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Mathematical Model for Analysis of Translational Displacements of Tooth Root

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Abstract. Analytical modeling of stress-strain state of a periodontal ligament in the case of the translational displacement of a tooth root was carried out. The tooth root was assumed as a rigid body. The boundary conditions corresponding to the translational displacement of the root and fixed external surface of the periodontal ligament in the dental alveolus were considered. The system of differential equations describing the periodontal ligament's plane-strain state induced by the translational motion of the tooth were used as the governing equations. An analytical solution was found for the governing equations in the explicit form. Comparative analysis of the concentrated force generated by the prescribed translational motion of the tooth root was performed using the obtained analytical solution and the model of an incompressible periodontal ligament in the form of a circular paraboloid and hyperboloid. The mathematical model developed in this paper can be used to analyze stresses and strains in the periodontal tissue during orthodontic movement.

Keywords: periodontal ligament, translational displacement, tooth root, stress-strain state, static load, analytical modelling.

AMS Subject Classification: 35Q74; 74L15; 92C10.

1 Introduction

Prevention and correction of malocclusion, as well as other dentoalveolar anomalies are the main orthodontics issues [26]. In particular, prediction of the initial displacements is an important problem [6]. Such movements occur under the action of short-time load and then, a tooth comes back to its original location after its removal [27,32]. Tooth is surrounded by periodontal ligament (PDL). PDL is composed of collagen fibers, the matrix phase with nerve endings and blood vessels [9, 15], and connects the tooth to the surrounding alveolar bone. Under normal conditions, there is no contact between the tooth root and the bone tissue. Any load acting on the crown of the tooth is transmitted to the alveolar bone via the PDL. Short-term (initial) and long-term (orthodontic) teeth motion are modeled in a linearly elastic (bilinear elastic), viscoelastic, hyperelastic or multiphase PDL [1, 10, 16, 17, 20, 22]. The same continuous models are used to calculate the stress-strain state of the PDL under the action of the various load types.

Orthodontic movement of a tooth is the result of a biological response of the dentoalveolar bone at the stresses and strains of the PDL [12]. Several authorities [7, 14, 18] indicate that the initial tooth displacements are defined by the high elasticity of the PDL tissue in comparison with the bones and teeth. Analysis of the initial mobility of single-root and multi-root teeth was carried out in several finite-element studies [5, 7, 19, 20, 31]. In mathematical modeling of the stress-strain state of the system "tooth–PDL", in most cases, the root of the tooth is approximated by a circular (elliptical) paraboloid or hyperboloid [3, 21, 26]. Analytical approaches for determination of the centres of resistance and/or rotation as well as for assessment of the stress-strain state of the PDL with the initial displacements of the tooth root in the shape of a cone, circular and elliptic paraboloid were presented in the literature [8, 18, 21, 26]. These approaches are generally based on simplified assumptions, for instance, in papers [8, 21, 26], the total strain derived from thickness of the PDL along the normal to the surface of the tooth is assumed on the basis of the incompressibility of the PDL tissue. In the study [3], the stress-strain state of PDL was determined assuming the dependence of a displacement of the internal surface of the PDL on the spatial coordinates is known.

A more rigorous approach for studying the initial displacement of the tooth root in the shape of a cone was realized in [18]. In this study the centre of rotation and the initial displacements were determined based on the expression for the potential in the polar coordinate system associated with the vertex of the cone cross-section. Expression for the potential was formulated on the basis of the stress function satisfying the equation of compatibility for the plane strain state. Relationship between the potential and the system of normal and shear forces as well as bending moments applied to the external surface of the cross-section of the root using the equations of equilibrium of the tooth root was obtained. Cross-section of the tooth root was placed in a plane passing through the longitudinal axis of the tooth. Integration constants were determined using the boundary conditions for the displacements in the polar coordinate system. One of the lacks of study [18] has been that the PDL has not been taken into account, the root of the tooth is rigidly fixed to the bone tooth alveolus.

The aim of this study is to develop a practical approach for assessment PDL stress-strain state in the case of the plane-strain state when the cross-section plane is perpendicular to the longitudinal axis of the tooth. To simulate the behavior of the PDL we shell use a linear elastic model.

2 Tooth Root Equilibrium in PDL

A tooth has a significantly high stiffness in comparison with the stiffness of the PDL tissue, so the tooth root was assumed to be a rigid body in this research. The system of forces acting on a tooth root produces its translational displacement in the PDL. In practice, this displacement may be achieved by simultaneous action of a concentrated load and a couple applied to the tooth [11, 25, 30]. Figure 1 shows a load acting on the tooth root, as well as its geometrical dimensions and the coordinate system associated with the tooth.

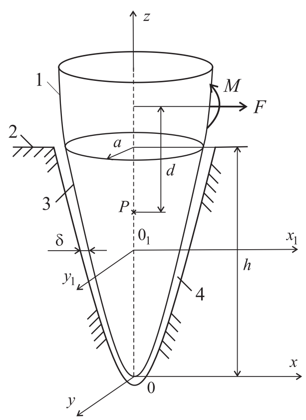

Figure 1. Model of initial translational motion of tooth root in PDL: 1 – tooth crown; 2 – dental alveolus; 3 – tooth root; 4 – periodontal slit of constant thickness normal to root surface; F – concentrated force acting on the tooth crown; $M = Fd$ – couple (couple is orthogonal to the xy-plane); d – distance between application point of force and centre of rotation P; a – radius of tooth root cross-section at level of alveolar crest; h – height of tooth root; δ – width of PDL in horizontal plane $x_1 0_1 y_1$.

Considering the equilibrium position of cross-section of the tooth root in the plane $x_1 0_1 y_1$ parallel to the plane $x0y$ in the case of the plane-strain state (it was assumed that the tooth root cross-section in any plane perpendicular to the longitudinal axis of the tooth $0z$ has a circular shape). With the effects of force and couple the tooth root was displaced horizontally on the magnitude u_0 in the direction of the $0x$- axis. The positions of the root cross-section before loading and after the translational displacement in the plane $x_1 0_1 y_1$ as well as the geometric dimensions of the cross-section are represented in Figure 2. Note that the thickness of the periodontal ligament along the tooth root height is not constant and varies in the range 0.2 ± 0.1 mm [24, 31]. In a cross-section perpendicular to the longitudinal axis of the tooth root, the periodontal ligament has an almost constant thickness. In addition, the stress-strain state of the periodontal ligament with variable thickness was investigated in study [29]. This paper has shown that non-uniform thickness of the periodontal ligament has virtually no effect on the stress state of the periodontal ligament tissue.

The equilibrium equations for displacements (u, v) of the PDL were assumed as follows [28]:

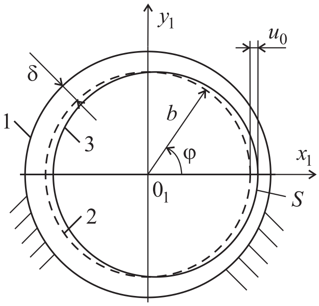

Figure 2. Positions of tooth root in sectional plane before and after displacement on magnitude of u_0: 1 – external contour of PDL fixed on dental alveolus surface; 2 – position of cross-section of tooth root before acting load; 3 – position of cross-section of tooth root after load application; u_0 – displacement tooth root in 0_1x_1-axis direction; φ – polar angle.

$$
\begin{aligned}
&A_{11}u_r(r_0,\varphi)+A_{12}u_\varphi(r_0,\varphi)=0, \qquad A_{21}u_r(r_0,\varphi)+A_{22}u_\varphi(r_0,\varphi)=0,\\
&A_{11}=r_0\frac{\partial^2}{\partial r_0^2}+\frac{\partial}{\partial r_0}-\frac{1}{r_0}+\frac{\mu}{\lambda+2\mu}\frac{1}{r_0}\frac{\partial^2}{\partial\varphi^2},\\
&A_{12}=\frac{\lambda+\mu}{\lambda+2\mu}\frac{\partial^2}{\partial\varphi\partial r_0}-\frac{\lambda+3\mu}{\lambda+2\mu}\frac{1}{r_0}\frac{\partial}{\partial\varphi},\\
&A_{21}=\frac{\lambda+\mu}{\lambda+2\mu}\frac{\partial^2}{\partial\varphi\partial r_0}+\frac{\lambda+3\mu}{\lambda+2\mu}\frac{1}{r_0}\frac{\partial}{\partial\varphi},\\
&A_{22}=\frac{\mu r_0}{\lambda+2\mu}\frac{\partial^2}{\partial r_0^2}+\frac{\mu}{\lambda+2\mu}\frac{\partial}{\partial r_0}-\frac{\mu}{\lambda+2\mu}\frac{1}{r_0}+\frac{1}{r_0}\frac{\partial^2}{\partial\varphi^2},
\end{aligned}
\tag{2.1}
$$

where $u_r(r_0,\varphi)$, $u_\varphi(r_0,\varphi)$ are the radial and circular PDL displacements, respectively, and λ, μ are the Lamé coefficients.

Dimensionless displacements and coordinates can be introduced as

$$
u(r,\varphi)=\frac{u_r(r_0,\varphi)}{h}, \quad v(r,\varphi)=\frac{u_\varphi(r_0,\varphi)}{h}, \quad r=\frac{r_0}{h},
$$

where h is the height of the tooth root. Then equations (2.1) will be rewritten in the dimensionless form as

$$
\begin{aligned}
&a_{11}u(r,\varphi)+a_{12}v(r,\varphi)=0, \qquad a_{21}u(r,\varphi)+a_{22}v(r,\varphi)=0,\\
&a_{11}=r(1-\nu)\frac{\partial^2}{\partial r^2}+(1-\nu)\frac{\partial}{\partial r}-\frac{1-\nu}{r}+\frac{1-2\nu}{2}\frac{1}{r}\frac{\partial^2}{\partial\varphi^2},\\
&a_{12}=\frac{1}{2}\frac{\partial^2}{\partial\varphi\partial r}-\frac{3-4\nu}{2}\frac{1}{r}\frac{\partial}{\partial\varphi}, \qquad a_{21}=\frac{1}{2}\frac{\partial^2}{\partial\varphi\partial r}+\frac{3-4\nu}{2}\frac{1}{r}\frac{\partial}{\partial\varphi},\\
&a_{22}=\frac{(1-2\nu)r}{2}\frac{\partial^2}{\partial r^2}+\frac{1-2\nu}{2}\frac{\partial}{\partial r}-\frac{1-2\nu}{r}+\frac{1-\nu}{r}\frac{\partial^2}{\partial\varphi^2},
\end{aligned}
\tag{2.2}
$$

where ν is the Poisson's ratio of PDL.

The boundary conditions for the displacements u and v are taken as

$$
\begin{aligned}
&u(b,\varphi)=u_0\cos(\varphi), \quad v(b,\varphi)=u_0\sin(\varphi),\\
&u(b_1,\varphi)=v(b_1,\varphi)=0, \quad b_1=b+\delta.
\end{aligned}
\tag{2.3}
$$

The solution of the boundary problem (2.2), (2.3) can be found in the explicit form [13] of

$$u(r,\varphi) = \Big(c_1 r^2(1-4\nu) + \frac{c_2}{r^2} - \frac{4c_3(1-\nu)}{3-4\nu} + c_3\ln(r) + c_4\Big)\cos(\varphi), \qquad (2.4)$$

$$v(r,\varphi) = \Big(c_1 r^2(5-4\nu) + \frac{c_2}{r^2} + c_3 - c_3\ln(r) - c_4\Big)\sin(\varphi). \qquad (2.5)$$

Taking into account the boundary conditions (2.3), coefficients can be found accordingly as

$$c_1 = \frac{b^2 u_0(3-4\nu)\ln(b/b_1)}{(b^2-b_1^2)(b_1^2-b^2+(b_1^2+b^2)(3-4\nu)^2\ln(b/b_1))},$$

$$c_2 = \frac{b^2 b_1^2 u_0(b^2-b_1^2-b_1^2(3-4\nu)^2\ln(b/b_1))}{(b^2-b_1^2)(b_1^2-b^2+(b_1^2+b^2)(3-4\nu)^2\ln(b/b_1))},$$

$$c_3 = \frac{2b^2 u_0(3-4\nu)}{b_1^2-b^2+(b^2+b_1^2)(3-4\nu)^2\ln(b/b_1)},$$

$$c_4 = \frac{b^2 u_0((b^2-b_1^2)(7-8\nu)+2(3-4\nu)(b_1^2\ln(b)-b^2\ln(b_1)))}{(b^2-b_1^2)(b_1^2-b^2+(b^2+b_1^2)(3-4\nu)^2\ln(b/b_1))},$$

appearing in (2.5).

Equations (2.4) and (2.5) yield values for displacements of the periodontal points in the horizontal plane when moving the tooth root at the magnitude of $u0$.

3 Stress-Strain State of PDL

Foregoing solutions (2.4) and (2.5) depend on the constant u_0 which can be determined using the equilibrium equations

$$\int_S (\sigma_{rr}n_1 + \sigma_{r\varphi}n_2)dS - \frac{F}{h} = 0, \quad \int_S (\sigma_{r\varphi}n_1 + \sigma_{\varphi\varphi}n_2)dS = 0,$$
$$\int_S \left((\sigma_{r\varphi}n_1 + \sigma_{\varphi\varphi}n_2)\cos(\varphi) - (\sigma_{rr}n_1 + \sigma_{r\varphi}n_2)\sin(\varphi)\right) r dS = 0 \qquad (3.1)$$

for the tooth root, where σ_{rr}, $\sigma_{r\varphi}$, $\sigma_{\varphi\varphi}$ are the components of the stress tensor, F and M are the concentrated force and couple, respectively, acting on the tooth, $n_1 = -\cos(\varphi)$, $n_2 = -\sin(\varphi)$ are the components of the unit normal to the external contour of the PDL; S is the contour 3 (the circle) of the tooth root as shown in Figure 2.

The first two equations describe the balance of forces and stress resultants in the direction of the O_1x_1- and O_1y_1-axes, respectively, and the third equation guarantees the balance of the stress-couples.

The components of the stress tensor have the following form

$$\sigma_{rr} = \lambda\theta + 2\mu\frac{\partial u}{\partial r}, \quad \sigma_{r\varphi} = \mu\left(\frac{1}{r}\frac{\partial u}{\partial\varphi} + r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right),$$
$$\sigma_{\varphi\varphi} = \lambda\theta + \frac{2\mu}{r}\left(\frac{\partial v}{\partial\varphi} + u\right), \quad \theta = \frac{1}{r}\left(\frac{\partial(ur)}{\partial r} + \frac{\partial v}{\partial\varphi}\right). \qquad (3.2)$$

Using the first equation (3.1), one obtains

$$Ku_0=\frac{F}{Eh^2},\ K=\frac{4\pi\left(\left(\left(3-4\nu\right)b_1^4+b^4\right)\left(3-4\nu\right)\ln\left(b_1/b\right)-\left(b_1^2-b^2\right)^2\right)}{\left(1+\nu\right)\left(\left(b_1^2-b^2\right)^2-\left(b_1^4-b^4\right)\left(3-4\nu\right)^2\ln\left(b_1/b\right)\right)},$$

where E is the modulus of elasticity of the PDL. It may be checked that the second and third equations (3.1) are satisfied identically.

In order to validate the proposed analytical model, the force required to displace the tooth root on the magnitude of $u_0 = 0.2\ \mu m$ in the PDL was calculated. Its elastic properties are characterized by the constants $E = 680$ kPa and $\nu = 0.49$ [21]. The geometrical dimensions of the tooth root were specified by the values $h = 13$ mm and $a = 3.9$ mm, and the thickness of the PDL normal to the external surface of the tooth root is $\delta_0 = 0.229$ mm [21]. The external and internal surfaces of the PDL may be presented with the circular hyperboloid or paraboloid

$$f_{02}(x,y,z) = z - \frac{h}{a\sqrt{1+p^2}-p}\left(\sqrt{x^2+y^2+(ap)^2}-ap\right), \tag{3.3}$$

$$f_2(x,y,z) = z + n_{2z}\delta_0 - \frac{h}{a\sqrt{1+p^2}-p} \times\left(\sqrt{(x+n_{2x}\delta_0)^2+(y+n_{2y}\delta_0)^2+(ap)^2}-ap\right) = 0, \tag{3.4}$$

$$f_{01}(x,y,z) = z - \frac{h}{a^2}(x^2+y^2) = 0, \tag{3.5}$$

$$f_1(x,y,z) = z + n_{1z}\delta_0 - \frac{h}{a^2}\left((x+n_{1x}\delta_0)^2+(y+n_{1y}\delta_0)^2\right) = 0, \tag{3.6}$$

where (n_{ix}, n_{iy}, n_{iz}) are the components of the unit normal to the external surface of the PDL (here, the subscripts $i = 1$ and $i = 2$ correspond to PDL in the form of a circular paraboloid and hyperboloid, respectively), p is the parameter characterizing the rounding rate of the apex (the tooth root passes through a circular cone at $p = 0$); in this case $p = 0.4$. Functions (3.3), (3.4) and (3.5), (3.6) correspond to the internal and external surfaces of the PDL in the form of a circular hyperboloid and paraboloid, respectively.

The results of the load calculations for the root cross-section located in the middle of the dental alveolus at the distance of 6.5 mm from the apex are presented in Table 1. In addition, the load magnitudes defined in [2, 26] for a tooth root in the form of the circular paraboloid and hyperboloid at the same geometrical characteristics of the tooth root and the elastic constants for the PDL are also given in Table 1. Note that in studies [2, 26], the analysis of the stress-strain state of the PDL caused by the translation motion of root in horizontal direction was carried out under the assumption of incompressibility of the periodontal tissues.

Table 1 shows that the magnitudes of the load calculated under assumption of the plane strain state for the displacement of the tooth root in the shape of the circular paraboloid or hyperboloid are slightly different from the load, calculated under the assumption of the periodontium incompressibility.

Table 1. Magnitudes of force required for translational displacement of tooth root with circular cross-section in PDL at value of 0.2 microns (magnitude in parentheses are force obtained in [2, 26] under the assumption of the periodontal tissue incompressibility).

The shape of the PDL	Radius b, mm	Radius b_1, mm	The force magnitude F, N
Circular hyperboloid	2.4212	2.6567	0.86 (0.93 [2])
Circular paraboloid	2.7577	2.9918	1.0 (1.07 [26])

4 Influence of Poisson's Ratio on Stresses in PDL

Different values of Poisson's ratio defined on basis of *in-vivo* and *in-vitro* experiments are given in papers [24]. According to these studies, researchers used high values of Poisson's ratio, in particular, 0.45 or 0.49, as well as values in the range of 0.28–0.35. It should be noted that the influence of Poisson's ratio on the stress-strain state of the PDL was not investigated. Figure 3 shows the displacement u_0 versus Poisson's ratio in the range from 0.3 to 0.49. Load acting on the tooth was equaled to 1 N, and the material constants and dimensions of the tooth root were taken as above.

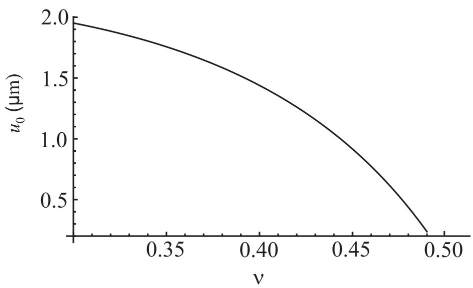

Figure 3. Initial displacement u_0 vs Poison's ratio.

Figure 3 shows the nonlinear dependence of the initial displacement upon the Poisson's ratio. When increasing the ratio from 0.3 to 0.49, the displacement u_0 is reduced significantly (more than 4 times).

The normal and shear stresses arising in the PDL can be found from equations (3.2):

$$\begin{aligned}
\sigma_{rr} &= DPb^2\cos(\varphi)((b_1^2-b^2)(b_1^2+r^2(2\nu-3)) \\
&\quad + (4\nu-3)(b_1^4(4\nu-3)-r^4)\ln(b/b_1), \\
\sigma_{r\varphi} &= DPb^2\sin(\varphi)((b_1^2-b^2)(b_1^2+r^2(1-2\nu)) \\
&\quad + (4\nu-3)(b_1^4(4\nu-3)-r^4)\ln(b/b_1), \\
\sigma_{\varphi\varphi} &= DPb^2\cos(\varphi)((b^2-b_1^2)(b_1^2-r^2(1-2\nu)) \\
&\quad - (4\nu-3)(b_1^4(4\nu-3)+3r^4)\ln(b/b_1), \\
D &= \frac{1}{2\pi r^3\left((b^4+b_1^4(3-4\nu))\,(4\nu-3)\ln(b/b_1)-(b_1^2-b^2)^2\right)}.
\end{aligned}$$

Dependencies of the stresses on the radial coordinate for different Poisson's ratios in the cross-section of the tooth root located at the distance of 6.5 mm from the apex are shown in Figures 4–6. The magnitude of the load was 1 N, and the radial coordinate was varied from 2.4212 mm to 2.6567 mm.

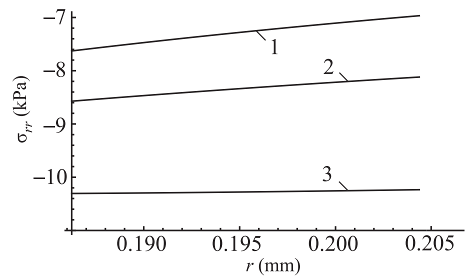

Figure 4. Stress σ_{rr} vs radial coordinate (circular coordinate $\varphi = 0$): curves 1, 2, 3 correspond to values $\nu = 0.30; 0.40; 0.49$, respectively.

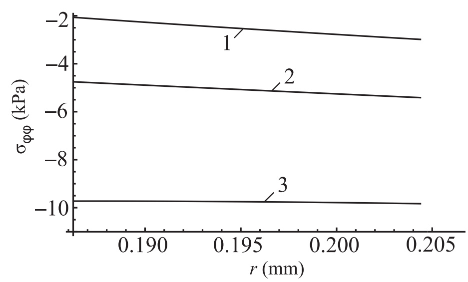

Figure 5. Stress $\sigma_{\varphi\varphi}$ vs radial coordinate (circular coordinate $\varphi = 0$): curves 1, 2, 3 correspond to values $\nu = 0.30; 0.40; 0.49$, respectively.

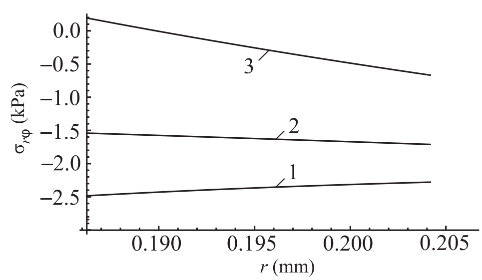

Figure 6. Stress $\sigma_{r\varphi}$ vs radial coordinate (circular coordinate $\varphi = \pi/2$): curves 1, 2, 3 correspond to values $\nu = 0.30; 0.40; 0.49$, respectively.

It is seen from Figures 4–6 that dependencies of the normal and shear stresses on the radial coordinates are nonlinear. Furthermore, the stresses are significantly influenced by Poisson's ratio. The values of the normal stresses $\sigma_{\varphi\varphi}$ for $\nu = 0.30$ and $\nu = 0.49$ vary almost five times. Shear stresses in the periodontal tissues are positive for $\nu = 0.49$, while the shear stresses for $\nu = 0.30$ or $\nu = 0.40$ at the same radial coordinate are negative. It should be noted that the dependencies of the normal and shear strains arising in the PDL have a form similar to those shown in Figures 4–6.

5 Conclusions

The mathematical model of the plane-strain state of the PDL in the case of the translational displacement of the tooth root with the circular cross-section was proposed.

It is based on the model of linear elastic isotropic medium and allows to assess the stress-strain state of the PDL at low loads on the tooth root circular cross-section. Further development of the proposed simulation can be associated with the use of a viscoelastic model for the describing the stress relaxation and time-dependent properties of periodontal tissue. Also, to improve the model one can use the elliptical cross-sections of the tooth root and PDL. Another step in the study may be a 3-D simulation of the single-root tooth motion in the PDL under the action of short-term and long-term loads. This model can be implemented in the form of real computing means, and it can be tested using the method of finite elements for the real geometric shapes of teeth and in clinical conditions.

Mathematical functions for the radial and circular displacements of the PDL for the case of translational displacement of the tooth root in the horizontal direction were derived. Displacement of the PDL points analyzed with the developed functions allowed us to determine stresses in the periodontal tissue. The equilibrium equations for the tooth were developed using a practical formula coupling the translational displacement of the tooth and the concentrated force required to perform this displacement.

Comparative analysis of the concentrated forces computed on the basis of both the proposed analytical model and the assumption of incompressibility of PDL has shown a sufficiently good agreement of results. Eventually proposed model of the plane-strain can be used to evaluate strains and stresses in the PDL regions which are far from the apex and the alveolar crest. The obtained results can be used to determine magnitude of a load for orthodontic tooth movement resulting in the optimal stresses [29], as well as to simulate bone remodeling on the basis of changes of stresses and strains in the PDL during orthodontic movement [4, 23]. The outcomes of this research can be used to assess the load ranges for orthodontic procedures on the basis of the optimal magnitudes stresses for PDL.

Acknowledgments

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