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Relaxed η-proximal Operator for Solving a Variational-Like Inclusion Problem

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Abstract. In this paper, we introduce a new resolvent operator and we call it relaxed η-proximal operator. We demonstrate some of the properties of relaxed η-proximal operator. By applying this concept, we consider and study a variational -like inclusion problem with a nonconvex functional. Based on relaxed η-proximal operator, we define an iterative algorithm to approximate the solutions of a variational-like inclusion problem and the convergence of the iterative sequences generated by the algorithm is also discussed. Our results can be treated as refinement of many previously known results. An example is constructed in support of Theorem 1.

Keywords: algorithm, nonconvex, proximal, relaxed, solution.

AMS Subject Classification: 47A10; 47J22.

1 Introduction

Variational inequality theory was introduced by Hartmann and Stampacchia [19] in 1966 as a tool for the study of partial differential equations with applications principally drawn from mechanics. Variational inclusions involving two or more variables are of great importance and natural extensions of various types of variational inequalities existing in the literature. Variational inclusions have wide range of applications in industry, mathematical finance, economics and in several branches of applied sciences, see [5,7,8,9,10,14,15,16,17,18,20,25,26,27, 29] and references therein. It is carefully observed that the projection method and its variant forms can not be applied for solving variational inclusions. This fact gives rise to use the techniques based on proximal operators. For recent development of the subject, we refer to [2, 4, 11, 22, 23].

By the use of proximal operators, one may develop powerful and efficient iterative algorithms for solving several classes of variational inclusions (variational-like inclusions and other related problems). In fact, optimization algorithms are called proximal algorithms and are generally applicable but well-suited to problems of substantial recent interest involving large or higher dimensional datasets.

In this paper, a different interpretation of proximal operator i.e., a relaxed η-proximal operator is introduced and we prove some of its characteristics. Based on relaxed η-proximal operator, we define an iterative algorithm for solving a variational-like inclusion problem. In support of Theorem 1, we construct an example.

2 Preliminaries

Let H be a real Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$. Let $CB(H)$ be the family of all nonempty bounded closed subsets of H. Let $\widetilde{D}(\cdot,\cdot)$ be the Hausdörff metric on $CB(H)$ defined by

$$\widetilde{D}(A,B)=\max\left\{\sup_{x\in A} d(x,B),\sup_{y\in B} d(A,y)\right\},\quad \forall A,B\in CB(H),$$

where $d(x,B)=\inf\limits_{y\in B} d(x,y)$ and $d(A,y)=\inf\limits_{x\in A} d(x,y)$.

Let us recall the required definitions.

Definition 1. Let $T:H\longrightarrow CB(H)$ be a set-valued mapping, $g,R:H\longrightarrow H$ and $\eta:H\times H\longrightarrow H$ be single-valued mappings. Then

(*i*) T is said to be $\widetilde{D}$-Lipschitz continuous if, there exist a constant $\delta_T>0$ such that
$$\widetilde{D}(T(x),T(y))\le \delta_T\|x-y\|,\quad \forall x,y\in H.$$

(*ii*) R is said to be η-relaxed Lipschitz continuous if, there exists a constant $\alpha>0$ such that
$$\langle R(x)-R(y),\eta(x,y)\rangle\le -\alpha\|x-y\|^2,\quad \forall x,y\in H.$$

(*iii*) η is said to be Lipschitz continuous if, there exists a constant $\tau>0$ such that
$$\|\eta(x,y)\|\le \tau\|x-y\|,\quad \forall x,y\in H.$$

(*iv*) η is said to be strongly monotone if, there exists a constant $\delta>0$ such that
$$\langle \eta(x,y),x-y\rangle\ge \delta\|x-y\|^2,\quad \forall x,y\in H.$$

(*v*) g is said to be strongly monotone if, there exists a constant $\xi>0$ such that
$$\langle g(x)-g(y),x-y\rangle\ge \xi\|x-y\|^2,\quad \forall x,y\in H.$$

(vi) g is said to be Lipschitz continuous if, there exist a constant $\lambda_g > 0$ such that
$$\|g(x) - g(y)\| \leq \lambda_g \|x - y\|, \quad \forall x, y \in H.$$

(vii) R is said to be strongly monotone with respect to g if, there exists a constant $\delta_R > 0$ such that
$$\langle R(g(x)) - R(g(y)), g(x) - g(y)\rangle \geq \delta_R \|x - y\|^2, \quad \forall x, y \in H.$$

Definition 2. [33] A functional $f : H \times H \longrightarrow \mathbb{R}$ is said to be 0-diagonally quasi-concave (in short, 0-DQCV) in x if, for any finite set $\{x_1, \cdots, x_n\} \subset H$ and for any $y = \sum_{i=1}^{n} \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$,
$$\min_{1\leq i\leq n} f(x_i, y) \leq 0.$$

Definition 3. Let $\eta : H \times H \longrightarrow H$ be a mapping and $\phi : H \longrightarrow \mathbb{R} \cup \{\infty\}$ be a proper functional. A vector $f^* \in H$ is called an η-subgradient of ϕ at $x \in dom\phi$ if,
$$\langle f^*, \eta(y, x)\rangle \leq \phi(y) - \phi(x), \quad \forall y \in H.$$

Each ϕ can be associated with the following map $\partial_\eta \phi$, called η-subdifferential of ϕ at x, defined by
$$\partial_\eta \phi(x) = \begin{cases} f^* \in H : \langle f^*, \eta(y, x)\rangle \leq \phi(y) - \phi(x), \ \forall y \in H, & x \in dom\phi \\ \emptyset, & x \notin dom\phi. \end{cases}$$

Lemma 1. *[12] Let D be a nonempty convex subset of a topological vector space and $f : D \times D \longrightarrow (-\infty, \infty)$ be such that*

(i) *for each $x \in D$, $y \mapsto f(x, y)$ is lower semicontinuous on each compact subset of D;*

(ii) *for each finite set $\{x_1, \cdots, x_n\} \subset D$ and for each $y = \sum_{i=1}^{n} \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$, $\min_{1\leq i\leq n} f(x_i, y) \leq 0$;*

(iii) *there exists a nonempty compact convex subset D_0 of D and a nonempty compact subset K of D such that for each $y \in D \setminus K$, there is an $x \in Co\,(D_0 \cup \{y\})$ satisfying $f(x, y) > 0$.*

Then, there exists $\hat{y} \in D$ such that $f(x, \hat{y}) \leq 0$, for all $x \in D$.

Definition 4. Let $\phi : H \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, η-subdifferentiable (may not be convex) functional, $\eta : H \times H \longrightarrow H$, $R : H \longrightarrow H$ be the mappings and $I : H \longrightarrow H$ be an identity mapping. If for any given $z \in H$ and $\rho > 0$, there exists a unique point $x \in H$ satisfying
$$\langle (I - R)x - z, \eta(y, x)\rangle + \rho\phi(y) - \rho\phi(x) \geq 0, \quad \forall y \in H,$$

then the mapping $z \mapsto x$, denoted by $R^{\partial_\eta \phi}_{\rho,I}(z)$ is said to be relaxed η-proximal operator of ϕ. We have $z - (I - R)x \in \rho\partial_\eta \phi(x)$, it follows that

$$R^{\partial_\eta \phi}_{\rho,I}(z) = \left[(I - R) + \rho\partial_\eta \phi\right]^{-1}(z). \tag{2.1}$$

□

A comparison of relaxed η-proximal operator of ϕ (2.1) with some existing resolvent operators is as follows:

(*i*) If the mapping $R = 0$, then the relaxed η-proximal operator (2.1) reduces to the following resolvent operator introduced and studied by Ahmad et al. [1]

$$R^{\partial_n \phi}_{\rho,I}(z) = [I + \rho\partial_n \phi]^{-1}(z). \tag{2.2}$$

(*ii*) If $H = X$, a Banach space with its dual X^* and $(I - R) = J$, where $J : X \to X^*$ is a mapping, then relaxed η-proximal operator (2.1) reduces to the following resolvent operator introduced and studied by Siddiqi et al. [31]

$$R^{\partial_n \phi}_{\rho}(z) = [J + \rho\partial_n \phi]^{-1}(z). \tag{2.3}$$

(*iii*) If the mapping $R = 0$, $\partial_n \phi = \partial\phi$, where $\partial\phi$ is the subdifferential operator of ϕ, then the relaxed η-proximal operator (2.1) becomes

$$R^{\partial \phi}_{\rho,I}(z) = [I + \rho\partial \phi]^{-1}(z). \tag{2.4}$$

The details of resolvent operator (2.4) can be found in [6].

(*iv*) If the mapping $R = 0$, $\partial_n \phi = G$, where $G : H \to 2^H$ is a maximal monotone set-valued mapping. Then, the relaxed η-proximal operator (2.1) becomes the following classical resolvent operator

$$R^{G}_{\rho,I}(z) = [I + \rho G]^{-1}(z). \tag{2.5}$$

From the above discussion it follows that the relaxed η-proximal resolvent operator (2.1) includes many resolvent operators studied in recent past.

Now, we give some adequate conditions which guarantee the existence and Lipschitz continuity of the relaxed η-proximal operator $R^{\partial_\eta \phi}_{\rho,I}$.

Theorem 1. *Let H be a real Hilbert space and $\eta : H \times H \longrightarrow H$ be a strongly monotone with constant δ and Lipschitz continuous with constant τ such that $\eta(x, y) = -\eta(y, x)$, for all $x, y \in H$. Let $R : H \longrightarrow H$ be η-relaxed Lipschitz continuous mapping with constant α and $I : H \longrightarrow H$ be an identity mapping. Let $\phi : H \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous, η-subdifferential, proper functional which may not be convex and for any $z, x \in H$, the mapping $h(y, x) = \langle z - (I - R)x, \eta(y, x)\rangle$ is 0-DQCV in y. Then for any $\rho > 0$ and any $z \in H$, there exists a unique $x \in H$ such that*

$$\langle (I - R)x - z, \eta(y, x)\rangle + \rho\phi(y) - \rho\phi(x) \geq 0, \quad \forall y \in H, \tag{2.6}$$

i.e., $x = R^{\partial_\eta \phi}_{\rho,I}(z)$ and hence the relaxed η-proximal operator of ϕ is well-defined.

Proof. For any given $\rho > 0$ and $z \in H$, define a functional $f : H \times H \longrightarrow \mathbb{R} \cup \{+\infty\}$ by

$$f(y, x) = \langle z - (I - R)x, \eta(y, x)\rangle + \rho\phi(x) - \rho\phi(y), \quad \forall x, y \in H.$$

Using the continuity of the mappings I, R, η and lower semicontinuity of ϕ, we have for each $y \in H$, $x \mapsto f(y, x)$ is lower semicontinuous on H.

We claim that $f(y, x)$ satisfies the condition (ii) of Lemma 1. If it is false, then there exists a finite set $\{y_1, \cdots, y_n\} \subset H$ and $x_0 = \sum_{i=1}^{n} t_i y_i$ with $t_i \geq 0$ and $\sum_{i=1}^{n} t_i = 1$, we have

$$\langle z - (I - R)x_0, \eta(y_i, x_0)\rangle + \rho\phi(x_0) - \rho\phi(y_i) > 0, \quad \forall i = 1, 2, \cdots, n.$$

Since ϕ is η-subdifferentiable at x_0, there exists a point $f^*_{x_0} \in H$ such that

$$\phi(y_i) - \phi(x_0) \geq \langle f^*_{x_0}, \eta(y_i, x_0)\rangle, \quad \forall i = 1, 2, \cdots, n.$$

It follows that

$$\langle z-(I-R)x_0, \eta(y_i, x_0)\rangle > \rho\phi(y_i)-\rho\phi(x_0) \geq \rho\langle f^*_{x_0}, \eta(y_i, x_0)\rangle, \ \forall i = 1, 2, \cdots, n.$$

Thus, we have

$$\langle z - (I - R)x_0 - \rho f^*_{x_0}, \eta(y_i, x_0)\rangle > 0, \quad \forall i = 1, 2, \cdots, n. \tag{2.7}$$

On the other hand, by assumption $h(y, x) = \langle z - (I - R)x, \eta(y, x)\rangle$ is 0-DQCV in y, we have

$$\min_{1 \leq i \leq n} \langle z - (I - R)x_0 - \rho f^*_{x_0}, \eta(y_i, x_0)\rangle \leq 0,$$

which contradicts the inequality (2.7). Hence $f(y, x)$ satisfies condition (ii) of Lemma 1. We take a point $\bar{y} \in dom\phi$ and as ϕ is η-subdifferentiable at $\bar{y}$, there exists a point $f^*_{\bar{y}} \in H$ such that

$$\phi(x) - \phi(\bar{y}) \geq \langle f^*_{\bar{y}}, \eta(x, \bar{y})\rangle, \quad \forall x \in H.$$

Since η is strongly monotone with constant δ, Lipschitz continuous with constant τ and R is η-relaxed Lipschitz continuous with constant α, we have

$$\begin{aligned}
f(\bar{y}, x) &= \langle z - (I - R)x, \eta(\bar{y}, x)\rangle + \rho\phi(x) - \rho\phi(\bar{y}) \\
&\geq \langle \bar{y} - R(\bar{y}) - x + R(x), \eta(\bar{y}, x)\rangle + \langle z - \bar{y} + R(\bar{y}), \eta(\bar{y}, x)\rangle + \rho\langle f^*_{\bar{y}}, \eta(x, \bar{y})\rangle \\
&= \langle \bar{y} - x, \eta(\bar{y}, x)\rangle - \langle R(\bar{y}) - R(x), \eta(\bar{y}, x)\rangle + \langle z, \eta(\bar{y}, x)\rangle \\
&\quad + \langle R(\bar{y}) - \bar{y}, \eta(\bar{y}, x)\rangle + \langle f^*_{\bar{y}}, \eta(x, \bar{y})\rangle \\
&\geq \delta\|\bar{y}-x\|^2+\alpha\|\bar{y}-x\|^2-\tau\|z\|\|\bar{y}-x\|-\tau\left(\|R(\bar{y})\|+\|\bar{y}\|\right)\|\bar{y}-x\|-\rho\tau\|f^*_{\bar{y}}\|\|\bar{y}-x\| \\
&= \|\bar{y} - x\| \left[(\alpha + \delta)\|\bar{y} - x\| - \tau\left\{\|z\| + \|R(\bar{y})\| + \|\bar{y}\| + \rho\|f^*_{\bar{y}}\|\right\}\right].
\end{aligned}$$

Let $r = \frac{\tau}{(\alpha + \delta)}\left\{\|z\| + \|R(\bar{y})\| + \|\bar{y}\| + \rho\|f^*_{\bar{y}}\|\right\}$ and $K = \{x \in H : \|\bar{y} - x\| \leq r\}$. Then, $D_0 = \{\bar{y}\}$ and K are both weakly compact convex subsets of H, and for each $x \in H \setminus K$, there exists a $\bar{y} \in Co\,(D_0 \cup \{\bar{y}\})$ such that $f(\bar{y}, x) > 0$. Hence, all the conditions of Lemma 1 are satisfied. Then, there exists an $\bar{x} \in H$ such that $f(y, \bar{x}) \leq 0$, for all $y \in H$, i.e.,

$$\langle (I - R)\bar{x} - z, \eta(y, \bar{x})\rangle + \rho\phi(y) - \rho\phi(\bar{x}) \geq 0, \quad \forall y \in H.$$

Now, we show that $\bar{x}$ is a unique solution of problem (2.6). Suppose that $x_1, x_2 \in H$ are two arbitrary solutions of problem (2.6). Then, we have

$$\langle (I - R)x_1 - z, \eta(y, x_1)\rangle + \rho\phi(y) - \rho\phi(x_1) \geq 0, \quad \forall y \in H, \tag{2.8}$$
$$\langle (I - R)x_2 - z, \eta(y, x_2)\rangle + \rho\phi(y) - \rho\phi(x_2) \geq 0, \quad \forall y \in H. \tag{2.9}$$

Taking $y = x_2$ in (2.8) and $y = x_1$ in (2.9) and adding the resulting inequalities, we obtain

$$\langle (I - R)x_1 - z, \eta(x_2, x_1)\rangle + \langle (I - R)x_2 - z, \eta(x_1, x_2)\rangle \geq 0.$$

Since $\eta(x, y) = -\eta(y, x)$, for all $x, y \in H$, we can write

$$\begin{aligned} 0 &\leq \langle -(I - R)x_1 + z + (I - R)x_2 - z, \eta(x_1, x_2)\rangle \\ &\geq \langle x_1 - x_2, \eta(x_1, x_2)\rangle - \langle R(x_1) - R(x_2), \eta(x_1, x_2)\rangle. \end{aligned} \tag{2.10}$$

As η is strongly monotone with constant δ and R is η-relaxed Lipschitz continuous with constant α, from (2.10), it follows that

$$\delta\|x_1 - x_2\|^2 + \alpha\|x_1 - x_2\|^2 \leq \langle x_1 - x_2 - z, \eta(x_1, x_2)\rangle - \langle R(x_1) - R(x_2), \eta(x_1, x_2)\rangle \leq 0,$$

and hence we must have $x_1 = x_2$. This completes the proof.

The following example exhibits that all the conditions on η and R of Theorem 1 are satisfied.

Example 1. Let $H = \mathbb{R}$ and the mapping $\eta : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$\eta(x, y) = \begin{cases} (x - y) & if \;\; |xy| < \frac{c}{2}, \\ \frac{2}{c}|xy|(x - y) & if \;\; \frac{c}{2} \leq |xy| < c, \\ \frac{1}{c}(x - y) & if \;\; |xy| \geq c, \end{cases}$$

where $c > 0$ is any number. Then, it is easy to observe that

(*i*) $\langle \eta(x, y), x - y\rangle \geq |x - y|^2$, for all $x, y \in \mathbb{R}$, i.e., η is 1-strongly monotone;

(*ii*) $\eta(x, y) = -\eta(y, x)$, for all $x, y \in \mathbb{R}$;

(*iii*) $|\eta(x, y)| \leq \frac{1}{c}|x - y|$, for all $x, y \in \mathbb{R}$, i.e., η is $\frac{1}{c}$-Lipschitz continuous.

Let $R : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $R(x) = -kx$, for any $x \in \mathbb{R}$ and $k > 0$. Therefore,

$$\langle R(x) - R(y), \eta(x, y)\rangle = \begin{cases} -k(x - y)^2 & if \;\; |xy| < \frac{c}{2}, \\ -\frac{2k}{c}|xy|(x - y)^2 & if \;\; \frac{c}{2} \leq |xy| < c, \\ -\frac{k}{c}(x - y)^2 & if \;\; |xy| \geq c. \end{cases}$$

It can be easily seen that

$$\langle R(x)-R(y),\eta(x,y)\rangle \le -\frac{k}{c}|x-y|^2,\quad \forall x,y\in\mathbb{R},$$

i.e., R is η-relaxed Lipschitz continuous mapping with constant k/c.

Now, for any $x,z\in\mathbb{R}$, the mapping

$$\begin{aligned} h(y,x)&=\langle z-(I-R)x,\eta(y,x)\rangle\\ &=\langle z-(1+k)x,\eta(y,x)\rangle=(z-(1+k)x)\,\eta(y,x)\end{aligned}$$

is 0-DQCV in y. If it is false, then there exists a finite set $\{y_1,\cdots,y_m\}$ and $x_0=\sum\limits_{i=1}^{m}t_iy_i$ with $t_i\ge 0$ and $\sum\limits_{i=1}^{m}t_i=1$ such that for each $i=1,\cdots,m$,

$$0<h(y_i,x_0)=\begin{cases}(z-(1+k)x_0)\,(x_0-y_i) & if\ \ |xy|<\frac{c}{2},\\ \frac{2}{c}|x_0y_i|\,(z-(1+k)x_0)\,(x_0-y_i) & if\ \ \frac{c}{2}\le|xy|<c,\\ \frac{1}{c}\,(z-(1+k)x_0)\,(x_0-y_i) & if\ \ |xy|\ge c.\end{cases}$$

Here, we can see that $(z-(1+k)x_0)\,(x_0-y_i)>0$, for each $i=1,\cdots,m$, and therefore

$$0<\sum_{i=1}^{m}t_i\,(z-(1+k)x_0)\,(x_0-y_i)=(z-(1+k)x_0)\,(x_0-x_0)=0,$$

which is not possible. Therefore, for any $x,z\in\mathbb{R}$, the mapping $h(y,x)$ is 0-DQCV in y. Hence, η and R satisfy all the conditions of Theorem 1.

Theorem 2. *If all the conditions of Theorem* 1 *are satisfied, then the relaxed η-proximal operator $R^{\partial_\eta\phi}_{\rho,I}$ of ϕ is $\tau/(\alpha+\delta)$-Lipschitz continuous.*

Proof. By Theorem 1, the relaxed η-proximal operator $R^{\partial_\eta\phi}_{\rho,I}$ of ϕ is well-defined. For any given $z_1,z_2\in H$, let $x_1=R^{\partial_\eta\phi}_{\rho,I}(z_1)$ and $x_2=R^{\partial_\eta\phi}_{\rho,I}(z_2)$ be such that

$$\langle (I-R)x_1-z_1,\eta(y,x_1)\rangle\ge\rho\phi(x_1)-\rho\phi(y),\quad\forall y\in H,\tag{2.11}$$

$$\langle (I-R)x_2-z_2,\eta(y,x_2)\rangle\ge\rho\phi(x_2)-\rho\phi(y),\quad\forall y\in H.\tag{2.12}$$

Taking $y=x_2$ in (2.11) and $y=x_1$ in (2.12) and adding the resulting inequalities, we have

$$\langle (I-R)x_1-z_1,\eta(x_2,x_1)\rangle+\langle (I-R)x_2-z_2,\eta(x_1,x_2)\rangle\ge 0.\tag{2.13}$$

Since $\eta(x,y)=-\eta(y,x)$, η is strongly monotone with constant δ, Lipschitz continuous with constant τ and R is η-relaxed Lipschitz continuous with constant α, from (2.13), we get

$$\begin{aligned} &\langle x_1-x_2,\eta(x_2,x_1)\rangle+\langle R(x_2)-R(x_1),\eta(x_2,x_1)\rangle\ge\langle z_1-z_2,\eta(x_2,x_1)\rangle\\ \Rightarrow\quad &\langle x_2-x_1,\eta(x_2,x_1)\rangle-\langle R(x_2)-R(x_1),\eta(x_2,x_1)\rangle\le\langle z_2-z_1,\eta(x_2,x_1)\rangle,\end{aligned}$$

which implies that

$$(\delta+\alpha)\|x_2-x_1\|^2 \leq \langle z_2-z_1, \eta(x_2,x_1)\rangle \leq \tau\|z_2-z_1\|\|x_2-x_1)\|,$$

i.e.,

$$\|x_2-x_1\| \leq \frac{\tau}{(\alpha+\delta)}\|z_2-z_1\|.$$

Therefore, the relaxed η-proximal operator $R_{\rho,I}^{\partial_\eta \phi}$ of ϕ is $\frac{\tau}{(\alpha+\delta)}$-Lipschitz continuous. This completes the proof. □

3 Formulation of the Problem and Proximal Point Algorithm

Let $P, f, g : H \longrightarrow H$, $N, \eta : H \times H \longrightarrow H$ be the single-valued mappings, and $A, B, C, D : H \longrightarrow CB(H)$ be the set-valued mappings. Let $\phi : H \times H \longrightarrow \mathbb{R} \cup \{+\infty\}$ be such that for each fixed $x \in H$, $\phi(\cdot, x)$ is lower semicontinuous, η-subdifferential, proper functional on H (may not be convex) satisfying $g(H) \cap dom(\partial_\eta \phi(\cdot, x)) \neq \emptyset$, where $\partial_\eta \phi(\cdot, x)$ is the η-subdifferentiable of $\phi(\cdot, x)$. We consider the following variational-like inclusion problem:

Find $x \in H$, $u \in A(x)$, $v \in B(x)$, $w \in C(x)$ and $e \in D(x)$ such that $g(x) \in dom(\partial_\eta \phi(\cdot, x))$ and

$$\langle P(u) - (f(v) - N(w,e)), \eta(y, g(x))\rangle \geq \phi(g(x), x) - \phi(y, x), \quad \forall y \in H. \quad (3.1)$$

Special Cases:

(*i*) If $N \equiv 0$, $\eta(y, g(x)) = y - g(x)$ and $\phi(x, y) = \phi(x)$, then problem (3.1) reduces to the following problem which is to find $x \in H$, $u \in A(x)$ and $v \in B(x)$ such that

$$\langle P(u) - f(v), y - g(x)\rangle \geq \phi(g(x)) - \phi(y), \quad \forall y \in H. \quad (3.2)$$

Problem (3.2) is called set-valued nonlinear generalized variational inclusion problem which was introduced by Huang [21].

(*ii*) If $N \equiv 0$, P, f, g are identity mappings, A, B are single-valued mappings and $\phi(x, y) = \phi(x)$, then problem (3.1) coincides with the following problem of finding $x \in H$ such that

$$\langle A(x) - B(x), \eta(y, x)\rangle \geq \phi(x) - \phi(y), \quad \forall y \in H. \quad (3.3)$$

Problem (3.3) was considered and studied by Lee et al. [24].

For suitable choices of operators involved in the formulation of problem (3.1), one can obtain problems considered and studied by Ding and *Lou* [13], Salahuddin and Ahmad [30] and Verma [32], etc..

Definition 5. Let $f : H \longrightarrow H$ be a single-valued mapping, and $A : H \longrightarrow 2^H$ be a set-valued mapping. For all $x, y \in H$, the mapping $N(\cdot, \cdot) : H \times H \longrightarrow H$ is called

(i) relaxed Lipschitz continuous in the first argument with respect to A if, there exists a constant $r_1 > 0$ such that

$$\langle N(u_1,\cdot) - N(u_2,\cdot), x - y\rangle \leq -r_1\|x - y\|^2, \quad \forall u_1 \in A(x), u_2 \in A(y);$$

(ii) Lipschitz continuous in the first argument with respect to A if, there exists a constant $\lambda_{N_1} > 0$ such that

$$\|N(u_1,\cdot) - N(u_2,\cdot)\| \leq \lambda_{N_1}\|u_1 - u_2\|, \quad \forall u_1 \in A(x), u_2 \in A(y).$$

Similarly, we can define relaxed Lipschitz continuity of N in the second argument with respect to A and Lipschitz continuity of N in the second argument with respect to A.

Definition 6. Let $A : H \longrightarrow 2^H$ be a set-valued mapping, and $f : H \longrightarrow H$ be a single-valued mapping. Then, A is said to be

(i) relaxed Lipschitz continuous with respect to f if, there exists a constant $k > 0$ such that

$$\langle f(u_1) - f(u_2), x - y\rangle \leq -k\|x - y\|^2, \quad \forall u_1 \in A(x), u_2 \in A(y);$$

(ii) relaxed monotone with respect to f if, there exists a constant $c > 0$ such that

$$\langle f(u_1) - f(u_2), x - y\rangle \geq -c\|x - y\|^2, \quad \forall u_1 \in A(x), u_2 \in A(y).$$

We first transfer the variational-like inclusion problem (3.1) into a fixed point problem.

Theorem 3. (x, u, v, w, e) *is a solution of variational-like inclusion problem* (3.1) *if and only if (x,u,v,w,e) satisfies the following relation:*

$$g(x) = R_{\rho,I}^{\partial_\eta\phi(\cdot,x)}\left[(I - R)g(x) - \rho\{P(u) - (f(v) - N(w, e))\}\right],$$

where $x \in H$, $u \in A(x)$, $v \in B(x)$, $w \in C(x)$, $e \in D(x)$, $\rho > 0$ *and* $R_{\rho,I}^{\partial_\eta\phi(\cdot,x)} = [(I - R) + \rho\partial_\eta\phi(\cdot, x)]^{-1}$ *is the relaxed* η*-proximal operator of* $\phi(\cdot, x)$.

Proof. One can prove it easily by using the Definition 4 and hence we omit it. $\square$

Based on Theorem 3, we suggest the following iterative algorithm for approximating the solutions of variational-like inclusion problem (3.1).

Algorithm 1. Let $P, R, f, g : H \longrightarrow H$, $N, \eta : H \times H \longrightarrow H$ be the single-valued mappings such that $g(H) = H$, $A, B, C, D : H \longrightarrow CB(H)$ be the set-valued mappings, and $I : H \longrightarrow H$ be an identity mapping. Let $\phi : H \times H \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous, η-subdifferential, proper functional on H (may not be convex) satisfying $g(H) \cap dom(\partial_\eta\phi(\cdot, x)) \neq \emptyset$. For any $x_0 \in H$,

$u_0 \in A(x_0)$, $v_0 \in B(x_0)$, $w_0 \in C(x_0)$ and $e_0 \in D(x_0)$, as $g(H) = H$, there exists a point $x_1 \in H$ such that

$$g(x_1) = R_{\rho,I}^{\partial_\eta \phi(\cdot,x_0)} \left[(I-R)g(x_0) - \rho\{P(u_0) - (f(v_0) - N(w_0,e_0))\}\right].$$

By Nadler's Theorem [28], there exists $u_1 \in A(x_1)$, $v_1 \in B(x_1)$, $w_1 \in C(x_1)$ and $e_1 \in D(x_1)$ such that

$$\begin{aligned} &\|u_1 - u_0\| \le \widetilde{D}(A(x_1), A(x_0)), \quad \|v_1 - v_0\| \le \widetilde{D}(B(x_1), B(x_0)), \\ &\|w_1 - w_0\| \le \widetilde{D}(C(x_1), C(x_0)), \quad \|e_1 - e_0\| \le \widetilde{D}(D(x_1), D(x_0)). \end{aligned}$$

Let

$$g(x_2) = R_{\rho,I}^{\partial_\eta \phi(\cdot,x_1)} \left[(I-R)g(x_1) - \rho\{P(u_1) - (f(v_1) - N(w_1,e_1))\}\right].$$

Continuing the above scheme inductively, we can define the following iterative sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ and $\{e_n\}$ for solving variational-like inclusion problem (3.1) as follows:

$$\begin{aligned} &g(x_{n+1}) = R_{\rho,I}^{\partial_\eta \phi(\cdot,x_n)} \left[(I-R)g(x_n) - \rho\{P(u_n) - (f(v_n) - N(w_n,e_n))\}\right], \\ &u_n \in A(x_n), \quad \|u_{n+1} - u_n\| \le \widetilde{D}(A(x_{n+1}), A(x_n)), \\ &v_n \in B(x_n), \quad \|v_{n+1} - v_n\| \le \widetilde{D}(B(x_{n+1}), B(x_n)), \\ &w_n \in C(x_n), \quad \|w_{n+1} - w_n\| \le \widetilde{D}(C(x_{n+1}), C(x_n)), \\ &e_n \in D(x_n), \quad \|e_{n+1} - e_n\| \le \widetilde{D}(D(x_{n+1}), D(x_n)), \end{aligned}$$

where $\rho > 0$ is a constant and $n = 0, 1, 2, \cdots$.

Now, we prove the following existence and convergence result for variational-like inclusion problem (3.1).

Theorem 4. *Let $P, R, f, g : H \longrightarrow H$ and $N : H \times H \longrightarrow H$ be the single-valued mappings such that P is Lipschitz continuous with constant λ_P; g is Lipschitz continuous with constant λ_g and strongly monotone with constant ξ such that $g(H) = H$; R is Lipschitz continuous with constant λ_R, relaxed Lipschitz continuous with constant α and strongly monotone with respect to g with constant δ_R; f is Lipschitz continuous with constant λ_f; N is Lipschitz continuous in the first argument with respect to C with constant λ_{N_1}, Lipschitz continuous in the second argument with respect to D with constant λ_{N_2}, relaxed Lipschitz continuous with respect to C in the first argument with constant r_1, relaxed Lipschitz continuous with respect to D in the second argument with constant r_2. Let $A, B, C, D : H \longrightarrow CB(H)$ be $\widetilde{D}$-Lipschitz continuous mappings with constants δ_A, δ_B, δ_C and δ_D, respectively, B is relaxed Lipschitz continuous with respect to f with constant k, and A is relaxed monotone with respect to P with constant c. Let $I : H \longrightarrow H$ be an identity mapping and $\eta : H \times H \longrightarrow H$ be a strongly monotone with constant δ and Lipschitz continuous with constant τ such that $\eta(x,y) = -\eta(y,x)$, for all $x, y \in H$, and for any given $z \in H$, the mapping $h(y,x) = \langle z - (I-R)x, \eta(y,x)\rangle$ is 0-DQCV in y. Let*

$\phi : H \times H \longrightarrow \mathbb{R} \cup \{+\infty\}$ be such that for each $x \in H$, $\phi(\cdot, x)$ is lower semicontinuous, η-subdifferentiable, proper functional satisfying $g(x) \in dom(\partial_\eta \phi(\cdot, x))$, where $\partial_\eta \phi(\cdot, x)$ is the η-subdifferentiable of $\phi(\cdot, x)$. Suppose that there exist constants $\rho > 0$, $\mu > 0$ such that for each $z \in H$

$$\left\| R_{\rho,I}^{\partial_\eta \phi(\cdot, x_n)}(z) - R_{\rho,I}^{\partial_\eta \phi(\cdot, x_{n-1})}(z) \right\| \leq \mu \|x_n - x_{n-1}\|, \tag{3.4}$$

and the following condition is satisfied:

$$|\alpha + \delta| \leq \frac{\tau(t_1 + t_2 + t_3)\sqrt{2\xi^2 - 4\mu^2}}{2\mu^2 - \xi^2}, \tag{3.5}$$

where

$$t_1 = \sqrt{\lambda_g^2 - 2\delta_R^2 + \lambda_R^2 \lambda_g^2}, \quad t_2 = \sqrt{1 - 2\rho(k - c) + \rho^2 \left\{\lambda_f \delta_B + \lambda_P \delta_A\right\}^2},$$
$$t_3 = \sqrt{1 - 2\rho(r_1 + r_2) + \rho^2 \left\{\lambda_{N_1}^2 \delta_C^2 + \lambda_{N_2}^2 \delta_D^2\right\}}.$$

Then, the iterative sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ and $\{e_n\}$ generated by Algorithm 1 converge strongly to x, u, v, w and e, respectively and (x, u, v, w, e) is the solution of variational-like inclusion problem (3.1).

Proof. By Cauchy-Schwartz inequality and strongly monotonicity of g with constant ξ, we have

$$\begin{aligned} \|g(x_{n+1}) - g(x_n)\| \|x_{n+1} - x_n\| &\geq \langle g(x_{n+1}) - g(x_n), x_{n+1} - x_n \rangle \\ &\geq \xi \|x_{n+1} - x_n\|^2, \end{aligned}$$

which implies that

$$\|x_{n+1} - x_n\| \leq \frac{1}{\xi} \|g(x_{n+1}) - g(x_n)\|. \tag{3.6}$$

By Algorithm 1, we have

$$g(x_{n+1}) = R_{\rho,I}^{\partial_\eta \phi(\cdot, x_n)} \left[(I - R)g(x_n) - \rho\{P(u_n) - (f(v_n) - N(w_n, e_n))\}\right].$$

Hence, we have

$$\begin{aligned} \|g(x_{n+1}) - g(x_n)\| = \Big\| & R_{\rho,I}^{\partial_\eta \phi(\cdot, x_n)} \left[(I - R)g(x_n) - \rho\{P(u_n) - (f(v_n) - N(w_n, e_n))\}\right] \\ & - R_{\rho,I}^{\partial_\eta \phi(\cdot, x_{n-1})} \left[(I - R)g(x_{n-1}) - \rho\{P(u_{n-1}) - (f(v_{n-1}) - N(w_{n-1}, e_{n-1}))\}\right] \Big\|. \end{aligned}$$

Since $\|x + y\|^2 \leq 2\left(\|x\|^2 + \|y\|^2\right)$, using the Lipschitz continuity of relaxed

η-resolvent operator and condition (3.4), we have

$$\begin{aligned}
\frac{1}{2}\|g(x_{n+1})-g(x_n)\|^2 &\leq \Big\| R^{\partial_\eta\phi(\cdot,x_n)}_{\rho,I}[(I-R)g(x_n)-\rho\{P(u_n)-(f(v_n)-N(w_n,e_n))\}] \\
&\quad - R^{\partial_\eta\phi(\cdot,x_n)}_{\rho,I}[(I-R)g(x_{n-1})-\rho\{P(u_{n-1})-(f(v_{n-1})-N(w_{n-1},e_{n-1}))\}]\Big\|^2 \\
&\quad + \Big\| R^{\partial_\eta\phi(\cdot,x_n)}_{\rho,I}[(I-R)g(x_{n-1})-\rho\{P(u_{n-1})-(f(v_{n-1})-N(w_{n-1},e_{n-1}))\}] \\
&\quad - R^{\partial_\eta\phi(\cdot,x_{n-1})}_{\rho,I}[(I-R)g(x_{n-1})-\rho\{P(u_{n-1})-(f(v_{n-1})-N(w_{n-1},e_{n-1}))\}]\Big\|^2 \\
&\leq \frac{\tau^2}{(\alpha+\delta)^2}\Big\|[(I-R)g(x_n)-(I-R)g(x_{n-1})]-\rho\{P(u_n)-(f(v_n)-N(w_n,e_n))\} \\
&\quad + \rho\{P(u_{n-1})-(f(v_{n-1})-N(w_{n-1},e_{n-1}))\}\Big\|^2 + \mu^2\|x_n-x_{n-1}\|^2. \qquad (3.7)
\end{aligned}$$

We now evaluate

$$\begin{aligned}
&\|[(I-R)g(x_n)-(I-R)g(x_{n-1})]-\rho\{P(u_n)-(f(v_n)-N(w_n,e_n))\} \\
&\quad + \rho\{P(u_{n-1})-(f(v_{n-1})-N(w_{n-1},e_{n-1}))\}\| \\
&\leq \|g(x_n)-g(x_{n-1})-[R(g(x_n))-R(g(x_{n-1}))]\| + \|x_n-x_{n-1} \\
&\quad + \rho[f(v_n)-f(v_{n-1})]-\rho[P(u_n)-P(u_{n-1})]\| + \|x_n-x_{n-1} \\
&\quad + \rho[N(w_n,e_n)-N(w_{n-1},e_{n-1})]\|. \qquad (3.8)
\end{aligned}$$

Since g is Lipschitz continuous with constant λ_g, R is Lipschitz continuous with constant λ_R and strongly monotone with respect to g with constant δ_R, we have

$$\begin{aligned}
\|g(x_n)-g(x_{n-1})-[R(g(x_n))-R(g(x_{n-1}))]\|^2 &= \|g(x_n)-g(x_{n-1})\|^2 \\
&\quad - 2\langle g(x_n)-g(x_{n-1}), R(g(x_n))-R(g(x_{n-1}))\rangle + \|R(g(x_n))-R(g(x_{n-1}))\|^2 \\
&\leq \lambda_g^2\|x_n-x_{n-1}\|^2 - 2\delta_R^2\|x_n-x_{n-1}\|^2 + \lambda_R^2\lambda_g^2\|x_n-x_{n-1}\|^2 \\
&= \left(\lambda_g^2 - 2\delta_R^2 + \lambda_R^2\lambda_g^2\right)\|x_n-x_{n-1}\|^2. \qquad (3.9)
\end{aligned}$$

Since B is relaxed Lipschitz continuous with respect to f with constant k, A is relaxed monotone with respect to P with constant c, P is Lipschitz continuous with constant λ_P, f is Lipschitz continuous with constant λ_f, and A, B are $\widetilde{D}$-Lipschitz continuous with constants δ_A, δ_B, respectively, we have

$$\begin{aligned}
&\|x_n-x_{n-1}+\rho[f(v_n)-f(v_{n-1})]-\rho[P(u_n)-P(u_{n-1})]\|^2 \\
&\quad = \|x_n-x_{n-1}\|^2 + 2\rho\langle f(v_n)-f(v_{n-1}), x_n-x_{n-1}\rangle - 2\rho\langle P(u_n) \\
&\qquad - P(u_{n-1}), x_n-x_{n-1}\rangle + \rho^2\|f(v_n)-f(v_{n-1})-[P(u_n)-P(u_{n-1})]\|^2 \\
&\quad \leq \left[1-2\rho(k-c)+\rho^2\left\{\lambda_f\delta_B+\lambda_P\delta_A\right\}^2\right]\|x_n-x_{n-1}\|^2.
\end{aligned}$$

Since N is Lipschitz continuous in the first argument with respect to C with constant λ_{N_1}, Lipschitz continuous in the second argument with respect to D with constant λ_{N_2}, relaxed Lipschitz continuous with respect to C in the first argument with constant r_1, relaxed Lipschitz continuous with respect to D in the second argument with constant r_2, and C, D are $\widetilde{D}$-Lipschitz continuous with constants δ_C, δ_D, respectively, we have

$$\begin{aligned}
&\|x_n - x_{n-1} + \rho[N(w_n, e_n) - N(w_{n-1}, e_{n-1})]\|^2 \\
&= \|x_n - x_{n-1} + \rho[N(w_n, e_n) - N(w_{n-1}, e_n) + N(w_{n-1}, e_n) - N(w_{n-1}, e_{n-1})]\|^2 \\
&= \|x_n - x_{n-1} + \rho[N(w_n, e_n) - N(w_{n-1}, e_n)] + \rho[N(w_{n-1}, e_n) - N(w_{n-1}, e_{n-1})]\|^2 \\
&\le \|x_n - x_{n-1}\|^2 + \rho^2 \|N(w_n, e_n) - N(w_{n-1}, e_n)\|^2 + \rho^2 \|N(w_{n-1}, e_n) \\
&\quad - N(w_{n-1}, e_{n-1})\|^2 + 2\rho\langle N(w_n, e_n) - N(w_{n-1}, e_n), x_n - x_{n-1}\rangle \\
&\quad + 2\rho\langle N(w_{n-1}, e_n) - N(w_{n-1}, e_{n-1}), x_n - x_{n-1}\rangle \\
&\le \|x_n - x_{n-1}\|^2 + \rho^2 \lambda_{N_1}^2 \delta_C^2 \|x_n - x_{n-1}\|^2 + \rho^2 \lambda_{N_2}^2 \delta_D^2 \|x_n - x_{n-1}\|^2 \\
&\quad - 2\rho r_1 \|x_n - x_{n-1}\|^2 - 2\rho r_2 \|x_n - x_{n-1}\|^2 \\
&= \left[1 - 2\rho(r_1 + r_2) + \rho^2 \left\{\lambda_{N_1}^2 \delta_C^2 + \lambda_{N_2}^2 \delta_D^2\right\}\right] \|x_n - x_{n-1}\|^2. \qquad (3.10)
\end{aligned}$$

Combining (3.9) to (3.10) with (3.8), we have

$$\begin{aligned}
&\|[(I - R)g(x_n) - (I - R)g(x_{n-1})] - \rho\{P(u_n) - (f(v_n) - N(w_n, e_n))\} \\
&\quad + \rho\{P(u_{n-1}) - (f(v_{n-1}) - N(w_{n-1}, e_{n-1}))\}\| \\
&\quad \le \Big[\sqrt{\lambda_g^2 - 2\delta_R^2 + \lambda_R^2 \lambda_g^2} + \sqrt{1 - 2\rho(k - c) + \rho^2 \left\{\lambda_f \delta_B + \lambda_P \delta_A\right\}^2} \\
&\quad + \sqrt{1 - 2\rho(r_1 + r_2) + \rho^2 \left\{\lambda_{N_1}^2 \delta_C^2 + \lambda_{N_2}^2 \delta_D^2\right\}}\Big] \|x_n - x_{n-1}\|. \qquad (3.11)
\end{aligned}$$

Using (3.11), (3.7) becomes

$$\|g(x_{n+1}) - g(x_n)\|^2 \le \left[\frac{2\tau^2}{(\alpha + \delta)^2} \Theta(*)^2 + 2\mu^2\right] \|x_n - x_{n-1}\|^2, \qquad (3.12)$$

where

$$\begin{aligned}
\Theta(*) = &\sqrt{\lambda_g^2 - 2\delta_R^2 + \lambda_R^2 \lambda_g^2} + \sqrt{1 - 2\rho(k - c) + \rho^2 \left\{\lambda_f \delta_B + \lambda_P \delta_A\right\}^2} \\
&+ \sqrt{1 - 2\rho(r_1 + r_2) + \rho^2 \left\{\lambda_{N_1}^2 \delta_C^2 + \lambda_{N_2}^2 \delta_D^2\right\}}.
\end{aligned}$$

Using (3.12), (3.6) becomes

$$\begin{aligned}
\|x_{n+1} - x_n\| &\le \frac{1}{\xi}\Big(\frac{2\tau^2}{(\alpha + \delta)^2} \Theta(*)^2 + 2\mu^2\Big)^{1/2} \|x_n - x_{n-1}\| \\
&= \Theta(✠) \|x_n - x_{n-1}\|, \qquad (3.13)
\end{aligned}$$

where

$$\begin{aligned}
&\Theta(✠) = \frac{1}{\xi}\Big(\frac{2\tau^2}{(\alpha + \delta)^2} \Theta(*)^2 + 2\mu^2\Big)^{1/2}, \quad \Theta(*) = t_1 + t_2 + t_3, \\
&t_1 = \sqrt{\lambda_g^2 - 2\delta_R^2 + \lambda_R^2 \lambda_g^2}, \quad t_2 = \sqrt{1 - 2\rho(k - c) + \rho^2 \left\{\lambda_f \delta_B + \lambda_P \delta_A\right\}^2}, \\
&t_3 = \sqrt{1 - 2\rho(r_1 + r_2) + \rho^2 \left\{\lambda_{N_1}^2 \delta_C^2 + \lambda_{N_2}^2 \delta_D^2\right\}}.
\end{aligned}$$

Condition (3.5) implies that $0 < \Theta(\maltese) < 1$, it follows from (3.13) that $\{x_n\}$ is a Cauchy sequence in H and hence $x_n \to x$. Since the mapping A, B, C and D are $\widetilde{D}$-Lipschitz continuous and using Algorithm 1, it follows that $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ and $\{e_n\}$ are also Cauchy sequences, we can assume that $u_n \to u$, $v_n \to v$, $w_n \to w$ and $e_n \to e$. Since R, p, f, g, I and $N(\cdot,\cdot)$ are continuous mappings and by using Algorithm 1, we have

$$g(x) = R_{\rho,I}^{\partial_\eta \phi(\cdot,x)} \left[(I-R)g(x) - \rho\{P(u) - (f(v) - N(w,e))\}\right].$$

It can be easily proved by using the techniques of *Ahmad et al.* [3] that $d(u, A(x)) = 0$. Since $A(x) \in CB(H)$, it follows that $u \in A(x)$. Similarly, we can show that $v \in B(x)$, $w \in C(x)$ and $e \in D(x)$. By Theorem 3, we conclude that (x, u, v, w, e) is the solution of variational-like inclusion problem (3.1). This completes the proof.

4 Conclusions

It is well known that the proximal gradient methods are the generalized forms of projection methods. The proximal gradient methods play an important role in analysis and to find the solution of variational inclusion problems and equivalent problems.

Most of the splitting methods are based on the resolvent operators of the form $[I + \lambda M]^{-1}$, where M is a set-valued monotone mapping, λ is a positive constant and I is the identity mapping.

Due to interesting applications of above discussed concept, in this paper, we introduce a generalized proximal operator i.e., a relaxed η-proximal operator and prove some of its properties. Finally, this new concept is applied to solve a variational-like inclusion problem. We remark that our results may be further considered in higher dimensional spaces.

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Appendix: Verification of $\Theta(\maltese) < 1$

The condition (3.5) states that

$$|\alpha+\delta| < \tau\theta(*)\sqrt{2\xi^2-4\mu^2}/(2\mu^2-\xi^2), \quad \theta(*) = t_1+t_2+t_3,$$

where $t_1 = \sqrt{\lambda_g^2 - 2\delta_R^2 + \lambda_R^2\lambda_g^2}$, $t_2 = \sqrt{1-2\rho(k-c)+\rho^2\left\{\lambda_f\delta_B+\lambda_P\delta_A\right\}^2}$, $t_3 = \sqrt{1-2\rho(r_1+r_2)+\rho^2\left\{\lambda_{N_1}^2\delta_C^2+\lambda_{N_2}^2\delta_D^2\right\}}$. Squaring both sides of the above inequality, we obtain

$$\begin{aligned} & (\alpha+\delta)^2(2\mu^2-\xi^2)^2 < \tau^2\theta(*)^2(2\xi^2-4\mu^2) \\ \Rightarrow\quad & (\alpha+\delta)^2(2\mu^2-\xi^2)^2 < \tau^2\theta(*)^2\{-2(2\mu^2-\xi^2)\} \\ \Rightarrow\quad & (\alpha+\delta)^2(2\mu^2-\xi^2) < -2\tau^2\theta(*)^2, \end{aligned}$$

which implies that

$$\begin{aligned} & 2\tau^2\theta(*)^2 + (\alpha+\delta)^2(2\mu^2-\xi^2) < 0 \\ \Rightarrow\quad & 2\tau^2\theta(*)^2 + 2\mu^2(\alpha+\delta)^2 - \xi^2(\alpha+\delta)^2 < 0 \\ \Rightarrow\quad & 2\tau^2\theta(*)^2 + 2\mu^2(\alpha+\delta)^2 < \xi^2(\alpha+\delta)^2. \end{aligned}$$

Dividing the above inequality by $(\alpha+\delta)^2$, we obtain

$$\frac{2\tau^2\theta(*)^2}{(\alpha+\delta)^2} + 2\mu^2 < \xi^2.$$

The above inequality implies that

$$\sqrt{\frac{2\tau^2}{(\alpha+\delta)^2}\theta(*)^2 + 2\mu^2} < 1,$$

which implies that

$$\Theta(\maltese) = \frac{1}{\xi}\sqrt{\frac{2\tau^2}{(\alpha+\delta)^2}\theta(*)^2 + 2\mu^2} < 1.$$