axis, see [4, §2.4, Theorem 2.4.11] and [13]. In this sense, the analogue of the Riemann hypothesis holds for $Z_C(s)$. Moreover, the following statement is true.

Theorem 5. *There exists a positive number* B *such that the functions* $Z_C(s)$ *and* $M(s)$, *for* $t > B$, $0 < \sigma < 1/2$, *satisfy the inequality*

$$\mathrm{Re}\frac{Z'_C(s)}{Z_C(s)} < \mathrm{Re}\frac{M'(s)}{M(s)} < 0.$$

Note that a part of Theorem 5 is proved in [9], namely,

$$\mathrm{Re}\frac{Z_C'(s)}{Z_C(s)} < 0$$

for $-c \le \sigma \le 1/2$ and $t \ge t_0 > 0$, where $c > 0$ is an arbitrary constant, and t_0 is a constant depending on c.

A couple of corollaries follow from Theorem 5 for functions $Z_C(s)$ and $M(s)$.

Corollary 3. For a fixed and sufficiently large t, the function $|M(\sigma + it)|$ is decreasing for $0 < \sigma < 1/2$, and is increasing for $1/2 < \sigma < 1$.

Corollary 4. For a fixed and sufficiently large t, the function $|Z_C(\sigma + it)|$ is decreasing for $0 < \sigma < 1/2$.

Proofs of Corollaries 3 and 4 are the same as proofs of Corollaries 1 and 2, and Theorem 5 is proved in Section 3.

2 Proof of the Theorem 4

Before the proof of Theorem 4, we state one lemma.

Lemma 1. (*a*) *Let f be a holomorphic function in an open domain D and not identically zero. Let us also suppose* $\mathrm{Re}\frac{f'(s)}{f(s)} < 0$ *for all $s \in D$ such that $f(s) \neq 0$. Then $|f(s)|$ is strictly decreasing with respect to σ in D, i.e., for each $s_0 \in D$, there exists $\delta > 0$ such that $|f(s)|$ is strictly monotonically decreasing with respect to σ on the horizontal interval from $s_0 - \delta$ to $s_0 + \delta$.*
(*b*) *Conversely, if $|f(s)|$ is decreasing with respect to σ in D, then* $\mathrm{Re}\frac{f'(s)}{f(s)} \le 0$ *for all $s \in D$ such that $f(s) \neq 0$.*

The proof of the lemma is given in [11].

Remark 1. Of course, the analogous results hold for monotonically increasing $|f(s)|$ and $\mathrm{Re}\frac{f'(s)}{f(s)} > 0$.

Now we prove Theorem 4.
Proof of Theorem 4. First we prove that

$$\mathrm{Re}\frac{Z_{\mathrm{PSL}(2,\mathbb{Z})}'(s)}{Z_{\mathrm{PSL}(2,\mathbb{Z})}(s)} < \mathrm{Re}\frac{\Xi'(s)}{\Xi(s)}, \quad t > C_1 > 0,\ 0 < \sigma < 1/4.$$

From the equality $\Xi(s) = Z_{\mathrm{PSL}(2,\mathbb{Z})}(s) Z_{id}(s) Z_{ell}(s) Z_{par}(s)$, we find that

$$\begin{aligned}\frac{\Xi'(s)}{\Xi(s)} &= \frac{Z_{\mathrm{PSL}(2,\mathbb{Z})}'(s)}{Z_{\mathrm{PSL}(2,\mathbb{Z})}(s)} + \frac{Z_{id}'(s)}{Z_{id}(s)} + \frac{Z_{ell}'(s)}{Z_{ell}(s)} + \frac{Z_{par}'(s)}{Z_{par}(s)} \\ &=: \frac{Z_{\mathrm{PSL}(2,\mathbb{Z})}'(s)}{Z_{\mathrm{PSL}(2,\mathbb{Z})}(s)} + U(s).\end{aligned}$$

Hence, to complete the proof it is sufficient to show that

$$\mathrm{Re}U(s) > 0, \quad t > C_1 > 0,\, 0 < \sigma < 1/4.$$

By (1.1), we obtain

$$\begin{aligned} U(s) =& a_0 + \frac{1}{4}\left(\Psi\left(\frac{s}{2}+\frac{1}{2}\right) - \Psi\left(\frac{s}{2}\right)\right) + \frac{2}{9}\left(\Psi\left(\frac{s}{3}+\frac{2}{3}\right) - \Psi\left(\frac{s}{3}\right)\right) \\ & + \frac{1}{3}\Psi_2(s) - \frac{7}{6}\Psi(s) - \Psi\left(s+\frac{1}{2}\right) - 2\frac{\zeta'}{\zeta}(2s), \end{aligned} \tag{2.1}$$

where $a_0 = \frac{1}{6}\log 2\pi + \log\frac{\pi}{2} = 0.757\ldots$, $\Psi(s) = \Gamma'(s)/\Gamma(s)$ and $\Psi_2(s) = \Gamma_2'(s)/\Gamma_2(s)$.

To prove the inequality $\mathrm{Re}U(s) > 0$, we need to investigate the behavior of the functions $\Psi(s)$, $\Psi_2(s)$ and $\zeta'(2s)/\zeta(2s)$ in the region $0 < \sigma < 1/4$ and $t > C_1 > 0$. For the function $\Psi(s)$, the estimate [10]

$$\Psi(s) = \log s - \frac{1}{2s} + O\left(\frac{1}{|s|^2}\right), \quad |s| \to \infty,\, |\arg s| \le \pi - \delta < \pi,$$

holds. From this, we deduce that

$$\mathrm{Re}\Psi(s) = \log t + O\left(\frac{1}{t}\right), \quad t \to \infty,\, |\arg s| < \pi. \tag{2.2}$$

It is known [21] that, for $-s \notin \mathbb{N}$

$$\begin{aligned} \frac{\Gamma_2'(s+1)}{\Gamma_2(s+1)} &= \Psi_2(s+1) = \frac{1-\log 2\pi}{2} + (\gamma_0+1)s - \sum_{k=1}^{\infty}\left(\frac{k}{k+s} - 1 + \frac{s}{k}\right) \\ &= -\frac{1+\log 2\pi}{2} + s - s\Psi(s). \end{aligned}$$

This and (2.2) show that

$$\begin{aligned} \mathrm{Re}\Psi_2(s) &= -\frac{3+\log 2\pi}{2} + \sigma + (1-\sigma)\mathrm{Re}\Psi(s-1) + t\mathrm{Im}\Psi(s-1) \\ &= -\frac{3+\log 2\pi}{2} + \sigma + (1-\sigma)\log t + t\left(\pi - \arctan\left(\frac{t}{\sigma-1}\right)\right) + O\left(\frac{1}{t}\right) \\ &= -\frac{3+\log 2\pi}{2} + \sigma + (1-\sigma)\log t + t\arctan\left(\frac{t}{\sigma}\right) + O\left(\frac{1}{t}\right) \end{aligned} \tag{2.3}$$

for $0 < \sigma < 1/4$ and $t > C_1 > 0$.

From the formula [2]

$$\xi(s) = \xi(0)\prod_{\rho}\left(1 - \frac{s}{\rho}\right),$$

we obtain that

$$\frac{\xi'}{\xi}(s) = \sum_{\rho}\frac{1}{s-\rho},$$

where the summation runs over all non-trivial zeros of the Riemann zeta-function taken in conjugate pairs and in order of increasing imaginary parts. If $\rho = \beta + i\gamma$, then

$$\mathrm{Re}\frac{\xi'}{\xi}(s) = \sum_{\beta+i\gamma} \frac{\sigma-\beta}{(\sigma-\beta)^2+(t-\gamma)^2}.$$

If we assume the Riemann hypothesis, i.e., $\beta = 1/2$, then $\mathrm{Re}\xi(s)'/\xi(s) > 0$ for $\sigma > 1/2$, and $\mathrm{Re}\xi(s)'/\xi(s) < 0$ for $\sigma < 1/2$.

On the other hand, from the equation

$$\xi(s) = (s-1)\pi^{-s/2}\Gamma\left(s/2+1\right)\zeta(s)$$

we get

$$\frac{\xi'}{\xi}(s) = \frac{\zeta'}{\zeta}(s) + \frac{1}{2}\Psi\left(\frac{s}{2}+1\right) - \frac{1}{2}\log\pi + \frac{1}{s-1}.$$

This yields that, for $\sigma > 1/2$,

$$\mathrm{Re}\frac{\zeta'(s)}{\zeta(s)} > \frac{1}{2}\log t - \frac{1}{2}\log 2\pi + O\left(\frac{1}{t}\right), \tag{2.4}$$

and, for $\sigma < 1/2$,

$$-\mathrm{Re}\frac{\zeta'(s)}{\zeta(s)} > \frac{1}{2}\log t - \frac{1}{2}\log 2\pi + O\left(\frac{1}{t}\right). \tag{2.5}$$

In view of (2.1), (2.2), (2.3) and (2.5), we find that for $t \to \infty$,

$$\begin{aligned}
\mathrm{Re}U(s) &= a_0 - \frac{13}{6}\log t + \frac{1}{3}\mathrm{Re}\Psi_2(s) - 2\mathrm{Re}\frac{\zeta'}{\zeta}(2s) + O\left(\frac{1}{t}\right)\\
&= \log\frac{\pi}{2} + \frac{\sigma}{3} - \frac{1}{2} - \frac{2\sigma+11}{6}\log t + \frac{t}{3}\arctan\left(\frac{t}{\sigma}\right) - 2\mathrm{Re}\frac{\zeta'}{\zeta}(2s) + O\left(\frac{1}{t}\right)\\
&> \frac{t}{3}\arctan\left(\frac{t}{\sigma}\right) - \frac{5+2\sigma}{6}\log t + c(\sigma) + O\left(\frac{1}{t}\right),
\end{aligned} \tag{2.6}$$

where $a_0 = \frac{1}{6}\log 2\pi + \log\frac{\pi}{2}$ and $c(\sigma) = \log\frac{1}{2} + \frac{\sigma}{3} - \frac{1}{2}$. This shows that there exists a constant $C_1 > 0$ such that $\mathrm{Re}U(s)$ is positive for $t > C_1$ and $0 < \sigma < 1/4$. Hence, for $t > C$ and $0 < \sigma < 1/4$,

$$\mathrm{Re}\frac{Z'_{\mathrm{PSL}(2,\mathbb{Z})}(s)}{Z_{\mathrm{PSL}(2,\mathbb{Z})}(s)} < \mathrm{Re}\frac{\Xi'(s)}{\Xi(s)}.$$

We note that the restriction of $\sigma < 1/4$ is due to the zeros of the function $\zeta(2s)$.

Now we prove that

$$\mathrm{Re}\frac{\Xi'(s)}{\Xi(s)} < 0$$

for $t > C_1$ and $0 < \sigma < 1/2$. The function $\Xi(s)$ is an entire function of order two. It has a canonical product expansion [14], [18]

$$\Xi(s) = e^{as^2+bs+c}s^n \prod_{\hat\rho}\left(1 - \frac{s}{\hat\rho}\right)e^{s/\hat\rho+(1/2)(s/\hat\rho)^2}, \tag{2.7}$$

where $\hat\rho$ runs over the nonzero roots of $\Xi(s)$, and a, b, c, and n are constants. This implies

$$\frac{\Xi'(s)}{\Xi(s)} = 2as + b + \frac{n}{s} + \sum_{\hat\rho}\frac{s^2}{\hat\rho^2(s-\hat\rho)}$$
$$= 2as + b + \frac{n}{s} + \sum_{\hat\rho}\left(\frac{s}{\hat\rho^2} + \frac{1}{\hat\rho} + \frac{1}{s-\hat\rho}\right).$$

If $\hat\rho = 1/2 + ir_n$, $n \ge 0$, then the latter sum splits into two parts: for those $\hat\rho$ for which the numbers $1/2 + ir_n$ are real, and for those $\hat\rho$ for which the numbers $1/2+ir_n$ are complex. There are only a finite number of real numbers $1/2+ir_n$. Then

$$\mathrm{Re}\left(\frac{\Xi'(s)}{\Xi(s)}\right) = 2a\sigma + b + \frac{n\sigma}{\sigma^2+t^2} + \sum_{n>n_0}\frac{\sigma(1/4-r_n^2)+tr_n}{(1/4-r_n^2)^2+r_n^2}$$
$$+ \sum_{n>n_0}\frac{1/2}{1/4+r_n^2} + \sum_{n>n_0}\frac{\sigma-1/2}{(\sigma-1/2)^2+(t-r_n)^2}$$
$$+ \sum_{0\le n\le n_0}\left(\frac{\sigma}{(1/2+ir_n)^2} + \frac{1}{1/2+ir_n} + \frac{\sigma-1/2-ir_n}{\sigma-1/2-ir_n+t^2}\right). \tag{2.8}$$

We see that the sum

$$\sum_{n>n_0}\frac{\sigma(1/4-r_n^2)+tr_n}{(1/4-r_n^2)^2+r_n^2} = \frac{\sigma(1/4-r_{n_0+1}^2)+tr_{n_0+1}}{(1/4-r_{n_0+1}^2)^2+r_{n_0+1}^2} + \sum_{n>n_0+1}\frac{\sigma(1/4-r_n^2)+tr_n}{(1/4-r_n^2)^2+r_n^2}$$

is positive and unbounded as $t \to \infty$. Then, from equation (2.8), it follows that there exists a number $C > 0$ such that

$$\mathrm{Re}\frac{\Xi'(s)}{\Xi(s)} > 0$$

for $t > C$ and $1/2 < \sigma < 1$. By a note after Lemma 1, for fixed $t > C$, the function $|\Xi(\sigma+it)|$ is monotonically increasing as a function of σ, $0 < \sigma < 1/2$. In view of the functional equation $\Xi(s) = \Xi(1-s)$ and $\Xi(\overline{s}) = \overline{\Xi(s)}$, the function $|\Xi(\sigma+it)|$ is monotonically decreasing for $t > C$ as a function of σ, $1/2 < \sigma < 1$. So, the real part of its logarithmic derivative is negative, and the second assertion of the theorem holds.

The statement that if

$$\mathrm{Re}\frac{Z'_{\mathrm{PSL}(2,\mathbb{Z})}(s)}{Z_{\mathrm{PSL}(2,\mathbb{Z})}(s)} < 0$$

for $t > C_1$ and $0 < \sigma < 1/4$, then the Riemann hypothesis is true, follows straightforward from Lemma 1 and the fact that the function $Z_{\mathrm{PSL}(2,\mathbb{Z})}(s)$ has zeros $s = \rho/2$, where ρ are non-trivial zeros of $\zeta(s)$.

Recall that

$$U(s) = a_0 + \frac{1}{4}\left(\Psi\left(\frac{s}{2}+\frac{1}{2}\right) - \Psi\left(\frac{s}{2}\right)\right) + \frac{2}{9}\left(\Psi\left(\frac{s}{3}+\frac{2}{3}\right) - \Psi\left(\frac{s}{3}\right)\right)$$
$$+ \frac{1}{3}\Psi_2(s) - \frac{7}{6}\Psi(s) - \Psi\left(s+\frac{1}{2}\right) - 2\frac{\zeta'}{\zeta}(2s),$$

where $a_0 = \frac{1}{6}\log 2\pi + \log\frac{\pi}{2} = 0.757\dots$.

Corollary 5. If $0 < \sigma < 1/4$, then

$$-\mathrm{Re}\frac{Z'_{\mathrm{PSL}(2,\mathbb{Z})}(s)}{Z_{\mathrm{PSL}(2,\mathbb{Z})}(s)} > \mathrm{Re}(U(s))$$
$$> \frac{t}{3}\arctan\left(\frac{t}{\sigma}\right) - \frac{5+2\sigma}{6}\log t + c(\sigma) + O\left(\frac{1}{t}\right)$$

holds. If $1/2 < \sigma < 1$, then

$$-\mathrm{Re}\frac{Z'_{\mathrm{PSL}(2,\mathbb{Z})}(s)}{Z_{\mathrm{PSL}(2,\mathbb{Z})}(s)} < \mathrm{Re}(U(s))$$
$$< \frac{t}{3}\arctan\left(\frac{t}{\sigma}\right) - \frac{5+2\sigma}{6}\log t + c(\sigma) + O\left(\frac{1}{t}\right)$$

holds, where $c(\sigma) = \log\frac{1}{2} + \frac{\sigma}{3} - \frac{1}{2}$ and $t \to \infty$.

Proof. The first part of the corollary follows from the fact $\mathrm{Re}\,(\Xi'/\Xi(s)) < 0$, $0 < \sigma < 1/2$, and inequality (2.6). The second part is obtained analogically. □

3 Proof of Theorem 5

Proof of Theorem 5. Recall that the Selberg zeta-function attached to compact Riemann surfaces satisfies the functional equation $M(s) = M(1-s)$, where

$$M(s) = Z_C(s)\exp\left(2\pi(g-1)\int_0^{1/2-s} v\tan\pi v\,dv\right).$$

The function $M(s)$ is an entire function of order two, and it has the same form of canonical product expansion (2.7) as the function $\Xi(s)$. So, for $t > t_0 > 0$, the function $|M(s)|$ is monotonically decreasing with respect to $0 < \sigma < 1/2$.

Let

$$l(s) = \exp\left(\int_0^{1/2-s} v\tan\pi v\,dv\right).$$

To complete the proof, we need to show that

$$\mathrm{Re}\left(\frac{l'(s)}{l(s)}\right) > 0$$

for $0 < \sigma < 1/2$, and $t > \hat{t}_0$. By elementary calculation, we obtain

$$\begin{aligned}&\mathrm{Re}\left(\frac{l'(s)}{l(s)}\right) = \mathrm{Re}\left\{\left(s-\frac{1}{2}\right)\tan\pi\left(\frac{1}{2}-s\right)\right\}\\&= \frac{t(1-e^{-4\pi t})}{e^{-4\pi t}-2e^{-2\pi t}\cos 2\pi\sigma+1} + \left(\sigma-\frac{1}{2}\right)\frac{2e^{-2\pi t}\sin 2\pi\sigma}{e^{-4\pi t}-2e^{-2\pi t}\cos 2\pi\sigma+1}\\&= t(1+o(1)),\end{aligned}$$

as $t \to \infty$. Taking $B = \max(t_0, \hat{t}_0)$ completes the proof.

In the same way the following corollary follows.

Corollary 6. If $0 < \sigma < 1/2$, then

$$-\mathrm{Re}\frac{Z_C'(s)}{Z_C(s)} > 2\pi(g-1)\cdot t\cdot\left(1+O\left(e^{-2\pi t}\right)\right), \quad t\to\infty,$$

holds. If $1/2 < \sigma < 1$, then

$$-\mathrm{Re}\frac{Z_C'(s)}{Z_C(s)} < 2\pi(g-1)\cdot t\cdot\left(1+O\left(e^{-2\pi t}\right)\right), \quad t\to\infty,$$

holds.

Proof. Proof is the same as that for Corollary 5. □

4 Some remarks on the Riemann zeta-function

In this section, we present some remarks on the Riemann zeta-function $\zeta(s)$, which could have been obtained proving Theorems 4 and 5.

Let, as above, $\rho = \beta + i\gamma$ be non-trivial zeros of $\zeta(s)$. Recall that

$$\frac{\xi'}{\xi}(s) = \sum_{\rho}\frac{1}{s-\rho},$$

where the summation is over all non-trivial zeros of the Riemann zeta-function taken in conjugate pairs in order of increasing imaginary parts. Also,

$$\frac{\xi'}{\xi}(s) = \frac{\zeta'}{\zeta}(s) + \frac{1}{2}\Psi\left(\frac{s}{2}+1\right) - \frac{1}{2}\log\pi + \frac{1}{s-1}.$$

Comparing the latter equalities with

$$\frac{\zeta'}{\zeta}(s) = b - \frac{1}{s-1} - \frac{1}{2}\Psi\left(\frac{s}{2}+1\right) + \sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right),$$

where $b = \log 2\pi - 1 - \gamma_0/2$, we have [19] that

$$\sum_{\rho} \frac{1}{\rho} = 1 + \frac{\gamma_0}{2} - \frac{1}{2}\log 4\pi.$$

The inequalities (2.4) and (2.5) give the bounds for the real part of the logarithmic derivative of the Riemann zeta-function in the half-planes $\sigma < 1/2$ and $\sigma > 1/2$, respectively. Assuming the Riemann hypothesis, allows to construct more precise bounds. For this we need some lemmas.

Lemma 2. *Let $N(T)$ be the number of zeros of $\zeta(s)$ in the rectangle $0 < \sigma < 1$, $0 < t < T$. Then, as $T \to \infty$,*

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + R(T),$$

where $R(T) = O(\log T)$. If the Riemann hypothesis is true, then $R(T) = O\left(\frac{\log T}{\log\log T}\right)$.

The proof of the lemma can be found, for example, in [19].

Lemma 3. *For $t > 1$, the inequality*

$$\arctan t < \frac{\pi}{2} - \frac{1}{2t}$$

holds.

Proof. We have that

$$\frac{\pi}{2} = \int_0^\infty \frac{dx}{1+x^2} = \int_0^t \frac{dx}{1+x^2} + \int_t^\infty \frac{dx}{1+x^2} > \arctan t + \int_t^\infty \frac{dx}{x^2+x^2}$$
$$= \arctan t + \frac{1}{2t}.$$

□

Lemma 4. *Let $\rho_1 = 1/2 + i\gamma_1$, $\gamma_1 = 14.134725\ldots$, be the first non-trivial zero of $\zeta(s)$. Then*

$$\sum_{\gamma>0} \frac{1}{1/4 + (\gamma - t)^2} > \frac{1}{2}\log\frac{t}{\gamma_1} + O\left(\frac{1}{t}\right)$$

and

$$\sum_{\gamma>0} \frac{1}{1/4 + (\gamma + t)^2} > \frac{1}{8\pi t}\log\frac{t}{2\pi} + O\left(\frac{1}{t}\right)$$

as $t \to \infty$.

Proof. By Lemma 2, summing by parts, we get

$$\sum_{\gamma>0}\frac{1}{1/4+(\gamma-t)^2}$$
$$=\int_{\gamma_1}^{\infty}\frac{1}{1/4+(u-t)^2}\,d\left(\frac{u}{2\pi}\log\frac{u}{2\pi}-\frac{u}{2\pi}+R(u)\right)+O\left(\frac{1}{t^2}\right)$$
$$=\frac{1}{2\pi}\int_{\gamma_1}^{\infty}\frac{\log(u/2\pi)\,du}{1/4+(u-t)^2}+O\left(\int_{\gamma_1}^{\infty}\log u\,d\left(\frac{1}{1/4+(u-t)^2}\right)\right)+O\left(\frac{1}{t^2}\right)$$
$$=\frac{1}{2\pi}\int_{\gamma_1}^{t}\frac{\log(u/2\pi)\,du}{1/4+(u-t)^2}+\frac{1}{2\pi}\int_{t}^{\infty}\frac{\log(u/2\pi)\,du}{1/4+(u-t)^2}+O\left(\frac{1}{t}\right)$$
$$>\frac{1}{2\pi}\log\frac{\gamma_1}{2\pi}\int_{\gamma_1}^{t}\frac{du}{1/4+(u-t)^2}+\frac{1}{2\pi}\log\frac{t}{2\pi}\int_{t}^{\infty}\frac{du}{1/4+(u-t)^2}+O\left(\frac{1}{t}\right)$$
$$=\frac{\arctan(2(t-\gamma_1))}{\pi}\log\frac{\gamma_1}{2\pi}+\frac{1}{2}\log\frac{t}{2\pi}+O\left(\frac{1}{t}\right)$$
$$>\frac{1}{2}\log\frac{t}{\gamma_1}+O\left(\frac{1}{t}\right),\quad t\to\infty,$$

where the last inequality was obtained using that $-\pi/2\le\arctan v\le\pi/2$. This proves the first part of the lemma.

Similar arguments and Lemma 3 show that

$$\sum_{\gamma>0}\frac{1}{1/4+(\gamma+t)^2}=\frac{1}{2\pi}\int_{\gamma_1}^{t}\frac{\log(u/2\pi)du}{1/4+(u+t)^2}+\frac{1}{2\pi}\int_{t}^{\infty}\frac{\log(u/2\pi)du}{1/4+(u+t)^2}+O\left(\frac{1}{t}\right)$$
$$>\frac{1}{2\pi}\log\frac{\gamma_1}{2\pi}\left(2\arctan 4t-2\arctan(2(t-\gamma_1))\right)+\frac{1}{2\pi}\left(\pi-2\arctan 4t\right)\log\frac{t}{2\pi}$$
$$+O\left(\frac{1}{t}\right)>\frac{1}{8\pi t}\log\frac{t}{2\pi}+O\left(\frac{1}{t}\right),\quad t\to\infty.$$

□

It is well known that

$$\operatorname{Re}\frac{\zeta'}{\zeta}(s)=\sum_{\rho}\frac{\sigma-\beta}{(\sigma-\beta)^2+(t-\gamma)^2}-\frac{\sigma-1}{(\sigma-1)^2+t^2}$$
$$-\frac{1}{2}\operatorname{Re}\left(\Psi\left(\frac{s}{2}+1\right)\right)+\frac{1}{2}\log\pi.$$

Assume the Riemann hypothesis. Then, in view of Lemma 4, we obtain

$$\sum_{\gamma}\frac{1}{(\sigma-1/2)^2+(t-\gamma)^2}>\sum_{\gamma}\frac{1}{1/4+(t-\gamma)^2}$$
$$=\sum_{\gamma>0}\frac{1}{1/4+(t-\gamma)^2}+\sum_{\gamma>0}\frac{1}{1/4+(t+\gamma)^2}$$
$$>\frac{1}{2}\log\frac{t}{\gamma_1}+\frac{1}{8\pi t}\log\frac{t}{2\pi}+O\left(\frac{1}{t}\right),\quad t\to\infty.$$

Using this and (2.2), we find that

$$-\mathrm{Re}\frac{\zeta'}{\zeta}(s) > -\frac{1}{2}\left(\sigma-\frac{3}{2}\right)\log t - \frac{\sigma-1/2}{8\pi t}\log\frac{t}{2\pi} - \frac{1}{2}\log 2\pi + \frac{\sigma-1/2}{2}\log\gamma_1 + O\left(\frac{1}{t}\right),$$

for $0<\sigma<1/2$, and

$$-\mathrm{Re}\frac{\zeta'}{\zeta}(s) < -\frac{1}{2}\left(\sigma-\frac{3}{2}\right)\log t - \frac{\sigma-1/2}{8\pi t}\log\frac{t}{2\pi} - \frac{1}{2}\log 2\pi + \frac{\sigma-1/2}{2}\log\gamma_1 + O\left(\frac{1}{t}\right)$$

for $1/2<\sigma<1$ as $t\to\infty$.

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