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Positive Solutions of the Semipositone Neumann Boundary Value Problem

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Abstract. In this paper we consider the Neumann boundary value problem at resonance

$$-u''(t) = f\big(t, u(t)\big), \quad 0 < t < 1, \quad u'(0) = u'(1) = 0.$$

We assume that the nonlinear term satisfies the inequality $f(t,z) + \alpha^2 z + \beta(t) \geq 0$, $t \in [0,1]$, $z \geq 0$, where $\beta : [0,1] \to \mathbf{R}_+$, and $\alpha \neq 0$. The problem is transformed into a non-resonant positone problem and positive solutions are obtained by means of a Guo–Krasnosel'skiĭ fixed point theorem.

Keywords: Neumann boundary conditions, resonance, semipositone.

AMS Subject Classification: 34B15; 34B18.

1 Introduction

We study the Neumann boundary value problem

$$-u''(t) = f\big(t, u(t)\big), \quad 0 < t < 1, \tag{1.1}$$
$$u'(0) = u'(1) = 0, \tag{1.2}$$

with a sign-changing nonlinearity.

We will make the assumptions precise in the next section, we only mention now that the continuous function $f : [0,1] \times \mathbf{R}_+ \to \mathbf{R}$ satisfies the inequality $f(t,z) \geq -\alpha^2 z - \beta(t)$ in $[0,1] \times \mathbf{R}_+$, for some constant $\alpha \neq 0$ and a non-negative valued function $\beta(t)$.

One of the most frequently mentioned papers that stimulated the discussion of semipositone problems is the paper [7] by Miciano and Shivaji. The authors of [7] used the bifurcation techniques to obtain multiple positive solutions for the Neumann problem. We only mention several among many results based on applications of a Guo–Krasnosel'skiĭ fixed point theorem and fixed point index computations. In [10], Sun and Wei obtained positive solutions of the non-local boundary value problem

$$-u''(t) = f\big(t, u(t)\big), \quad 0 < t < 1,$$
$$u(0) = \alpha u(\eta), \quad u(1) = \beta u(\eta),$$

where the right side is a continuous function with $f(t,u) + M \geq 0$ for some $M > 0$. Lu [5] obtained multiple positive solutions for singular semipositone periodic boundary value problems. It should be mentioned that, in [5], the nonhomogeneous term depends on the first order derivative. In this regard, the results of [5] are similar to those obtained by Ma [6] who studied a fourth order semipositone boundary value problem

$$u^{(4)}(t) = \lambda f\big(t, u(t), u'(t)\big), \quad 0 < t < 1,$$
$$u(0) = u'(0) = u''(1) = u'''(1) = 0.$$

Other interesting results for second order boundary value problems can be found in [1, 4, 9, 13]. Semipositone boundary value problems of higher order have been studied in [2, 6, 11, 12] just to name a few. It seems, however, that resonant semipositone problems for ordinary differential equations have not been studied as extensively as their "invertible" counterparts. Nkashama and Santanilla [8] obtained nonpositive and nonnegative solutions of the Neumann problem using generalized Ambrosetti-Prodi conditions. Since we are unaware of results based on cone-theoretic methods, we believe that our study of the Neumann problem provides new results. We only treat the most basic case of (1.1) with a continuous right side.

2 Properties of Green's Function

As a first step, we introduce $g(t,z) = f(t,z) + \alpha^2 z$ to transform (1.1) into

$$-u''(t) + \alpha^2 u(t) = g\big(t, u(t)\big), \quad t \in (0,1), \tag{2.1}$$

which we consider together with the boundary condition (1.2).

For $\beta \in C[0,1]$, the differential equation

$$-u''(t) + \alpha^2 u(t) = \beta(t), \quad 0 < t < 1,$$

satisfying the boundary condition (1.2) has a unique solution

$$u_0(t) = \int_0^1 G(t,s)\beta(s)\,ds \tag{2.2}$$

with the Green function

$$G(t,s) = \frac{1}{\alpha \sinh \alpha} \begin{cases} \cosh \alpha(1-t) \cosh \alpha s, & 0 \leq s \leq t \leq 1, \\ \cosh \alpha t \cosh \alpha(1-s), & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.3}$$

It is obvious that

$$G(t,s) \le G(s,s), \quad (t,s) \in [0,1] \times [0,1].$$

If $s \le t$, then

$$\begin{aligned} G(t,s) &= \frac{1}{\alpha \sinh \alpha} \cosh \alpha(1-t) \cosh \alpha s \\ &\ge \frac{1}{\alpha \sinh \alpha} \cosh \alpha(1-t) \cosh \alpha s \, \frac{\cosh \alpha(1-s)}{\cosh \alpha} \\ &\ge \frac{\cosh \alpha(1-t)}{\cosh \alpha} G(s,s). \end{aligned}$$

Similarly, for $t \le s$,

$$G(t,s) \ge \frac{\cosh \alpha t}{\cosh \alpha} \, G(s,s).$$

Combining the inequalities above, we obtain

$$q(t) G(s,s) \le G(t,s) \le G(s,s), \quad (t,s) \in [0,1] \times [0,1], \tag{2.4}$$

where

$$q(t) = \frac{1}{\cosh \alpha} \min\big\{\cosh \alpha t, \cosh \alpha(1-t)\big\}. \tag{2.5}$$

Also,

$$L = \max_{t \in [0,1]} \int_0^1 G(t,s)\, ds = \frac{1}{\alpha^2}. \tag{2.6}$$

For $0 < \gamma < 1/2$,

$$\int_\gamma^{1-\gamma} G(1-t,s)\, ds = \int_\gamma^{1-\gamma} G(1-t,1-s)\, ds = \int_\gamma^{1-\gamma} G(t,s)\, ds.$$

It suffices to consider

$$\begin{aligned} &\int_\gamma^{1-\gamma} G(t,s)\, ds \\ &\quad = \frac{1}{\alpha^2 \sinh \alpha} \begin{cases} (\sinh \alpha(1-\gamma) - \sinh \alpha\gamma) \cosh \alpha t, & 0 \le t \le \gamma, \\ \sinh \alpha - \sinh \alpha\gamma (\cosh \alpha t + \cosh \alpha(1-t)), & \gamma \le t \le 1/2, \end{cases} \end{aligned}$$

for $t \in [0, 1/2]$, since the above function is symmetric about $t = 1/2$. Since it is increasing in $[0,1/2]$,

$$C = \max_{t \in [0,1]} \int_\gamma^{1-\gamma} G(t,s)\, ds = \frac{1}{\alpha^2 \sinh \alpha} (\sinh \alpha - 2 \sinh \alpha\gamma \cosh \alpha/2). \tag{2.7}$$

Lemma 1. *Let $\beta \in C[0,1]$ and $\beta(t) \ge 0$ in $[0,1]$, $\beta(\tau) > 0$ for some $\tau \in [0,1]$. Then the inequality*

$$q(t) \ge \mu u_0(t), \quad t \in [0,1], \tag{2.8}$$

holds for

$$\mu = \frac{\alpha \sinh \alpha}{\cosh^2 \alpha \int_0^1 \beta(s)\, ds}. \tag{2.9}$$

Proof. Note that

$$u_0(t) = \int_0^1 G(t,s)\beta(s)\,ds \le G(t,t)\int_0^1 \beta(s)\,ds.$$

Hence

$$\begin{aligned} q(t) &= \frac{1}{\cosh\alpha}\min\bigl\{\cosh\alpha t, \cosh\alpha(1-t)\bigr\} \\ &\ge \min\left\{\frac{\cosh\alpha t}{\cosh\alpha}, \frac{\cosh\alpha(1-t)}{\cosh\alpha}\right\}\frac{1}{\cosh\alpha}\max\bigl\{\cosh\alpha t, \cosh\alpha(1-t)\bigr\} \\ &= \frac{1}{\cosh^2\alpha}\cosh\alpha t\cosh\alpha(1-t) = \frac{\alpha\sinh\alpha}{\cosh^2\alpha}G(t,t) \\ &= \mu G(t,t)\int_0^1 \beta(s)\,ds \ge \mu u_0(t) \end{aligned}$$

for all $t \in [0,1]$. □

Suppose that the function f in (1.1) satisfies

(A) $f \in C([0,1] \times \mathbf{R}_+, \mathbf{R})$;

(B) there exists a function $\beta \in C[0,1]$, $\beta(t) \ge 0$ in $[0,1]$, $\beta(\tau) > 0$ for some $\tau \in [0,1]$, and $\alpha \in \mathbf{R}$, $\alpha \ne 0$, such that

$$f(t,z) + \alpha^2 z + \beta(t) \ge 0, \quad (t,z) \in [0,1] \times \mathbf{R}_+.$$

We turn our attention to the equation

$$-v''(t) + \alpha^2 v(t) = f_p\bigl(t, v(t) - u_0(t)\bigr), \quad t \in (0,1), \tag{2.10}$$

where

$$f_p(t,z) = \begin{cases} f(t,z) + \alpha^2 z + \beta(t), & (t,z) \in [0,1] \times (0,\infty), \\ f(t,0) + \beta(t), & (t,z) \in [0,1] \times (-\infty,0], \end{cases}$$

and impose the boundary conditions (1.2).

Definition 1. A positive solution of the boundary value problem (1.1), (1.2) is a function $u \in C^2[0,1]$ satisfying (1.1), (1.2) and such that $u(t) > 0$ in $[0,1]$.

The next lemma discusses the relationship between the problems (1.1), (1.2) and (2.10), (1.2) by means of a "shift" $u \mapsto u + u_0$ applied to the equation (2.1).

Lemma 2. *The function u is a positive solution of the boundary value problem* (1.1), (1.2) *if and only if the function $v = u + u_0$, where u_0 is given by* (2.2), *is a solution of the boundary value problem* (2.10), (1.2) *satisfying $v(t) > u_0(t)$ in* $(0,1)$.

In the Banach space $\mathcal{B} = C[0,1]$ endowed with usual max-norm, we consider the operator

$$Tv(t) = \int_0^1 G(t,s) f_p\big(s, v(s) - u_0(s)\big)\, ds, \tag{2.11}$$

where $G(t,s)$ is given by (2.3). By (A), $T : \mathcal{B} \to \mathcal{B}$ is completely continuous.

Using the function q defined by (2.5), we introduce the cone

$$\mathcal{C} = \big\{ v \in \mathcal{B} : v(t) \geq q(t)\|v\|,\ t \in [0,1] \big\}.$$

By (2.4), $T : \mathcal{C} \to \mathcal{C}$. One can easily confirm that a fixed point of T in $\mathcal{C}$ is a solution of (2.10), (1.2), and conversely. In particular, for $0 < \gamma < 1/2$,

$$v(t) \geq \rho\|v\|, \quad t \in [\gamma, 1-\gamma], \tag{2.12}$$

where

$$\rho = \min_{t \in [\gamma, 1-\gamma]} q(t) = \frac{\cosh \alpha\gamma}{\cosh \alpha}.$$

The following is a fixed point theorem due to Guo and Krasnosel'skiĭ.

Theorem 1. **[3]** *Let $\mathcal{B}$ be a Banach space and let $\mathcal{C} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume that Ω_1, Ω_2 are open with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$T \colon \mathcal{C} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{C}$$

be a completely continuous operator such that either

(i) $\|Tu\| \leq \|u\|,\ u \in \mathcal{C} \cap \partial\Omega_1,$ *and* $\|Tu\| \geq \|u\|,\ u \in \mathcal{C} \cap \partial\Omega_2,$ *or*

(ii) $\|Tu\| \geq \|u\|,\ u \in \mathcal{C} \cap \partial\Omega_1,$ *and* $\|Tu\| \leq \|u\|,\ u \in \mathcal{C} \cap \partial\Omega_2.$

Then T has a fixed point in $\mathcal{C} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3 Positive Solutions

To make use of Theorem 1, we introduce, following [11], the "height" functions $\phi, \psi : \mathbf{R}_+ \to \mathbf{R}_+$ defined by

$$\phi(r) = \max\big\{ f_p\big(t, z - u_0(t)\big) : t \in [0,1],\ z \in [0,r] \big\}$$
$$\psi(r) = \min\big\{ f_p\big(t, z - u_0(t)\big) : t \in [\gamma, 1-\gamma],\ z \in [\rho r, r] \big\}, \quad 0 < \gamma < 1/2.$$

We present our main results.

Theorem 2. *Assume that* (A) *and* (B) *hold. Suppose that there exist $r, R > 0$ such that $\frac{1}{\mu} < r < R$, where $\mu > 0$ satisfies* (2.8), (2.9), *and*

(C) $\phi(r) \leq \alpha^2 r$ *and* $\psi(R) \geq \dfrac{\alpha^2 \sinh \alpha}{\sinh \alpha - 2\sinh \alpha\gamma \cosh \alpha/2} R.$

Then the boundary value problem (1.1), (1.2) *has at least one positive solution.*

Proof. Let

$$\Omega_1 = \{v \in \mathcal{B} : \|v\| < r\} \quad \text{and} \quad \Omega_2 = \{v \in \mathcal{B} : \|v\| < R\}.$$

For $v \in \mathcal{C} \cap \partial\Omega_1$, by Lemma 1, we have

$$v(s) - u_0(s) \geq q(s)\|v\| - u_0(s) \geq (\mu r - 1)u_0(s) > 0, \ s \in [0,1].$$

This implies that $f_p(s, v(s) - u_0(s)) \leq \phi(r)$, for $s \in [0,1]$, $0 \leq v(s) \leq r$. Thus, by (2.6) and (C),

$$\begin{aligned}
\|Tv\| &= \max_{t\in[0,1]} \int_0^1 G(t,s) f_p\big(s, v(s) - u_0(s)\big)\, ds \\
&\leq \max_{t\in[0,1]} \int_0^1 G(t,s)\, ds\, \phi(r) = L\phi(r) \\
&= \frac{1}{\alpha^2}\phi(r) \leq r.
\end{aligned}$$

That is, $\|Tv\| \leq \|v\|$ for all $v \in \mathcal{C} \cap \partial\Omega_1$.

Let $v \in \mathcal{C} \cap \partial\Omega_2$. Since $R > r$, we have $v(s) - u_0(s) \geq (\mu R - 1)u_0(s) \geq 0$, $s \in [0,1]$. Then, for all $s \in [\alpha, 1-\alpha]$, we have, recalling (2.12),

$$R \geq v(s) \geq q(s)\|v\| \geq \rho R.$$

Hence $f_p(s, v(s) - u_0(s)) \geq \psi(R)$, for $s \in [\gamma, 1-\gamma]$, $\gamma R \leq v(s) \leq R$. Then, by (2.7) and (C),

$$\begin{aligned}
\|Tv\| &= \max_{t\in[0,1]} \int_0^1 G(t,s) f_p\big(s, v(s) - u_0(s)\big)\, ds \\
&\geq \max_{t\in[0,1]} \int_\gamma^{1-\gamma} G(t,s) f_p\big(s, v(s) - u_0(s)\big)\, ds \\
&\geq \max_{t\in[0,1]} \int_\gamma^{1-\gamma} G(t,s)\, ds\, \psi(R) = C\psi(R) \\
&= \frac{1}{\alpha^2 \sinh\alpha}\left(\sinh\alpha - 2\sinh\alpha\gamma \cosh\alpha/2\right)\psi(R) \geq R.
\end{aligned}$$

That is, $\|Tv\| \geq \|v\|$ for all $v \in \mathcal{C} \cap \partial\Omega_2$.

By Theorem 1, there exists a fixed point $v_0 \in \mathcal{C}$ of (2.11), which, equivalently, is a positive solution of the positone problem (2.10), (1.2). Moreover, $u(t) = v_0(t) - u_0(t) \geq (\mu r - 1)u_0(t) > 0$ in $[0,1]$. By Lemma 2, u is a positive solution of the sign-changing problem (1.1), (1.2). □

The next result can be shown along similar lines.

Theorem 3. *Assume that* (A) *and* (B) *hold. Suppose that there exist* $r, R > 0$ *such that* $\frac{1}{\mu} < r < R$, *where* $\mu > 0$ *satisfies* (2.8), (2.9), *and*

(D) $\phi(R) \leq \alpha^2 R$ *and* $\psi(r) \geq \dfrac{\alpha^2 \sinh\alpha}{\sinh\alpha - 2\sinh\alpha\gamma\cosh\alpha/2}\, r.$

Then the boundary value problem (1.1), (1.2) *has at least one positive solution.*

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