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On Existence of Solutions for Nonlinear q-difference Equations with Nonlocal q-integral Boundary Conditions

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Abstract. In this paper, we discuss the existence of solutions for nonlinear q-difference equations with nonlocal q-integral boundary conditions. The first part of the paper deals with some existence and uniqueness results obtained by means of standard tools of fixed point theory. In the second part, sufficient conditions for the existence of extremal solutions for the given problem are established. The results are well illustrated with the aid of examples.

Keywords: q-difference equations, nonlocal q-integral boundary conditions, existence, extremal solutions, monotone sequence.

AMS Subject Classification: 05A30; 39A13.

1 Introduction

In recent years, there has emerged a great interest in the subject of q-calculus (also known as quantum calculus). The concept of q-calculus distinguishes itself from the classical one in the sense that it does not require the notion of limit.

The idea of limit (up-to our thinking) means that the world can be divided up to infinity. On the other hand, the modern science relying mainly on observation indicates that the world is organized in a different manner and depends on size of its constituents. This kind of consideration about the world gave birth to new types of calculus such as h-calculus (the origin of numerical analysis and computer modeling) and q-calculus, based respectively on finite difference principle and finite difference re-scaling. Eulers identities for q-exponential functions and q-binomial formula due to Gauss were the first few results in the field of q-calculus. This led to remarkable discovery of Heines formula for a q-hypergeometric function as a generalization of the hypergeometric series and its connection to the Ramanujan product formula, relation between Eulers identities and the Jacobi Triple product identity in the 19th century. The systematic research on q-difference equations owes to Jackson [27], Carmichael [18], Mason [33] and Adams [2] in the first quarter of 20th century. For some real applications of q-calculus, we refer the reader to the models [32,37] and the references cited therein. An important characteristic of q-difference equations is that they are always completely controllable and hence appear in the q-optimal control problems [15]. The q-analogue of continuous variational calculus is termed as variational q-calculus in which the extra-parameter q may be physical or economical in its nature. The number of results on q-calculus and their applications to different areas, for example, approximation theory can be found in articles [11, 12, 13, 30, 31].

The variational calculus on the q-uniform lattice studies the q-Euler equations and its applications to the isoperimetric and Lagrange problems and commutation equations. In other words, it suffices to solve the q-Euler-Lagrange equation for finding the extremum of the functional involved rather than solving the Euler-Lagrange equation [14]. As a matter of fact, there are numerous applications of q-difference equations in a variety of disciplines such as special functions, super-symmetry, operator theory, combinatorics, etc. For details, we refer the reader to a series of books [8, 9, 21, 22, 28] and papers [1, 3, 16, 34] and the references cited therein.

Besides the traditional treatment of quantum calculus, many interesting questions and problems concerning the theory of initial and boundary value problems of q-difference equations either remained open or were partially answered. In recent years, this aspect of q-difference equations has been addressed by several researchers and an account of recent development of the topic can be found in the papers [4, 5, 6, 7, 10, 17, 20, 23, 26, 35]. However, there are indeed several aspects of boundary value problems of q-difference equations that need to be developed. For instance, q-difference equations with nonlocal and integral boundary conditions are not explored in detail. It is imperative to note that integral boundary conditions are used to regularize ill-posed parabolic backward problems in time partial differential equations, see, e.g. mathematical models for bacterial self-regularization [19].

In this paper, we study a nonlinear boundary value problem of q-difference equations with nonlocal q-integral boundary condition given by

$$\begin{cases} D_q u(t) = f(t, u(t), u(\phi(t))), \quad t \in I_q, \\ u(0) = \lambda \int_0^\eta g(s, u(s)) d_q s + \mu, \end{cases} \tag{1.1}$$

where $D_q u(t)$ denote the q-derivative of u at $t \in I_q$, $f \in C(I_q \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $g \in C(I_q \times \mathbb{R}, \mathbb{R})$, $\phi \in C(I_q, I_q)$, $\eta \in I_q$, λ, $\mu \in \mathbb{R}$, $I_q = \{q^n : n \in \mathbb{N}\} \cup \{0, 1\}$, $q \in (0, 1)$ is a fixed constant.

Now let us recall some basic concepts of q-calculus [8, 28]. The q-derivative of a real valued function f is defined as

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad D_q f(0) = \lim_{t \to 0} D_q f(t).$$

The q-integral of a function f is defined as

$$\int_a^x f(t) d_q t := (1-q) \sum_{n=0}^{\infty} q^n [x f(x q^n) - a f(a q^n)], \quad x \in [a, b],$$

and for $a = 0$, we denote

$$I_q f(x) = \int_0^x f(t) d_q t = \sum_{n=0}^{\infty} x(1-q) q^n f(x q^n),$$

provided the series converges. If $a \in [0, b]$ and f is defined on the interval $[0, b]$, then

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t.$$

Similarly, we have

$$I_q^0 f(t) = f(t), \quad I_q^n f(t) = I_q I_q^{n-1} f(t), \quad n \in \mathbb{N}.$$

Observe that

$$D_q I_q f(x) = f(x),$$

and if f is continuous at $x = 0$, then $I_q D_q f(x) = f(x) - f(0)$. In q-calculus, the product rule and integration by parts formula are

$$\begin{aligned} &D_q(gh)(t) = D_q g(t) h(t) + g(qt) D_q h(t), \\ &\int_0^x f(t) D_q g(t)\, dqt = \Big[f(t) g(t)\Big]_0^x - \int_0^x D_q f(t) g(qt)\, d_q t. \end{aligned}$$

The rest of the paper is organized as follows. In Section 2, we present an auxiliary lemma and a comparison result, which are vital in establishing our main work. Section 3 contains some existence and uniqueness results which are based on Banach's contraction mapping principle, Leray-Schauder nonlinear alternative and Krasnoselskii's fixed point theorem. In Section 4, we discuss the existence of extremal solutions for the given problem. We emphasize that we have investigated a new class of problems consisting of q-difference equations with a nonlinearity of the form $f(t, u(t), u(\phi(t)))$ and nonlocal q-integral boundary conditions with two parameters. Our results are new in the given setting and are well illustrated with the aid of examples.

2 Preliminaries

Definition 1. *We say that $u(t)$ is a lower solution of problem* (1.1) *if*

$$\begin{cases} D_q u(t) \le f(t, u(t), u(\phi(t))), & t \in I_q, \\ u(0) \le \lambda \int_0^\eta g(s, u(s)) d_q s + \mu, \end{cases}$$

and it is an upper solution of (1.1) *if the above inequalities are reversed [29].*

Lemma 1. *Let $\lambda\eta \neq 1$ and $y \in C(I_q, \mathbb{R})$. Then the linear q-integral boundary value problem*

$$\begin{cases} D_q u(t) = y(t), & t \in I_q, \\ u(0) = \lambda \int_0^\eta u(s) d_q s + \mu \end{cases} \tag{2.1}$$

has a unique solution

$$u(t) = \int_0^t y(s)\, d_q s + \frac{\lambda}{1-\lambda\eta} \int_0^\eta (\eta - qs) y(s)\, d_q s + \frac{\mu}{1-\lambda\eta}.$$

Proof. Integrating both sides of the equation $D_q u(t) = y(t)$ and applying the condition $u(0) = \lambda \displaystyle\int_0^\eta u(s) d_q s + \mu$, we have

$$u(t) = I_q y(t) + u(0) = \int_0^t y(s) d_q s + \lambda \int_0^\eta u(s)\, d_q s + \mu. \tag{2.2}$$

Letting $\displaystyle\int_0^\eta u(s) d_q s = B$, and integrating both sides of (2.2), we get

$$\begin{aligned} B = \int_0^\eta u(t) d_q t &= \int_0^\eta \int_0^t y(s) d_q s d_q t + \int_0^\eta (\lambda B + \mu) d_q t \\ &= \int_0^\eta \int_0^t y(s) d_q s d_q t + \eta(\lambda B + \mu). \end{aligned}$$

Changing the order of integration, we obtain

$$\begin{aligned} B &= \int_0^\eta \int_{qs}^\eta y(s) d_q t d_q s + \eta(\lambda B + \mu) \\ &= \int_0^\eta (\eta - qs) y(s) d_q s + \eta(\lambda B + \mu), \end{aligned}$$

which yields

$$B = \frac{1}{1-\lambda\eta} \int_0^\eta (\eta - qs) y(s) d_q s + \frac{\eta\mu}{1-\lambda\eta}.$$

Substituting the value of B in (2.2), we obtain the unique solution of problem (2.1). This completes the proof. □

Remark 1. If $t \in I_q \backslash \{0\}$ and I_q contains a neighborhood of the point t such that f is differentiable at t, then $\lim_{q\to 1} D_q f(t) = f'(t)$. If $t = 0$ and $f'(0)$ exists, then $D_q f(0) = f'(0)$. However, $D_q f(0)$ may exist for a function f without being differentiable or even continuous at zero. For more details, see Section 1.3 in [8]. Thus our results are different from the similar ones for classical ordinary differential equations in aforementioned sense.

Lemma 2. (*Comparison Result*) *If there exists a nonnegative constant* λ *satisfying* $0 < \lambda\eta < 1$ *such that*

$$\begin{cases} D_q u(t) \geq 0, & t \in I_q, \\ u(0) \geq \lambda \int_0^\eta u(s)\, d_q s. \end{cases}$$

Then $u(t) \geq 0,\ \forall t \in I_q$.

Proof. By Lemma 1, the conclusion of Lemma 2 is obvious, so we omit the proof. □

3 Some Existence Results

In the sequel, we denote by $\mathcal{C} = C(I_q, \mathbb{R})$ the space of all continuous functions from $I_q \to \mathbb{R}$ equipped with the norm $\sup_{t\in I_q} |u(t)| = \|u\|$.

In view of Lemma 1, we define an operator $\mathcal{G} : C(I_q, \mathbb{R}) \to C(I_q, \mathbb{R})$ associated with the problem (1.1) as follows

$$\begin{aligned} \mathcal{G}u(t) = &\int_0^t f(s, u(s), u(\phi(s)))d_q s + \frac{\lambda}{1-\lambda\eta} \int_0^\eta (\eta - qs) f(s, u(s), u(\phi(s)))d_q s \\ &+ \frac{1}{1-\lambda\eta}\Big(\int_0^\eta [g(s, u(s)) - \lambda u(s)]d_q s + \mu\Big). \end{aligned} \tag{3.1}$$

Now we are in a position to show the existence and uniqueness of solutions continuous at 0 for the problem (1.1).

Theorem 1. *Assume that* $f : I_q\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ *and* $g : I_q\times\mathbb{R} \to \mathbb{R}$ *are continuous functions such that* $\sup_{t\in I_q} |f(t, 0, 0)| = M_1,\ \sup_{t\in I_q} |g(t, 0)| = M_2$ *for given positive constants* $M_j, j = 1, 2,$ *and that there exists* q*-integrable functions* $L_i : I_q \to \mathbb{R}^+, i = 1, 2$ *such that*

(S_1) $|f(t, u(t), u(\phi(t))) - f(t, v(t), v(\phi(t)))| \leq L_1(t)|u - v|, \quad t \in I_q,\ u, v \in \mathbb{R}.$

(S_2) $|g(t, u(t)) - g(t, v(t))| \leq L_2(t)|u - v|, \quad t \in I_q,\ u, v \in \mathbb{R}.$

If $A_1 < 1$, *then the problem (1.1) has a unique solution on* I_q, *where*

$$A_1 = \int_0^1 L_1(s)d_q s + \frac{1}{|1-\lambda\eta|} \int_0^\eta \Big(|\lambda|[(\eta - qs)L_1(s) + 1] + L_2(s)\Big)d_q s. \tag{3.2}$$

Proof. In the first step, using the given hypotheses, we show that $\mathcal{G}B_\rho \subset B_\rho$, where $\mathcal{G}$ is defined by (3.1), $B_\rho = \{x \in \mathcal{C} : \|x\| \le \rho\}$ and $\rho \ge A_2/(1-A_1)$ with A_1 given by (3.2) and

$$A_2 = \frac{1}{|1-\lambda\eta|}\Big(M_1(|1-\lambda\eta|) + \frac{M_1|\lambda|\eta^2}{1+q} + M_2\eta + |\mu|\Big). \tag{3.3}$$

For $u \in B_\rho,\ t \in I_q$, we have

$$\begin{aligned}
|f(t,u(t),u(\phi(t)))| &\le |f(t,u(t),u(\phi(t))) - f(t,0,0)| + |f(t,0,0)| \\
&\le L_1(t)\|u\| + M_1 \le L_1(t)\rho + M_1, \\
|g(t,u(t))| &\le |g(t,u(t)) - g(t,0)| + |g(t,0)| \le L_2(t)\|u\| + M_2 \\
&\le L_2(t)\rho + M_2.
\end{aligned}$$

Then

$$\begin{aligned}
|\mathcal{G}u(t)| &\le \int_0^t \Big(L_1(s)\rho + M_1\Big) d_qs + \frac{|\lambda|}{|1-\lambda\eta|}\int_0^\eta (\eta - qs)\Big(L_1(s)\rho + M_1\Big) d_qs \\
&+ \frac{1}{|1-\lambda\eta|}\Big(\int_0^\eta [(L_2(s) + |\lambda|)\rho + M_2] d_qs + |\mu|\Big) \le \rho A_1 + A_2 \le \rho.
\end{aligned}$$

This shows that $\mathcal{G}B_\rho \subset B_\rho$.

Now, for $u, v \in \mathcal{C}$ and $t \in I_q$, we obtain

$$\begin{aligned}
\|(\mathcal{G}u) - (\mathcal{G}v)\| &\le \sup_{t\in I_q}\Big\{\int_0^t |f(s,u(s),u(\phi(s))) - f(s,v(s),v(\phi(s)))| d_qs \\
&+ \frac{|\lambda|}{|1-\lambda\eta|}\int_0^\eta (\eta - qs)|f(s,u(s),u(\phi(s))) - f(s,v(s),v(\phi(s)))| d_qs \\
&+ \frac{1}{|1-\lambda\eta|}\int_0^\eta \Big(|g(s,u(s)) - g(s,v(s))| + |\lambda||u(s) - v(s)|\Big) d_qs\Big\} \\
&\le \|u - v\| \sup_{t\in I_q}\Big\{\int_0^t L_1(s) d_qs + \frac{|\lambda|}{|1-\lambda\eta|}\int_0^\eta (\eta - qs) L_1(s) d_qs \\
&+ \frac{1}{|1-\lambda\eta|}\int_0^\eta (L_2(s) + |\lambda|) d_qs\Big\} = A_1\|u - v\|,
\end{aligned}$$

where we have used (3.2). As $A_1 \in (0,1)$ by the given assumption, therefore $\mathcal{G}$ is a contraction. Hence Banach's contraction principle applies and the problem (1.1) has a unique solution. □

Corollary 1. *In case $L_i(t) = L_i, i = 1,2$ (L_i are constants), then the assumptions $(S_1), (S_2)$ and A_1 in the condition $A_1 < 1$ take the following form:*

$(\bar{S}_1)$ $|f(t,u(t),u(\phi(t))) - f(t,v(t),v(\phi(t)))| \le L_1|u - v|, \quad t \in I_q,\ u, v \in \mathbb{R},$

$(\bar{S}_2)$ $|g(t,u(t)) - g(t,v(t))| \le L_2|u - v|, \quad t \in I_q,\ u, v \in \mathbb{R}$

and

$$\bar{A}_1 = L_1\Big[1 + \frac{|\lambda|\eta^2}{|1-\lambda\eta|(1+q)}\Big] + \frac{(|\lambda| + L_2)\eta}{|1-\lambda\eta|}.$$

Next we show the existence of solutions via Leray-Schauder nonlinear alternative.

Theorem 2. *(Nonlinear alternative for single valued maps) [24]. Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) F has a fixed point in $\overline{U}$, or

(ii) there is a $u \in \partial U$ (the boundary of U in C) and $\kappa \in (0,1)$ with $u = \kappa F(u)$.

Theorem 3. *Let $f : I_q \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : I_q \times \mathbb{R} \to \mathbb{R}$ be continuous functions. In addition, we assume that*

(S_3) *there exist functions $m_1, m_2, n_1, n_2 \in L^1(I_q, \mathbb{R}^+)$, and nondecreasing functions $\psi_i : \mathbb{R}^+ \to \mathbb{R}^+ (i = 1, 2)$ such that*

$$|f(t, u(t), u(\phi(t)))| \le m_1(t)\psi_1(\|u\|) + m_2(t),$$
$$|g(t, u(t))| \le n_1(t)\psi_2(\|u\|) + n_2(t),$$

for each $(t, u(t), u(\phi(t))) \in I_q \times \mathbb{R} \times \mathbb{R},\ (t, u(t)) \in I_q \times \mathbb{R}$;

(S_4) *there exists a number $M > 0$ such that*

$$\frac{M[1 - |\lambda|\eta(|1-\lambda\eta|)^{-1}]}{\psi_1(M)\nu_1 + \nu_2 + \psi_2(M)\omega_1 + \omega_2 + |\mu|(|1-\lambda\eta|)^{-1}} > 1,$$

where

$$\nu_i = \int_0^1 m_i(s)d_qs + \frac{|\lambda|}{|1-\lambda\eta|}\int_0^\eta (\eta - qs)m_i(s)d_qs, \tag{3.4}$$

$$\omega_i = \frac{1}{|1-\lambda\eta|}\int_0^\eta n_i(s)d_qs,\ i = 1, 2. \tag{3.5}$$

Then the boundary value problem (1.1) has at least one solution on I_q.

Proof. We split the proof into several steps. In the first step, it will be shown that the operator $\mathcal{G}$ maps bounded sets into bounded sets in $C(I_q, \mathbb{R})$. For that, let $B_r = \{u \in C(I_q, \mathbb{R}) : \|u\| \le r\}$ be a bounded set in $C(I_q, \mathbb{R})$ for some $r > 0$. Notice that the operator $\mathcal{G} : C(I_q, \mathbb{R}) \to C(I_q, \mathbb{R})$ defined by (3.1) is continuous. Then, in view of the assumption (S_3), we have

$$\begin{aligned}
|\mathcal{G}u(t)| &\leq \int_0^t [m_1(s)\psi_1(\|u\|)+m_2(s)]d_qs \\
&+\frac{|\lambda|}{|1-\lambda\eta|}\int_0^\eta (\eta-qs)[m_1(s)\psi_1(\|u\|)+m_2(s)]d_qs \\
&+\frac{1}{1-\lambda\eta}\Big(\int_0^\eta [[n_1(s)\psi_2(\|u\|)+n_2(s)]+|\lambda|\|u\|]d_qs+|\mu|\Big), \\
&\leq \psi_1(r)\Big\{\int_0^1 m_1(s)d_qs+\frac{|\lambda|}{|1-\lambda\eta|}\int_0^\eta (\eta-qs)m_1(s)d_qs\Big\} \\
&+\int_0^1 m_2(s)d_qs+\frac{|\lambda|}{|1-\lambda\eta|}\int_0^\eta(\eta-qs)m_2(s)d_qs \\
&+\frac{\psi_2(r)}{|1-\lambda\eta|}\int_0^\eta n_1(s)d_qs+\frac{1}{|1-\lambda\eta|}\int_0^\eta n_2(s)d_qs+\frac{|\lambda|r\eta+|\mu|}{|1-\lambda\eta|},
\end{aligned}$$

which, by using (3.4) and (3.5), yields

$$\|\mathcal{G}u\| \leq \psi_1(r)\nu_1+\nu_2+\psi_2(r)\omega_1+\omega_2+\frac{|\lambda|r\eta+|\mu|}{|1-\lambda\eta|}\leq r.$$

Now we show that $\mathcal{G}$ maps bounded sets into equicontinuous sets of $C(I_q,\mathbb{R})$. Let $t_1,t_2\in I_q$ with $t_1<t_2$ and $x\in B_r$, where B_r is a bounded set of $C(I_q,\mathbb{R})$. Then

$$\begin{aligned}
&\|(\mathcal{G}u)(t_2)-(\mathcal{G}u)(t_1)\| \\
\leq\ & \Big|\int_0^{t_2}(t_2-qs)\,f(s,u(s),u(\phi(s)))d_qs-\int_0^{t_1}(t_1-qs)\,f(s,u(s),u(\phi(s)))d_qs\Big| \\
\leq\ & \Big|\int_0^{t_1}(t_2-t_1)[m_1(s)\psi_1(r)+m_2(s)]d_qs \\
&+\int_{t_1}^{t_2}(t_2-qs)\,[m_1(s)\psi_1(r)+m_2(s)]d_qs\Big|,
\end{aligned}$$

which tends to zero independently of $u\in B_r$ as $t_2-t_1\to 0$. Hence the Arzelá-Ascoli theorem applies and that $\mathcal{G}: C(I_q,\mathbb{R})\to C(I_q,\mathbb{R})$ is completely continuous.

Now let $\kappa\in(0,1)$ and let $u=\kappa\mathcal{G}u$. Then for $t\in I_q$, we have

$$\begin{aligned}
|u(t)| &\leq \psi_1(\|u\|)\Big\{\int_0^1 m_1(s)d_qs+\frac{|\lambda|}{|1-\lambda\eta|}\int_0^\eta(\eta-qs)m_1(s)d_qs\Big\} \\
&+\int_0^1 m_2(s)d_qs+\frac{|\lambda|}{|1-\lambda\eta|}\int_0^\eta(\eta-qs)m_2(s)d_qs \\
&+\frac{\psi_2(\|u\|)}{|1-\lambda\eta|}\int_0^\eta n_1(s)d_qs+\frac{1}{|1-\lambda\eta|}\int_0^\eta n_2(s)d_qs+\frac{|\lambda|\|u\|\eta+|\mu|}{|1-\lambda\eta|},
\end{aligned}$$

which implies that

$$\frac{\|u\|[1-|\lambda|\eta(|1-\lambda\eta|)^{-1}]}{\psi_1(\|u\|)\nu_1+\nu_2+\psi_2(\|u\|)\omega_1+\omega_2+|\mu|(|1-\lambda\eta|)^{-1}}\leq 1.$$

By the condition (S_4), we can find M such that $\|u\| \neq M$. We define

$$V = \{u \in C(I_q, \mathbb{R}) : \|u\| < M\}.$$

Note that the operator $\mathcal{G} : \overline{V} \to C(I_q, \mathbb{R})$ is continuous and completely continuous. From the choice of V, there does not exist any $u \in \partial V$ satisfying $u = \kappa \mathcal{G} u$ for some $\kappa \in (0, 1)$. Hence, by the Leray-Schauder alternative (Theorem 2), we can deduce that the operator $\mathcal{G}$ has a fixed point $u \in \overline{V}$ which is a solution of the problem (1.1). This completes the proof. □

Our next existence results is based on Krasnoselskii's fixed point theorem.

Lemma 3. *(Krasnoselskii) [36]. Let Y be a closed, bounded, convex and nonempty subset of a Banach space X. Let W_1, W_2 be the operators such that (i) $W_1 x + W_2 y \in Y$ whenever $x, y \in Y$; (ii) W_1 is compact and continuous; (iii) W_2 is a contraction. Then there exists $y \in Y$ such that $y = W_1 y + W_2 y$.*

Theorem 4. *Let $f : I_q \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : I_q \times \mathbb{R} \to \mathbb{R}$ be continuous satisfying the condition (S_1) and (S_2). Further, the following conditions hold:*

- (S_5) *there exists functions $\sigma_i \in C(I_q, \mathbb{R}^+)$ and nondecreasing functions $\chi_i \in C(I_q, \mathbb{R}^+)$, $i = 1, 2$ with*

$$|f(t, u(t), u(\phi(t)))| \leq \sigma_1(t)\chi_1(|u|), \ \ |g(t, u(t))| \leq \sigma_2(t)\chi_2(|u|)$$

  for $(t, u(t), u(\phi(t))) \in I_q \times \mathbb{R} \times \mathbb{R}$, $(t, u(t)) \in I_q \times \mathbb{R}$;

- (S_6) *there exists a constant $\overline{r}$ with*

$$\begin{aligned} \overline{r} \ &\geq \Big(1 - \frac{|\lambda|\eta}{|1 - \lambda\eta|}\Big)^{-1} \Big[\|\sigma_1\|\chi_1(\overline{r})\Big\{1 + \frac{|\lambda|\eta^2}{|1 - \lambda\eta|(1 + q)}\Big\} \\ &+ \frac{1}{|1 - \lambda\eta|}\Big\{\|\sigma_2\|\chi_2(\overline{r})\eta + \mu\Big\}\Big], \end{aligned} \tag{3.6}$$

  where $\|\sigma_i\| = \sup_{t \in I_q} |\sigma_i(t)|$.

If

$$\frac{1}{|1 - \lambda\eta|} \int_0^\eta \Big(|\lambda|[(\eta - qs)((L_1(s) + L_2(s))) + 1] + L_3(s)\Big) d_q s < 1, \tag{3.7}$$

then the boundary value problem (1.1) has at least one solution on I_q.

Proof. Let us consider a set $B_{\overline{r}} = \{u \in C(I_q, \mathbb{R}) : \|u\| \leq \overline{r}\}$ and define the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ on $B_{\overline{r}}$ as

$$\begin{aligned} (\mathcal{G}_1 u)(t) \ &= \ \int_0^t f(s, u(s), u(\phi(s))) d_q s, \\ (\mathcal{G}_2 u)(t) \ &= \ \frac{\lambda}{1 - \lambda\eta} \int_0^\eta (\eta - qs) f(s, u(s), u(\phi(s))) d_q s \\ &+ \frac{1}{1 - \lambda\eta}\Big(\int_0^\eta [g(s, u(s)) - \lambda u(s)] d_q s + \mu\Big), \quad t \in I_q. \end{aligned}$$

For $u, v \in B_{\overline{r}}$, we can obtain

$$\begin{aligned}|(\mathcal{G}_1u+\mathcal{G}_2v)(t)| \;\leq\; & \Big(1-\frac{|\lambda|\eta}{|1-\lambda\eta|}\Big)^{-1}\Big[\|\sigma_1\|\chi_1(\overline{r})\Big\{1+\frac{|\lambda|\eta^2}{|1-\lambda\eta|(1+q)}\Big\} \\ & +\frac{1}{|1-\lambda\eta|}\Big\{\|\sigma_2\|\chi_2(\overline{r})\eta+\mu\Big\}\Big] \leq \overline{r}.\end{aligned}$$

Thus, $(\mathcal{G}_1u+\mathcal{G}_2v)(t) \in B_{\overline{r}}$. It follows from (S_1) and (3.7) that $\mathcal{G}_2$ is a contraction. Continuity of f implies that the operator $\mathcal{G}_1$ is continuous. Also, $\mathcal{G}_1$ is uniformly bounded on $B_{\overline{r}}$ as $\|\mathcal{G}_1u\| \leq \chi_1(\overline{r})\|\sigma_1\|$. Now, for any $x \in B_{\overline{r}}$, and $t_1, t_2 \in I_q$ with $t_1 < t_2$, we have

$$\begin{aligned}& |(\mathcal{G}_1u)(t_2)-(\mathcal{G}_1u)(t_1)| \\ \leq\; & \Big|\int_0^{t_2}(t_2-qs)\,f(s,u(s),u(\phi(s)))d_qs-\int_0^{t_1}(t_1-qs)\,f(s,u(s),u(\phi(s)))d_qs\Big| \\ \leq\; & \chi_1(\overline{r})\|\sigma_1\|\Big|\int_0^{t_1}(t_2-t_1)d_qs+\int_{t_1}^{t_2}(t_2-qs)\,d_qs\Big|,\end{aligned}$$

which is independent of u and tends to zero as $t_2 \to t_1$. Thus, $\mathcal{G}_1$ is equicontinuous. So $\mathcal{G}_1$ is relatively compact on $B_{\overline{r}}$. Hence, it follows by the Arzelá-Ascoli Theorem that $\mathcal{G}_1$ is compact on $B_{\overline{r}}$. Thus all the assumptions of Lemma 3 are satisfied. So the conclusion of Lemma 3 implies that the problem (1.1) has at least one solution on I_q. This completes the proof. □

3.1 Examples

Example 1. Consider the problem

$$\begin{cases}D_{\frac{1}{3}}u(t)=f(t,u(t),u(\phi(t))), & t\in[0,1]_{\frac{1}{3}},\\ u(0)=\frac{1}{12}\int_0^{\frac{1}{2}}g(s,u(s)d_qs.\end{cases} \tag{3.8}$$

Here $q=1/3,\ f(t,u(t),u(\phi(t)))=\dfrac{1}{5}\tan^{-1}u(t)+\dfrac{1}{3}\dfrac{u(\frac{1}{3}t)\sin^2 t}{1+u(\frac{1}{3}t)}+(t+1)^2,\ \lambda=1/12,\ \eta=1/2(\notin[0,1]_{\frac{1}{3}}),\ \mu=0,\ g(t,u(t))=\dfrac{1}{4}\cos(u(t))+\sqrt{t^2+1}$. With the given data, $L_1=1/5, L_2=1/3$ and $\bar{A}_1=\frac{247}{345}<1$. Clearly all the conditions of Corollary 1 are satisfied. Hence, it follows by the conclusion of Corollary 1 that the problem (3.8) has a unique solution.

Example 2. Let us consider Example 1 with

$$\begin{aligned}& f(t,u(t),u(\phi(t)))=\frac{1}{6}\frac{|u(t)|^2}{(1+|u(t)|^2)}+\frac{|u(\frac{1}{4}t)|^3}{1+|u(\frac{1}{4}t)|^3}+1, \\ & g(t,u(t))=\frac{1}{3}\sin(u(t))+\frac{1}{8}.\end{aligned} \tag{3.9}$$

Clearly $|f(t,u(t),u(\phi(t)))| \leq 7/6+1,\ |g(t,u(t))| \leq \dfrac{1}{3}\|u\|+\dfrac{1}{8}$. Letting $m_1(t)=1=m_2(t), \psi_1(\|u\|)=7/6, n_1(t)=1/3, \psi_2(\|u\|)=\|u\|,\ n_2=1/8$, we find that

$\nu_1 = 187/184 = \nu_2,\ \omega_1 = 4/23,\ \omega_2 = 3/46$. By the condition (S_4), it is found that $M > 2503/864$. Thus the hypotheses of Theorem 3 hold and consequently there exists a solution for the problem (3.8) with $f(t, u(t), u(\phi(t)))$ and $g(t, u(t))$ given by (3.9).

4 Extremal solutions

In the sequel, we need the following known result.

Theorem 5. *[25] Let $[a, b]$ be an order interval in a subset Y of an ordered Banach space X and let $Q : [a, b] \to [a, b]$ be a nondecreasing mapping. If each sequence $\{Qx_n\} \subset Q([a, b])$ converges, whenever $\{x_n\}$ is a monotone sequence in $[a, b]$, then the sequence of Q-iteration of a converges to the least fixed point x_* of Q and the sequence of Q-iteration of b converges to the greatest fixed point x^* of Q. Moreover,*

$$x_* = \min\{y \in [a, b] :\ y \geq Qy\} \qquad \text{and} \qquad x^* = \max\{y \in [a, b] :\ y \leq Qy\}.$$

Theorem 6. *(Extremal solutions). Assume that $u_0, v_0 \in C(I_q, \mathbb{R})$ are lower and upper solutions of (1.1) respectively, and that $u_0(t) \leq v_0(t),\ \forall t \in I_q$. Further, the following conditions hold:*

(H_1) the function $f \in C(I_q \times \mathbb{R}^2, \mathbb{R})$ is nondecreasing with respect to second and third variables;

(H_2) there exists a nonnegative constant λ satisfying $\lambda\eta < 1$ such that

$$g(t, u) - g(t, v) \geq \lambda(u - v)$$

for $u_0(t) \leq v \leq u \leq v_0(t),\ \forall t \in I_q$.

Then there exist extremal solutions $u_*, v^* \in [u_0, v_0]$ for the nonlinear q-integral boundary value problem (1.1) *which can be obtained via the explicit iterative sequences:*

$$\begin{cases} u_{n+1}(t) &= \displaystyle\int_0^t f(s, u_n(s), u_n(\phi(s)))d_qs \\ &\quad + \displaystyle\frac{\lambda}{1-\lambda\eta}\int_0^\eta (\eta - qs) f(s, u_n(s), u_n(\phi(s)))d_qs \\ &\quad + \displaystyle\frac{1}{1-\lambda\eta}\Big(\int_0^\eta [g(s, u_n(s)) - \lambda u_n(s)]d_qs + \mu\Big), \\ v_{n+1}(t) &= \displaystyle\int_0^t f(s, v_n(s), v_n(\phi(s)))d_qs \\ &\quad + \displaystyle\frac{\lambda}{1-\lambda\eta}\int_0^\eta (\eta - qs) f(s, v_n(s), v_n(\phi(s)))d_qs \\ &\quad + \displaystyle\frac{1}{1-\lambda\eta}\Big(\int_0^\eta [g(s, v_n(s)) - \lambda v_n(s)]d_qs + \mu\Big). \end{cases} \tag{4.1}$$

Moreover, the sequences $\{u_n\}, \{v_n\}$ converge to u_*, v^* respectively satisfying the relation:

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq u_* \leq v^* \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0.$$

Proof. We consider the linear q-integral boundary value problem

$$\begin{cases} D_q u(t) = f(t, \sigma(t), \sigma(\phi(t))), & t \in I_q, \\ u(0) = \int_0^\eta [g(s, \sigma(s)) + \lambda(u - \sigma)(s)] d_q s + \mu. \end{cases} \tag{4.2}$$

It follows from Lemma 1 that the linear problem (4.2) has a unique solution

$$\begin{aligned} u(t) &= \int_0^t f(s, \sigma(s), \sigma(\phi(s))) d_q s + \frac{\lambda}{1-\lambda\eta} \int_0^\eta (\eta - qs) f(s, \sigma(s), \sigma(\phi(s))) d_q s \\ &\quad + \frac{1}{1-\lambda\eta}\Big(\int_0^\eta [g(s, \sigma(s)) - \lambda\sigma(s)] d_q s + \mu\Big). \end{aligned}$$

For any $\sigma \in [u_0, v_0]$, we define an operator $\mathcal{G}$ with $u(t) = \mathcal{G}\sigma(t)$. Then the operator $\mathcal{G}$ is nondecreasing and $\mathcal{G} : [u_0, v_0] \to [u_0, v_0]$.

Indeed, we set $u_1 = \mathcal{G}u_0,\ v_1 = \mathcal{G}v_0$. Then u_1, v_1 are well defined and respectively satisfy

$$\begin{cases} D_q u_1(t) = f(t, u_0(t), u_0(\phi(t))), & t \in I_q, \\ u_1(0) = \int_0^\eta [g(s, u_0(s)) + \lambda(u_1 - u_0)(s)] d_q s + \mu, \end{cases} \tag{4.3}$$

and

$$\begin{cases} D_q v_1(t) = f(t, v_0(t), v_0(\phi(t))), & t \in I_q, \\ v_1(0) = \int_0^\eta [g(s, v_0(s)) + \lambda(v_1 - v_0)(s)] d_q s + \mu. \end{cases} \tag{4.4}$$

Noting that u_0 is a lower solution of problem (1.1), we let $w = u_1 - u_0$. Then, it follows from (4.3) that

$$\begin{cases} D_q w(t) \geq 0, & t \in I_q, \\ w(0) \geq \lambda \int_0^\eta w(s) d_q s. \end{cases} \tag{4.5}$$

By Lemma 2, we have $w(t) \geq 0, \forall t \in I_q$, which implies that $\mathcal{G}u_0 \geq u_0$. Similarly, applying the definition of upper solution and (4.4), we can obtain $\mathcal{G}v_0 \leq v_0$. Thus $\mathcal{G} : [u_0, v_0] \to [u_0, v_0]$.

Setting $e = v_1 - u_1$ and making use of (4.3) and (4.4) with the conditions $(H_1), (H_2)$, we obtain

$$\begin{aligned} D_q e(t) &= f(t, v_0(t), v_0(\phi(t))) - f(t, u_0(t), u_0(\phi(t))) \geq 0, \qquad (4.6) \\ e(0) &= \int_0^\eta [g(s, v_0(s)) + \lambda(v_1 - v_0)(s)] d_q s \\ &\quad - \int_0^\eta [g(s, u_0(s)) + \lambda(u_1 - u_0)(s)] d_q s \geq \lambda \int_0^\eta e(s) d_q s. \end{aligned}$$

Likewise, we get $e(t) \geq 0$ by means of Lemma 2. Thus, $\mathcal{G}u_0 \leq \mathcal{G}v_0$. This, together with $u_0 \leq \mathcal{G}u_0$ and $\mathcal{G}v_0 \leq v_0$, implies that $\mathcal{G}$ is nondecreasing and $\mathcal{G} : [u_0, v_0] \to [u_0, v_0]$.

Assume that $\{w_n\} \subset [u_0, v_0]$ is a monotone iterative sequence. Then, $u_0 \leq \mathcal{G}w_n \leq v_0$. By means of Arzelá-Ascoli theorem, one can show that the sequence $\{\mathcal{G}w_n\} \subset \mathcal{G}([u_0, v_0])$ converges.

With the aid of Theorem 5, it is easy to see that the sequence of $\mathcal{G}$-iteration with u_0 converges to the least fixed point u_* of $\mathcal{G}$ and the sequence of $\mathcal{G}$-iteration with v_0 converges to the greatest fixed point v^* of $\mathcal{G}$. This, in turn, implies that the nonlocal q-integral boundary value problem (1.1) has extremal solutions $u_*, v^* \in [u_0, v_0]$, which can be achieved by the corresponding iterative sequences $\{u_n\}, \{v_n\}$ defined in (4.1). Obviously, the following conclusion holds

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq u_* \leq v^* \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0.$$

This completes the proof. □

Example 3. Consider a nonlinear q-integral boundary value problem given by

$$\begin{cases} D_{\frac{1}{2}} u(t) = \dfrac{1}{4}t^3 + \dfrac{1}{8}tu^2(t) + \dfrac{t^2}{10}\dfrac{u(\frac{1}{2}t)}{1+u(\frac{1}{2}t)}, \quad t \in [0,1]_{\frac{1}{2}}, \\ u(0) = \frac{3}{14}\int_0^{\frac{1}{4}} (\dfrac{1}{7}u^2(s) + \frac{1}{14}e^{u(s)}) d_{\frac{1}{2}} s + \frac{1}{2}. \end{cases} \tag{4.7}$$

Here $q = \dfrac{1}{2}$, $\lambda = \frac{3}{14}$, $\eta = \dfrac{1}{4}$, $\mu = \dfrac{1}{2}$, $\phi(t) = \frac{1}{2}t$, $f(t, u(t), u(\phi(t))) = \dfrac{1}{4}t^3 + \dfrac{1}{8}tu^2(t) + \dfrac{t^2}{10}\dfrac{u(\frac{1}{2}t)}{1+u(\frac{1}{2}t)}$ and $g(t, u(t)) = \dfrac{1}{7}u^2(t) + \dfrac{1}{14}e^{u(t)}$.

To show the applicability of the conclusion of Theorem 6, we shall verify that all conditions of Theorem 6 are satisfied.

Take $u_0 = 0$, $v_0 = 1 + t$. It is easy to verify that u_0 and v_0 are lower and upper solutions of problem (4.7) respectively. Since $f(t, u, v) = \dfrac{1}{4}t^3 + \dfrac{1}{8}tu^2 + \dfrac{t^2}{10}\dfrac{v}{1+v}$, the condition (H_1) holds.

With $g(t, u) = \dfrac{1}{7}u^2 + \dfrac{1}{14}e^u$, $\eta = \dfrac{1}{4}$, and for $\lambda = \dfrac{3}{14}$, $\lambda\eta = \dfrac{3}{56} < 1$, the function g satisfies the condition (H_2). Thus, by Theorem 6, we conclude that the problem (4.7) has extremal solutions u_*, v^*, which can be obtained by means of the iterative sequences $\{u_n\}, \{v_n\}$ defined by the expressions in (4.1).

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