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Approximation of the Sections of the Set of Trajectories of the Control System Described by a Nonlinear Volterra Integral Equation

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Abstract. Approximation of the sections of the set of trajectories of the control system described by a nonlinear Volterra integral equation is studied. The admissible control functions are chosen from the closed ball of the space L_p, $p > 1$, with radius μ and centered at the origin. The set of admissible control functions is replaced by the set of control functions, which includes a finite number of control functions and generates a finite number of trajectories. It is proved that the sections of the set of trajectories can be approximated by the sections of the set of trajectories, generated by a finite number of control functions.

Keywords: nonlinear Volterra integral equation, control system, integral constraint, set of trajectories, approximation method.

AMS Subject Classification: 93C23; 45D05.

1 Introduction

Control systems arise in various fields of theory and applications. Depending on the type of equation which describes the system's behavior, control systems can be classified as linear or nonlinear control systems, control systems described by ordinary differential equations or control systems described by partial differential equations or control systems described by integral equations and etc. Control systems can also be classified by the character of control functions, say as control systems with geometric constraints on the controls, control systems with integral constraints on the control functions and control

systems with mixed type constraints on the control functions. In general, geometric constraints on the controls arise in cases, where control resource is not exhausted by consumption. But in the case, when control resource is exhausted by consumption, then integral constraint on the control functions is inevitable. These kinds of controls take place, if control resource is energy, fuel, finance, food and etc. (see, e.g. [5,7,16,20,24] and references therein). For example, the mathematical model of flying objects with rapidly changing mass is described as a control system with integral constraint on controls (see, [7, 16, 26]).

The theory of control systems with geometric constraints on the controls is a well investigated part of the control system's theory. The problems of existence of the optimal trajectories, controllability of the control systems, necessary and sufficient conditions for optimality of given processes, numerical methods for calculation of optimal trajectories and attainable sets of control systems described by an ordinary differential equation are studied in numerous papers (see, e.g. [6,9,12,17,18,23] and references therein). The same problems for the systems described by a Volterra type integral equation are considered in papers [1, 2, 3, 21, 25, 29] (see, also the references in these papers). Various properties, including numerical methods for construction of the attainable sets of the control systems described by an ordinary differential equations with integral constraints on the controls are studied in papers [5, 7, 10, 11, 16, 19, 20, 24]. Note that integral constraint on the controls is essentially different than geometric constraint. The systems with geometric constraint on the controls can be reduced to the suitable, differential or integral inclusion (see, [6,9,12,17]). However, the same technic is not always applicable for the system with integral constraint on the controls, since integral boundedness of control functions does not guarantee the geometric boundedness of the controls, and therefore in the investigation of the control systems with integral constraints on the controls arise additional difficulties.

The mathematical models of many processes in mechanics, physics, economy, biology and etc. are described by integral equations. The Volterra integral equations are an important tool for description of mathematical models of various types of complex processes (see, e.g. [4, 8, 15, 22, 27, 28] and references therein). Some properties of the set of trajectories of the control systems described by a nonlinear Volterra integral equation with integral constraint on the controls are discussed in [13] and [14]. Precompactness of the set of trajectories of the control systems described by a nonlinear Volterra integral equation with integral constraint on the control functions is considered in [14]. In [13] an approximation of the sections of the set of trajectories of the control systems described by a nonlinear Volterra integral equation with integral constraint on the controls is investigated, where the sections of the set of trajectories is approximated by the sections of the set of trajectories generated by the compact set of control functions. In this paper, developing the result obtained in [13], the sections of the set of trajectories are approximated by the sections of the set of trajectories which consists of a finite number of trajectories.

The paper is organized as follows. In Section 2 the basic conditions are formulated which satisfy the system (Conditions 2.A, 2.B and 2.C). The known result from [13] is given which is used in the following arguments (Theorem 1).

In Section 3, the sections of the set of trajectories are approximated by the sections of the set of trajectories, generated by the set of control functions which are piecewise constant and have mixed, i.e. integral and geometric constraints (Theorem 2).

In Section 4, an evaluation for the Hausdorff distance between the set of trajectories, generated by the piecewise constant and mixed constrained control functions and the set of trajectories, generated by the mixed constrained and piecewise constant control functions the norm of which are the node points of the given uniform mesh, is obtained (Proposition 3).

In Section 5, a new class of control functions is introduced which consists of a finite number of control functions. These control functions generate the set of trajectories consisting of a finite number of trajectories. The Hausdorff distance between the set of trajectories generated by the mixed constrained and piecewise constant control functions the norm of which are the node points of the given uniform mesh, and the set of trajectories consisting of a finite number of trajectories is evaluated (Proposition 4).

In Section 6, the main result of the paper is given. It is proved that the sections of the set of trajectories of the system can be approximated by the sections of the set of trajectories, which consists of a finite number of trajectories (Theorem 3).

The Hausdorff distance between the sets $E \subset \mathbb{R}^n$ and $F \subset \mathbb{R}^n$ is denoted by $h_n(E,F)$, where $\mathbb{R}^n$ is n-dimensional Euclidean space. The Hausdorff distance between the sets $G \subset C([a,b];\mathbb{R}^n)$ and $W \subset C([a,b];\mathbb{R}^n)$ is denoted by $h_C(G,W)$, where $C([a,b];\mathbb{R}^n)$ is the space of continuous functions $x(\cdot):[a,b]\to\mathbb{R}^n$ with norm $\|x(\cdot)\|_C = \max\{\|x(\xi)\| : \xi \in [a,b]\}$ and $\|x\|$ is the Euclidean norm of $x \in \mathbb{R}^n$. We set

$$B_n = \big\{x \in \mathbb{R}^n : \|x\| \le 1\big\}, \qquad B_C = \big\{x(\cdot) \in C\big([a,b];\mathbb{R}^n\big) : \big\|x(\cdot)\big\|_C \le 1\big\}.$$

2 Preliminaries

The control system described by a nonlinear Volterra integral equation

$$x(\xi) = f\big(\xi, x(\xi)\big) + \lambda \int_a^{\xi} K\big(\xi, s, x(s), u(s)\big)ds \tag{2.1}$$

is considered, where $x(s) \in \mathbb{R}^n$ is the state vector of the system, $u(s) \in \mathbb{R}^m$ is the control vector, $\xi \in [a,b]$, $\lambda \ge 0$ is a real number.

For given $p > 1$ and $\mu > 0$ we set

$$U_{p,\mu} = \big\{u(\cdot) \in L_p\big([a,b];\mathbb{R}^m\big) : \big\|u(\cdot)\big\|_p \le \mu\big\},$$

where $\|u(\cdot)\|_p = (\int_a^b \|u(s)\|^p\, ds)^{\frac{1}{p}}$, $L_p([a,b];\mathbb{R}^m)$ is the space of Lebesgue measurable functions $u(\cdot):[a,b]\to\mathbb{R}^m$ such that $\|u(\cdot)\|_p < \infty$. The set $U_{p,\mu} \subset L_p([a,b];\mathbb{R}^m)$ is called the set of admissible control functions and every function $u(\cdot) \in U_{p,\mu}$ is called admissible control function.

From now on, it will be assumed that the functions $f(\cdot):[a,b]\times\mathbb{R}^n\to\mathbb{R}^n$, $K(\cdot):[a,b]\times[a,b]\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$ and number $\lambda \in [0,\infty)$ given in equation (2.1) satisfy the following conditions:

2.A. The functions $f(\cdot) : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ and $K(\cdot) : [a,b] \times [a,b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are continuous;

2.B. There exist $L_0 \in [0,1)$, $L_1 \geq 0$, $H_1 \geq 0$, $L_2 \geq 0$, $H_2 \geq 0$, $L_3 \geq 0$ and $H_3 \geq 0$ such that

$$\big\|f(\xi,x_1) - f(\xi,x_2)\big\| \leq L_0 \|x_1 - x_2\|$$

for every $(\xi,x_1) \in [a,b] \times \mathbb{R}^n$, $(\xi,x_2) \in [a,b] \times \mathbb{R}^n$, and

$$\begin{aligned}\big\|K(\xi_1,s,x_1,u_1) - K(\xi_2,s,x_2,u_2)\big\| &\leq \big[L_1 + H_1\big(\|u_1\| + \|u_2\|\big)\big]\,|\xi_1 - \xi_2| \\ &+ \big[L_2 + H_2\big(\|u_1\| + \|u_2\|\big)\big]\,\|x_1 - x_2\| + \big[L_3 + H_3\big(\|x_1\| + \|x_2\|\big)\big]\,\|u_1 - u_2\|\end{aligned}$$

for every $(\xi_1,s,x_1,u_1) \in [a,b] \times [a,b] \times \mathbb{R}^n \times \mathbb{R}^m$, $(\xi_2,s,x_2,u_2) \in [a,b] \times [a,b] \times \mathbb{R}^n \times \mathbb{R}^m$;

2.C. $0 \leq \lambda(L_2\,(b-a) + 2H_2(b-a)^{\frac{p-1}{p}}\mu) < 1 - L_0$.

Note that if the function $K(\cdot) : [a,b] \times [a,b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz continuous, then it satisfies the conditions 2.A and 2.B.

Let us define the trajectory of the system (2.1) generated by an admissible control function $u(\cdot) \in U_{p,\mu}$. Let $u_*(\cdot) \in U_{p,\mu}$. A continuous function $x_*(\cdot) : [a,b] \to \mathbb{R}^n$ satisfying the equation

$$x_*(\xi) = f\big(\xi, x_*(\xi)\big) + \lambda \int_a^{\xi} K\big(\xi, s, x_*(s), u_*(s)\big)ds$$

for every $\xi \in [a,b]$ is said to be a trajectory of the system (2.1) generated by the admissible control function $u_*(\cdot) \in U_{p,\mu}$.

The conditions 2.A–2.C guarantee that every admissible control function generates a unique trajectory of the system (2.1) (see [14]). We denote by $\mathbf{X}_{p,\mu}$ the set of all trajectories of the system (2.1) generated by all admissible control functions $u(\cdot) \in U_{p,\mu}$. The set $\mathbf{X}_{p,\mu}$ is called the set of trajectories of the system (2.1).

For each fixed $\xi \in [a,b]$ we set

$$\mathbf{X}_{p,\mu}(\xi) = \big\{x(\xi) \in \mathbb{R}^n : x(\cdot) \in \mathbf{X}_{p,\mu}\big\}. \tag{2.2}$$

According to [14] the set of trajectories $\mathbf{X}_{p,\mu}$ is a precompact subset of the space $C\left([a,b];\mathbb{R}^n\right)$, and hence there exists $r_* > 0$ such that

$$\big\|x(\cdot)\big\|_C \leq r_* \tag{2.3}$$

for every $x(\cdot) \in \mathbf{X}_{p,\mu}$. Denote

$$R_* = \frac{\lambda\,(L_3 + 2r_*H_3)}{1 - L_0} \cdot \exp\left[\frac{\lambda(L_2(b-a) + 2H_2\mu\,(b-a)^{\frac{p-1}{p}})}{1 - L_0}\right], \tag{2.4}$$

where r_* is defined by (2.3).

The validity of the following proposition follows from conditions 2.A and 2.B.

Proposition 1. *Let $x(\cdot) \in \mathbf{X}_{p,\mu}$ and $x_*(\cdot) \in \mathbf{X}_{p,\mu}$ be arbitrary trajectories of the system (2.1) generated by the admissible control functions $u(\cdot) \in U_{p,\mu}$ and $u_*(\cdot) \in U_{p,\mu}$ respectively. Then*

$$\big\|x(\xi) - x_*(\xi)\big\| \le R_* \int_a^{\xi} \big\|u(s) - u_*(s)\big\| ds$$

for every $\xi \in [a,b]$.

For given $H > 0$, $K > 0$ we denote

$$U_{p,\mu}^{H} = \big\{u(\cdot) \in U_{p,\mu} : \big\|u(\xi)\big\| \le H \text{ for every } \xi \in [a,b]\big\},$$
$$U_{p,\mu}^{H,lip,K} = \big\{u(\cdot) \in U_{p,\mu}^{H} : u(\cdot) : [a,b] \to \mathbb{R}^m \text{ is Lipschitz continuous and Lipschitz's constant is not greater than } K\big\},$$

and let $\mathbf{X}_{p,\mu}^{H,lip,K}$ be the set of all trajectories of the system (2.1) generated by all control functions $u(\cdot) \in U_{p,\mu}^{H,lip,K}$. For given $\xi \in [a,b]$ we set

$$\mathbf{X}_{p,\mu}^{H,lip,K}(\xi) = \big\{x(\xi) \in \mathbb{R}^n : x(\cdot) \in \mathbf{X}_{p,\mu}^{H,lip,K}\big\}. \tag{2.5}$$

Theorem 1. **[13]** *For each $\varepsilon > 0$ there exist $H(\varepsilon) > 0$ and $K_*(\varepsilon) = K_*\big(\varepsilon, H(\varepsilon)\big) > 0$ such that for every $K > K_*(\varepsilon)$ the inequality*

$$h_n\big(\mathbf{X}_{p,\mu}(\xi), \mathbf{X}_{p,\mu}^{H(\varepsilon),lip,K}(\xi)\big) < \frac{\varepsilon}{4}$$

is verified for every $\xi \in [a,b]$, where $\mathbf{X}_{p,\mu}(\xi)$ is defined by (2.2).

3 The Set of Trajectories Generated by the Piecewise Constant Control Functions

Let $\Gamma = \{a = \xi_0, \xi_1, \dots, \xi_N = b\}$ be a uniform partition of the closed interval $[a,b]$ and $\Delta = \xi_{i+1} - \xi_i$, $i = 0, 1, \dots, N-1$. For given $H > 0$ and $K > 0$ we set

$$U_{p,\mu}^{H,\Gamma} = \big\{u(\cdot) \in U_{p,\mu}^{H} : u(\xi) = u_i,\ \xi \in [\xi_i, \xi_{i+1}),\ i = 0, 1, \dots, N-1\big\},$$
$$V_{p,\mu}^{H,\Gamma,K} = \big\{u(\cdot) \in U_{p,\mu}^{H} : u(\xi) = u_i,\ \xi \in [\xi_i, \xi_{i+1}),\ i = 0, 1, \dots, N-1,$$
$$\|u_{i+1} - u_i\| \le K\Delta,\ i = 0, 1, \dots, N-2\big\}$$

and let $\mathbf{X}_{p,\mu}^{H,\Gamma}$ and $\mathbf{Z}_{p,\mu}^{H,\Gamma,K}$ be the set of trajectories of the system (2.1) generated by the control functions $u(\cdot) \in U_{p,\mu}^{H,\Gamma}$ and $v(\cdot) \in V_{p,\mu}^{H,\Gamma,K}$ respectively. For given $\xi \in [a,b]$ we denote

$$\mathbf{X}_{p,\mu}^{H,\Gamma}(\xi) = \big\{x(\xi) \in \mathbb{R}^n : x(\cdot) \in \mathbf{X}_{p,\mu}^{H,\Gamma}\big\}, \tag{3.1}$$
$$\mathbf{Z}_{p,\mu}^{H,\Gamma,K}(\xi) = \big\{x(\xi) \in \mathbb{R}^n : x(\cdot) \in \mathbf{Z}_{p,\mu}^{H,\Gamma,K}\big\}. \tag{3.2}$$

Since $V_{p,\mu}^{H,\Gamma,K} \subset U_{p,\mu}^{H,\Gamma}$ then we obtain that

$$\mathbf{Z}_{p,\mu}^{H,\Gamma,K} \subset \mathbf{X}_{p,\mu}^{H,\Gamma} \tag{3.3}$$

for every $H > 0$, $K = 1, 2, \dots$ and uniform partition Γ of the closed interval $[a,b]$.

Proposition 2. *For every $H > 0$, $K > 0$ and uniform partition $\Gamma = \{a = \xi_0, \xi_1, \dots, \xi_N = b\}$ of the interval $[a, b]$ the inclusion*

$$\mathbf{X}_{p,\mu}^{H,lip,K} \subset \mathbf{Z}_{p,\mu}^{H,\Gamma,K} + R_* K (b-a) \Delta \cdot B_C$$

holds, and hence

$$\mathbf{X}_{p,\mu}^{H,lip,K}(\xi) \subset \mathbf{Z}_{p,\mu}^{H,\Gamma,K}(\xi) + R_* K (b-a) \Delta \cdot B_n$$

for every $\xi \in [a, b]$, where R_* is defined by (2.4), $\mathbf{X}_{p,\mu}^{H,lip,K}(\xi)$ *and* $\mathbf{Z}_{p,\mu}^{H,\Gamma,K}(\xi)$ *are defined by* (2.5) *and* (3.2) *respectively, $\Delta = \xi_{i+1} - \xi_i$, $i = 0, 1, \dots, N-1$.*

Proof. Let us choose an arbitrary $x(\cdot) \in \mathbf{X}_{p,\mu}^{H,lip,K}$ and let $u(\cdot) \in U_{p,\mu}^{H,lip,K}$ be the control function, generating $x(\cdot)$. We define new control function $u_*(\cdot) : [a, b] \to \mathbb{R}^m$, setting

$$\begin{aligned} u_*(\xi) &= \frac{1}{\Delta} \int_{\xi_i}^{\xi_{i+1}} u(s) ds, \quad \xi \in [\xi_i, \xi_{i+1}),\ i = 0, 1, \dots, N-1, \\ u_*(\xi_N) &= u_*(\xi_{N-1}). \end{aligned} \tag{3.4}$$

It is obvious, that $\|u_*(\xi)\| \le H$ for every $\xi \in [a, b]$. Let $\xi \in [\xi_i, \xi_{i+1})$, $i = 0, 1, \dots, N-1$. It follows from (3.4) and Hölder's inequality that

$$\big\|u_*(\xi)\big\| \le \frac{1}{\Delta} \int_{\xi_i}^{\xi_{i+1}} \big\|u(s)\big\| ds \le \frac{1}{\Delta^{\frac{1}{p}}} \left(\int_{\xi_i}^{\xi_{i+1}} \big\|u(s)\big\|^p ds \right)^{\frac{1}{p}}$$

and hence

$$\int_{\xi_i}^{\xi_{i+1}} \big\|u_*(s)\big\|^p ds \le \int_{\xi_i}^{\xi_{i+1}} \big\|u(s)\big\|^p ds.$$

Since the last inequality is satisfied for every $i = 0, 1, \dots, N-1$ and $u(\cdot) \in U_{p,\mu}^{H,lip,K} \subset U_{p,\mu}$ then, we have

$$\int_a^b \big\|u_*(s)\big\|^p ds \le \int_a^b \big\|u(s)\big\|^p ds \le \mu^p,$$

which means that $u_*(\cdot) \in U_{p,\mu}$. Since $\|u_*(\xi)\| \le H$ for every $\xi \in [a, b]$, we conclude that $u_*(\cdot) \in U_{p,\mu}^H$.

Since $u(\cdot) \in U_{p,\mu}^{H,lip,K}$, then $\|u(\xi_*) - u(\xi^*)\| \le K |\xi_* - \xi^*|$ for every $\xi_* \in [a, b]$ and $\xi^* \in [a, b]$. Let $i < N-1$. According to (3.4) we obtain

$$\begin{aligned} \big\|u_*(\xi_{i+1}) - u_*(\xi_i)\big\| &= \left\| \frac{1}{\Delta} \int_{\xi_{i+1}}^{\xi_{i+2}} u(s) ds - \frac{1}{\Delta} \int_{\xi_i}^{\xi_{i+1}} u(s) ds \right\| \\ &\le \frac{1}{\Delta} \int_{\xi_i}^{\xi_{i+1}} \big\|u(s+\Delta) - u(s)\big\| ds \le K\Delta. \end{aligned} \tag{3.5}$$

Since $u_*(\cdot) \in U^H_{p,\mu}$, then from (3.4) and (3.5) it follows that $u_*(\cdot) \in V^{H,\Gamma,K}_{p,\mu}$. Let $x_*(\cdot)$ be the trajectory of the system (2.1) generated by the control function $u_*(\cdot) \in V^{H,\Gamma,K}_{p,\mu}$. Then, $x_*(\cdot) \in \mathbf{Z}^{H,\Gamma,K}_{p,\mu}$ and by virtue of Proposition 1 we have

$$\big\|x(\xi) - x_*(\xi)\big\| \le R_* \int_a^{\xi} \big\|u(s) - u_*(s)\big\| ds \tag{3.6}$$

for every $\xi \in [a,b]$, where R_* is defined by (2.4).

Let us choose an arbitrary $\xi \in [a,b)$. Then there exists $i = 0,1,\ldots,N-1$ such that $\xi \in [\xi_i, \xi_{i+1})$. Lipschitz continuity of the control function $u(\cdot)$ and (3.4) imply that

$$\begin{aligned}\big\|u(\xi) - u_*(\xi)\big\| &= \frac{1}{\Delta}\left\| \int_{\xi_i}^{\xi_{i+1}} u(\xi) ds - \int_{\xi_i}^{\xi_{i+1}} u(s) ds \right\| \\ &\le \frac{1}{\Delta} \int_{\xi_i}^{\xi_{i+1}} \big\|u(\xi) - u(s)\big\| ds \le \frac{1}{\Delta} K \int_{\xi_i}^{\xi_{i+1}} |\xi - s|\, ds \le K\Delta. \end{aligned} \tag{3.7}$$

Since $\xi \in [a,b)$ is arbitrarily chosen, we obtain from (3.6) and (3.7) that

$$\big\|x(\xi) - x_*(\xi)\big\| \le R_* K (b-a) \Delta$$

for every $\xi \in [a,b]$ and consequently

$$\big\|x(\cdot) - x_*(\cdot)\big\|_C \le R_* K (b-a) \Delta, \tag{3.8}$$

where R_* is defined by (2.4). Thus, for arbitrarily chosen $x(\cdot) \in \mathbf{X}^{H,lip,K}_{p,\mu}$ it is possible to define $x_*(\cdot) \in \mathbf{Z}^{H,\Gamma,K}_{p,\mu}$ such that the inequality (3.8) is verified. This completes the proof. □

The following theorem characterizes the Hausdorff distance between the sections of sets of trajectories $\mathbf{X}_{p,\mu}$ and $\mathbf{X}^{H,\Gamma}_{p,\mu}$.

Theorem 2. *For each $\varepsilon > 0$ there exist $H(\varepsilon) > 0$ and $\Delta_*(\varepsilon) > 0$ such that for every uniform partition Γ of the closed interval $[a,b]$, where $\Delta < \Delta_*(\varepsilon)$, the inequality*

$$h_n\big(\mathbf{X}_{p,\mu}(\xi), \mathbf{X}^{H(\varepsilon),\Gamma}_{p,\mu}(\xi)\big) \le \frac{\varepsilon}{2}$$

is verified for every $\xi \in [a,b]$. Here Δ is the diameter of the partition Γ.

Proof. According to the Theorem 1 for given $\varepsilon > 0$ there exist $H(\varepsilon) > 0$ and $K(\varepsilon) > 0$ such that the inequality

$$h_n\big(\mathbf{X}_{p,\mu}(\xi), \mathbf{X}^{H(\varepsilon),lip,K(\varepsilon)}_{p,\mu}(\xi)\big) \le \frac{\varepsilon}{4} \tag{3.9}$$

is satisfied for every $\xi \in [a,b]$. By virtue of Proposition 2, for given $H(\varepsilon) > 0$, $K(\varepsilon) > 0$ and for every uniform partition $\Gamma = \{a = \xi_0, \xi_1, \ldots, \xi_N = b\}$ of the interval $[a,b]$ the inclusion

$$\mathbf{X}^{H(\varepsilon),lip,K(\varepsilon)}_{p,\mu}(\xi) \subset \mathbf{Z}^{H(\varepsilon),\Gamma,K(\varepsilon)}_{p,\mu}(\xi) + R_* K(\varepsilon)(b-a)\Delta \cdot B_n \tag{3.10}$$

holds for every $\xi \in [a,b]$, where $\Delta = \xi_{i+1} - \xi_i$, $i = 0,1,\dots,N-1$.

Let $\Delta_*(\varepsilon) = \frac{\varepsilon}{4R_*K(\varepsilon)(b-a)}$. (3.10) implies that if $\Delta < \Delta_*(\varepsilon)$, then

$$\mathbf{X}_{p,\mu}^{H(\varepsilon),lip,K(\varepsilon)}(\xi) \subset \mathbf{Z}_{p,\mu}^{H(\varepsilon),\Gamma,K(\varepsilon)}(\xi) + \frac{\varepsilon}{4}B_n \tag{3.11}$$

for every $\xi \in [a,b]$.

By virtue of (3.3) we have that for every uniform partition $\Gamma = \{a = \xi_0, \xi_1, \dots, \xi_N = b\}$ of the interval $[a,b]$ the inclusion

$$\mathbf{Z}_{p,\mu}^{H(\varepsilon),\Gamma,K(\varepsilon)}(\xi) \subset \mathbf{X}_{p,\mu}^{H(\varepsilon),\Gamma}(\xi) \tag{3.12}$$

is satisfied for every $\xi \in [a,b]$. (3.9), (3.11) and (3.12) imply that if $\Delta < \Delta_*(\varepsilon)$, then

$$\mathbf{X}_{p,\mu}(\xi) \subset \mathbf{X}_{p,\mu}^{H(\varepsilon),\Gamma}(\xi) + \frac{\varepsilon}{2}B_n \tag{3.13}$$

for every $\xi \in [a,b]$, where Δ is the diameter of the partition Γ. Since $\mathbf{X}_{p,\mu}^{H(\varepsilon),\Gamma}(\xi) \subset \mathbf{X}_{p,\mu}(\xi)$ for every $\xi \in [a,b]$, then (3.13) completes the proof. □

4 The Set of Control Functions with Norms from Uniform Mesh

Let $\Gamma = \{a = \xi_0, \xi_1, \dots, \xi_N = b\}$ be a uniform partition of the closed interval $[a,b]$, $H > 0$ and $\Lambda = \{0 = w_0, w_1, \dots, w_q = H\}$ be a uniform partition of the closed interval $[0,H]$, $\delta = w_{j+1} - w_j$, $j = 0,1,\dots,q-1$. Denote

$$U_{p,\mu}^{H,\Gamma,\Lambda} = \big\{u(\cdot) \in U_{p,\mu}^{H} : u(\xi) = u_i,\ \xi \in [\xi_i, \xi_{i+1}),\\ \|u_i\| \in \Lambda,\ i = 0,1,\dots,N-1\big\}.$$

The set of trajectories of the system (2.1) generated by the control functions $u(\cdot) \in U_{p,\mu}^{H,\Gamma,\Lambda}$ is denoted by $\mathbf{X}_{p,\mu}^{H,\Gamma,\Lambda}$. For given $\xi \in [a,b]$ we denote

$$\mathbf{X}_{p,\mu}^{H,\Gamma,\Lambda}(\xi) = \big\{x(\xi) \in \mathbb{R}^n : x(\cdot) \in \mathbf{X}_{p,\mu}^{H,\Gamma,\Lambda}\big\}. \tag{4.1}$$

Proposition 3. *For every $H > 0$, uniform partition $\Gamma = \{a = \xi_0, \xi_1, \dots, \xi_N = b\}$ of the interval $[a,b]$ and uniform partition $\Lambda = \{0 = w_0, w_1, \dots, w_q = H\}$ of the interval $[0,H]$ the inequality*

$$h_C\big(\mathbf{X}_{p,\mu}^{H,\Gamma}, \mathbf{X}_{p,\mu}^{H,\Gamma,\Lambda}\big) \le R_*(b-a)\delta$$

holds, and hence

$$h_n\big(\mathbf{X}_{p,\mu}^{H,\Gamma}(\xi), \mathbf{X}_{p,\mu}^{H,\Gamma,\Lambda}(\xi)\big) \le R_*(b-a)\delta$$

for every $\xi \in [a,b]$, where R_* is defined by (2.4), $\mathbf{X}_{p,\mu}^{H,\Gamma}(\xi)$ and $\mathbf{X}_{p,\mu}^{H,\Gamma,\Lambda}(\xi)$ are defined by (3.1) and (4.1) respectively, $\delta = w_{j+1} - w_j$, $j = 0,1,\dots,q-1$ is the diameter of the partition Λ.

Proof. Choose an arbitrary $x(\cdot) \in \mathbf{X}_{p,\mu}^{H,\Gamma}$ generated by the control function $u(\cdot) \in U_{p,\mu}^{H,\Gamma}$. Since $u(\cdot) \in U_{p,\mu}^{H,\Gamma}$, then $u(\xi) = u_i$ for every $\xi \in [\xi_i, \xi_{i+1})$, $i = 0, 1, \ldots, N-1$, and moreover

$$\Delta \sum_{i=0}^{N-1} \|u_i\|^p \leq \mu^p, \quad \|u_i\| \leq H \quad \text{for every } i = 0, 1, \ldots, N-1, \tag{4.2}$$

where $\Delta = \xi_{i+1} - \xi_i$, $i = 0, 1, \ldots, N-1$, is the diameter of the partition Γ. We assume that $u(b) = u(\xi_{N-1})$.

If $\|u_i\| < H$ $(i = 0, 1, \ldots, N-1)$, then there exists $w_{j_i} \in \Lambda$ such that

$$\|u_i\| \in [w_{j_i}, w_{j_i+1}). \tag{4.3}$$

Now, using control function $u(\cdot) \in U_{p,\mu}^{H,\Gamma}$ define a new control function $u_*(\cdot) : [a, b] \to \mathbb{R}^m$, setting

$$u_*(\xi) = \begin{cases} \dfrac{u_i}{\|u_i\|} w_{j_i}, & \text{if } 0 < \|u_i\| < H, \\ u_i, & \text{if } \|u_i\| = 0 \text{ or } \|u_i\| = H, \end{cases} \tag{4.4}$$

where $\xi \in [\xi_i, \xi_{i+1})$, $i = 0, 1, \ldots, N-1$, and $w_{j_i} \in \Lambda$ is defined by (4.3). In addition, we set that $u_*(b) = u_*(\xi_{N-1})$. It follows from (4.3) and (4.4) that $\|u_*(\xi)\| \leq \|u(\xi)\|$ for every $\xi \in [a, b]$, and hence (4.2) implies that $u_*(\cdot) \in U_{p,\mu}^{H,\Gamma,\Lambda}$. Let $x_*(\cdot)$ be the trajectory of the system (2.1) generated by the control function $u_*(\cdot)$. Then $x_*(\cdot) \in \mathbf{X}_{p,\mu}^{H,\Gamma,\Lambda}$ and according to the Proposition 1

$$\big\|x(\xi) - x_*(\xi)\big\| \leq R_* \int_a^\xi \big\|u(s) - u_*(s)\big\| ds \tag{4.5}$$

for every $\xi \in [a, b]$, where R_* is defined by (2.4).

By virtue of (4.3) and (4.4) it is not difficult to verify that

$$\big\|u(\xi) - u_*(\xi)\big\| \leq \delta \tag{4.6}$$

for every $\xi \in [a, b]$, where δ is the diameter of the partition Λ. (4.5) and (4.6) yield

$$\big\|x(\xi) - x_*(\xi)\big\| \leq R_*(b-a)\delta$$

for every $\xi \in [a, b]$, and consequently

$$\big\|x(\cdot) - x_*(\cdot)\big\|_C \leq R_*(b-a)\delta. \tag{4.7}$$

The inequality (4.7) implies that

$$\mathbf{X}_{p,\mu}^{H,\Gamma} \subset \mathbf{X}_{p,\mu}^{H,\Gamma,\Lambda} + R_*(b-a)\delta \cdot B_C. \tag{4.8}$$

The inclusion $\mathbf{X}_{p,\mu}^{H,\Gamma,\Lambda} \subset \mathbf{X}_{p,\mu}^{H,\Gamma}$ and (4.8) imply the proof of the theorem. □

5 Finite Number of Trajectories

Let $\sigma > 0$ be a given number and $S = \{u \in \mathbb{R}^m : \|u\| = 1\}$. Since $S \subset \mathbb{R}^m$ is a compact set, then it has a finite σ-net. Let $S_\sigma = \{s_1, s_2, \dots, s_g\}$ be a finite σ-net on S. We define new set of control functions, setting

$$U_{p,\mu}^{H,\Gamma,\Lambda,\sigma} = \big\{u(\cdot) \in U_{p,\mu}^{H,\Gamma,\Lambda} : u(\xi) = w_{j_i} s_{l_i},\ \xi \in [\xi_i, \xi_{i+1}),\ w_{j_i} \in \Lambda,\ s_{l_i} \in S_\sigma,\ i = 0, 1, \dots, N-1\big\}.$$

Note that the set of control functions $U_{p,\mu}^{H,\Gamma,\Lambda,\sigma}$ can be redefined as

$$U_{p,\mu}^{H,\Gamma,\Lambda,\sigma} = \Big\{u(\cdot) \in L_p\big([a,b]\,;\mathbb{R}^m\big) : u(\xi) = w_{j_i} s_{l_i},\ \xi \in [\xi_i, \xi_{i+1}),\ w_{j_i} \in \Lambda,$$
$$s_{l_i} \in S_\sigma,\ i = 0, 1, \dots, N-1,\ \Delta \sum_{i=0}^{N-1} w_{j_i}^p \le \mu^p\Big\},$$

where Δ is the diameter of the partition Γ.

It is obvious that the set $U_{p,\mu}^{H,\Gamma,\Lambda,\sigma}$ consists of a finite number of control functions. By $\mathbf{X}_{p,\mu}^{H,\Gamma,\Lambda,\sigma}$ we denote the set of trajectories of the system (2.1) generated by the control functions $u(\cdot) \in U_{p,\mu}^{H,\Gamma,\Lambda,\sigma}$. It is obvious that the set $\mathbf{X}_{p,\mu}^{H,\Gamma,\Lambda,\sigma}$ also consists of a finite number of trajectories. For given $\xi \in [a,b]$ we set

$$\mathbf{X}_{p,\mu}^{H,\Gamma,\Lambda,\sigma}(\xi) = \big\{x(\xi) \in \mathbb{R}^n : x(\cdot) \in \mathbf{X}_{p,\mu}^{H,\Gamma,\Lambda,\sigma}\big\}. \tag{5.1}$$

Proposition 4. *For every $H > 0$, uniform partition $\Gamma = \{a = \xi_0, \xi_1, \dots, \xi_N = b\}$ of the interval $[a,b]$, uniform partition $\Lambda = \{0 = w_0, w_1, \dots, w_q = H\}$ of the interval $[0,H]$ and σ-net S_σ, the inequality*

$$h_C\left(\mathbf{X}_{p,\mu}^{H,\Gamma,\Lambda}, \mathbf{X}_{p,\mu}^{H,\Gamma,\Lambda,\sigma}\right) \le R_* H (b-a)\sigma$$

holds, and hence

$$h_n\left(\mathbf{X}_{p,\mu}^{H,\Gamma,\Lambda}(\xi), \mathbf{X}_{p,\mu}^{H,\Gamma,\Lambda,\sigma}(\xi)\right) \le R_* H (b-a)\sigma$$

for every $\xi \in [a,b]$, where R_* is defined by (2.4), *$\mathbf{X}_{p,\mu}^{H,\Gamma,\Lambda}(\xi)$ and $\mathbf{X}_{p,\mu}^{H,\Gamma,\Lambda,\sigma}(\xi)$ are defined by* (4.1) *and* (5.1) *respectively.*

Proof. Let us choose an arbitrary $x(\cdot) \in \mathbf{X}_{p,\mu}^{H,\Gamma,\Lambda}$ generated by the control function $u(\cdot) \in U_{p,\mu}^{H,\Gamma,\Lambda}$. Since $u(\cdot) \in U_{p,\mu}^{H,\Gamma,\Lambda}$, then $\|u(\xi)\| = w_{j_i}$ for every $\xi \in [\xi_i, \xi_{i+1})$, $i = 0, 1, \dots, N-1$, $w_{j_i} \in \Lambda$, $\Delta \cdot \sum_{i=0}^{N-1} w_{j_i}^p \le \mu^p$, where $\Delta = \xi_{i+1} - \xi_i$, $i = 0, 1, \dots, N-1$, is the diameter of the partition Γ.

Since $\|u(\xi)\| = w_{j_i}$ for every $\xi \in [\xi_i, \xi_{i+1})$, $i = 0, 1, \dots, N-1$, then there exist $b_i \in S$, $i = 0, 1, \dots, N-1$, such that

$$u(\xi) = w_{j_i} b_i \quad \text{for every } \xi \in [\xi_i, \xi_{i+1}),\ i = 0, 1, \dots, N-1. \tag{5.2}$$

Since $b_i \in S$, $i = 0, 1, \dots, N-1$, then there exist $s_{l_i} \in S_\sigma$, $i = 0, 1, \dots, N-1$, such that

$$\|b_i - s_{l_i}\| \le \sigma, \quad i = 0, 1, \dots, N-1. \tag{5.3}$$

Define new control function $u_*(\cdot) : [a, b] \to \mathbb{R}^m$ setting

$$u_*(\xi) = w_{j_i} s_{l_i} \quad \text{for every } \xi \in [\xi_i, \xi_{i+1}),\ i = 0, 1, \ldots, N-1. \tag{5.4}$$

It is not difficult to verify that $u_*(\cdot) \in U_{p,\mu}^{H,\Gamma,\Lambda,\sigma}$. Let $x_*(\cdot)$ be the trajectory, generated by the control function $u_*(\cdot)$. Then $x_*(\cdot) \in \mathbf{X}_{p,\mu}^{H,\Gamma,\Lambda,\sigma}$ and according to the Proposition 1 we have

$$\big\|x(\xi) - x_*(\xi)\big\| \le R_* \int_a^{\xi} \big\|u(s) - u_*(s)\big\| ds \tag{5.5}$$

for every $\xi \in [a, b]$, where R_* is defined by (2.4). (5.2), (5.3) and (5.4) yield that $\big\|u(\xi) - u_*(\xi)\big\| \le H\sigma$ for every $\xi \in [a, b)$. From this inequality and (5.5) we obtain

$$\big\|x(\xi) - x_*(\xi)\big\| \le R_* H(b-a)\sigma$$

for every $\xi \in [a, b]$, and hence

$$\big\|x(\cdot) - x_*(\cdot)\big\|_C \le R_* H(b-a)\sigma. \tag{5.6}$$

(5.6) implies that

$$\mathbf{X}_{p,\mu}^{H,\Gamma,\Lambda} \subset \mathbf{X}_{p,\mu}^{H,\Gamma,\Lambda,\sigma} + R_*(b-a)H\sigma \cdot B_C. \tag{5.7}$$

Since $\mathbf{X}_{p,\mu}^{H,\Gamma,\Lambda,\sigma} \subset \mathbf{X}_{p,\mu}^{H,\Gamma,\Lambda}$, then from (5.7) we obtain the proof of the theorem. □

6 Approximation

The following theorem asserts that the sections of the set of trajectories $\mathbf{X}_{p,\mu}(\xi)$, $\xi \in [a, b]$, can be approximated by the sets $\mathbf{X}_{p,\mu}^{H,\Gamma,\Lambda,\sigma}(\xi)$, $\xi \in [a, b]$, which consists of a finite number of points.

Theorem 3. *For each $\varepsilon > 0$ there exist $H(\varepsilon) > 0$, $\Delta_*(\varepsilon) > 0$, $\delta_*(\varepsilon) > 0$ and $\sigma_*(\varepsilon) > 0$ such that for every uniform partition Γ of the closed interval $[a, b]$, uniform partition Λ of the closed interval $[0, H(\varepsilon)]$ and σ-net S_σ, where $\Delta < \Delta_*(\varepsilon)$, $\delta < \delta_*(\varepsilon)$, $\sigma < \sigma_*(\varepsilon)$, the inequality*

$$h_n\big(\mathbf{X}_{p,\mu}(\xi), \mathbf{X}_{p,\mu}^{H(\varepsilon),\Gamma,\Lambda,\sigma}(\xi)\big) \le \varepsilon$$

is satisfied for every $\xi \in [a, b]$. Here Δ is the diameter of the partition Γ, δ is the diameter of the partition Λ.

Proof. According to the Theorem 2, for given $\varepsilon > 0$ there exist $H(\varepsilon) > 0$ and $\Delta_*(\varepsilon) > 0$ such that for every uniform partition Γ of the closed interval $[a, b]$, where $\Delta < \Delta_*(\varepsilon)$, the inequality

$$h_n\big(\mathbf{X}_{p,\mu}(\xi), \mathbf{X}_{p,\mu}^{H(\varepsilon),\Gamma}(\xi)\big) \le \frac{\varepsilon}{2} \tag{6.1}$$

is verified for every $\xi \in [a,b]$, where Δ is the diameter of the partition Γ.

Now, by virtue of Proposition 3, for given $H(\varepsilon) > 0$, for every uniform partition Γ of the interval $[a,b]$ such that $\Delta < \Delta_*(\varepsilon)$, and for every uniform partition Λ of the interval $[0, H(\varepsilon)]$ the inequality

$$h_n\big(\mathbf{X}_{p,\mu}^{H(\varepsilon),\Gamma}(\xi), \mathbf{X}_{p,\mu}^{H(\varepsilon),\Gamma,\Lambda}(\xi)\big) \le R_*(b-a)\delta \tag{6.2}$$

holds for every $\xi \in [a,b]$, where R_* is defined by (2.4), $\mathbf{X}_{p,\mu}^{H,\Gamma}(\xi)$ and $\mathbf{X}_{p,\mu}^{H,\Gamma,\Lambda}(\xi)$ are defined by (3.1) and (4.1) respectively, Δ is the diameter of the partition Γ, δ is the diameter of the partition Λ. If $\delta < \delta_*(\varepsilon) = \frac{\varepsilon}{4R_*(b-a)}$, then it follows from (6.2) that for given $H(\varepsilon) > 0$, for every uniform partition Γ of the interval $[a,b]$ such that $\Delta < \Delta_*(\varepsilon)$, and for every uniform partition Λ of the interval $[0, H(\varepsilon)]$ such that $\delta < \delta_*(\varepsilon)$, the inequality

$$h_n\big(\mathbf{X}_{p,\mu}^{H(\varepsilon),\Gamma}(\xi), \mathbf{X}_{p,\mu}^{H(\varepsilon),\Gamma,\Lambda}(\xi)\big) \le \frac{\varepsilon}{4} \tag{6.3}$$

is satisfied for every $\xi \in [a,b]$.

From Proposition 4 we obtain that, for given $H(\varepsilon) > 0$, for every uniform partition Γ of the interval $[a,b]$ such that $\Delta < \Delta_*(\varepsilon)$, for every uniform partition Λ of the interval $[0,H]$ such that $\delta < \delta_*(\varepsilon)$, and for every σ-net S_σ the inequality

$$h_n\big(\mathbf{X}_{p,\mu}^{H(\varepsilon),\Gamma,\Lambda}(\xi), \mathbf{X}_{p,\mu}^{H(\varepsilon),\Gamma,\Lambda,\sigma}(\xi)\big) \le R_* H(\varepsilon)(b-a)\sigma \tag{6.4}$$

is verified for every $\xi \in [a,b]$, where R_* is defined by (2.4), $\mathbf{X}_{p,\mu}^{H,\Gamma,\Lambda}(\xi)$ and $\mathbf{X}_{p,\mu}^{H,\Gamma,\Lambda,\sigma}(\xi)$ are defined by (4.1) and (5.1) respectively, Δ is the diameter of the partition Γ, δ is the diameter of the partition Λ. If $\sigma < \sigma_*(\varepsilon) = \frac{\varepsilon}{4R_* H(\varepsilon)(b-a)}$, then it follows from (6.4) that for given $H(\varepsilon) > 0$, for every uniform partition Γ of the interval $[a,b]$ such that $\Delta < \Delta_*(\varepsilon)$, for every uniform partition Λ of the interval $[0, H(\varepsilon)]$ such that $\delta < \delta_*(\varepsilon)$, and for every σ-net S_σ such that $\sigma < \sigma_*(\varepsilon)$ the inequality

$$h_n\big(\mathbf{X}_{p,\mu}^{H(\varepsilon),\Gamma,\Lambda}(\xi), \mathbf{X}_{p,\mu}^{H(\varepsilon),\Gamma,\Lambda,\sigma}(\xi)\big) \le \frac{\varepsilon}{4} \tag{6.5}$$

is held for every $\xi \in [a,b]$.

(6.1), (6.3) and (6.5) imply that for given $\varepsilon > 0$ there exist $H(\varepsilon) > 0$, $\Delta_*(\varepsilon) > 0$, $\delta_*(\varepsilon) > 0$ and $\sigma_*(\varepsilon) > 0$ such that for every uniform partition Γ of the closed interval $[a,b]$, for every uniform partition Λ of the closed interval $[0, H(\varepsilon)]$, for every σ-net S_σ, where $\Delta < \Delta_*(\varepsilon)$, $\delta < \delta_*(\varepsilon)$, $\sigma < \sigma_*(\varepsilon)$, the inequality

$$h_n\big(\mathbf{X}_{p,\mu}(\xi), \mathbf{X}_{p,\mu}^{H(\varepsilon),\Gamma,\Lambda,\sigma}(\xi)\big) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon$$

is satisfied for every $\xi \in [a,b]$. $\square$

7 Conclusion

In the paper an approximation method for construction of the sections of the set of trajectories of the control system described by a nonlinear Volterra type

integral equation with integral constraint on the control functions is given. The convergence of the approximations is proved. Using the algorithm presented in [11], which allows to align the finite number of piecewise constant control functions from the set of controls $U_{p,\mu}^{H(\varepsilon),\Gamma,\Lambda,\sigma}$, and an arbitrary numerical method for calculation of the solution of nonlinear Volterra type integral equation, it is possible to carry out calculation of the section $\mathbf{X}_{p,\mu}^{H(\varepsilon),\Gamma,\Lambda,\sigma}(\xi)$. The approximate construction of the sections permits the solution of various types of optimal control problems arising in different fields of theory and applications.

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