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Asymptotic Integration of Fractional Differential Equations with Integrodifferential Right-Hand Side

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Abstract. In this paper we deal with the problem of asymptotic integration of a class of fractional differential equations of the Caputo type. The left-hand side of such type of equation is the Caputo derivative of the fractional order $r \in (n-1, n)$ of the solution, and the right-hand side depends not only on ordinary derivatives up to order $n-1$ but also on the Caputo derivatives of fractional orders $0 < r_1 < \cdots < r_m < r$, and the Riemann–Liouville fractional integrals of positive orders. We give some conditions under which for any global solution $x(t)$ of the equation, there is a constant $c \in \mathbb{R}$ such that $x(t) = ct^R + o(t^R)$ as $t \to \infty$, where $R = \max\{n-1, r_m\}$.

Keywords: Caputo fractional derivative, nonlinear equation, asymptotic property, desingularization.

AMS Subject Classification: 34E10; 34A08; 26A33.

1 Introduction

In the asymptotic theory of n-th order nonlinear ordinary differential equations

$$y^{(n)} = f\big(t, y, y', \ldots, y^{(n-1)}\big), \tag{1.1}$$

the classic problem is to establish some conditions for the existence of a solution approaching a polynomial of degree $1 \le m \le n-1$ as $t \to \infty$. The first paper concerning this problem was published by D. Caligo [9] in 1941.

The first paper on the nonlinear second order differential equations

$$y''(t) = f\big(t, y(t)\big) \tag{1.2}$$

was published by W.F. Trench [32] in 1963, and then by D.S. Cohen [10], T. Kusano and W.F. Trench [14,15], F.M. Dannan [13], A. Constantin [11,12], Yu.V. Rogovchenko [29], S.P. Rogovchenko [28], O.G. Mustafa and Yu.V. Rogovchenko [24], J. Tong [31], O. Lipovan [16] and others. In the proofs of their results the key role plays the Bihari inequality (see [4]) which is a generalization of the Gronwall inequality. Some results on the existence of solutions of the n-th order differential equation

$$y^{(n)}(t) = f\big(t, y(t)\big), \quad n > 1,\ t \ge t_0 > 0,$$

approaching a polynomial function of the degree m with $1 \le m \le n-1$, are proved by Ch.G. Philos, I.K. Purnaras and P.Ch. Tsamatos [25]. Their proofs are based on an application of the Schauder fixed point theorem. The paper by R.P. Agarwal, S. Djebali, T. Moussaoui and O.G. Mustafa [2] surveys the literature concerning the topic in asymptotic integration theory of ordinary differential equations. Several conditions, under which all solutions of the one-dimensional p-Laplacian equation

$$\big(|y'|^{p-1}y'\big)' = f(t, y, y'), \quad p > 1$$

are asymptotic to $a + bt$ as $t \to \infty$ for some real numbers a, b, are proved in [23], and some sufficient conditions for the existence of such solutions of the equation

$$\big(\Phi\big(y^{(n)}\big)\big)' = f(t, y), \quad n \ge 1,$$

where $\Phi\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with a locally Lipschitz inverse, satisfying $\Phi(0) = 0$, are given in the paper [22].

The problem of asymptotic integration for a class of linear fractional differential equations of the Riemann–Liouville type is studied in the papers by D. Băleanu, O.G. Mustafa and R.P. Agarwal [7,8], where some conditions for the existence of at least one solution of this type of equations, approaching a linear function as $t \to \infty$, are given. In [7] a result on the existence of a solution of the equation

$${}_0D_t^{\alpha}\big[tx' - x + x(0)\big] + a(t)x = 0, \quad t > 0,$$

(${}_0D_t^{\alpha}$ is the Riemann–Liouville derivative of the order $\alpha \in (0,1)$), approaching a function $ct + d + o(1)$ for $t \to \infty$ is proved. In the paper [8], some results for the existence of a solution of the equations

$${}_0^{i}\mathcal{O}_t^{1+\alpha}x + a(t)x = 0, \quad t > 0,$$

approaching a function $a + bt^{\alpha} + O(t^{\alpha-1})$ for $i = 1$, and a function $bt^{\alpha} + O(t^{\alpha-1})$ for $i = 2, 3$ as $t \to \infty$, where ${}_0^{1}\mathcal{O}_t^{1+\alpha} := {}_0D^{\alpha} \circ \frac{\mathrm{d}}{\mathrm{d}t}$, ${}_0^{2}\mathcal{O}_t^{1+\alpha} := \frac{\mathrm{d}}{\mathrm{d}t} \circ {}_0D_t^{\alpha}$ and ${}_0^{3}\mathcal{O}_t^{1+\alpha} := {}_0D_t^{\alpha} \circ (t\frac{\mathrm{d}}{\mathrm{d}t} - id_{RL^{\alpha}((0,+\infty),\mathbb{R})})$ with

$$RL^{\alpha}\big((0,+\infty),\mathbb{R}\big) = \Big\{f \in C\big((0,\infty),\mathbb{R}\big) \;\Big|\; \lim_{t\to 0^+}\big[t^{1-\alpha}f(t)\big] \in \mathbb{R}\Big\},$$

$\alpha \in (0,1)$. In the proofs of all these results a fixed point method is applied.

The problem of the asymptotic integration for the equation

$$x^{\Delta\Delta} + f(t,u) = 0$$

on a time scale $\mathbb{T}$ is studied in the paper [3].

In the paper [5], a sufficient condition for all solutions of the equation

$$u''(t) + f\big(t, u(t), u'(t)\big) + \sum_{i=1}^{m} r_i(t) \int_0^t (t-s)^{\alpha_i - 1} f_i\big(\tau, u(\tau), u'(\tau)\big) \mathrm{d}\tau = 0$$

to be asymptotic to a straight line is proved.

The problem of the asymptotic integration for a class of sublinear fractional differential equations is investigated by D. Băleanu and O.G. Mustafa in [6], where a condition for the existence of a solution with the asymptotic behavior $o(t^\alpha)$ for a convenient $0 < \alpha < 1$ as $t \to \infty$, is proved.

In the paper [21] (see also [20]), the fractional differential equation with Caputo derivative

$${}^{C}D_a^r x(t) = f\big(t, x(t), x'(t), \dots, x^{(n-1)}(t)\big), \quad t \ge a \ge 1$$

for $n-1 < r < n \ni \mathbb{N}$ is considered, and a sufficient condition for the existence of a constant $c \in \mathbb{R}$, such that all solutions $x(t)$ of the above equation behave like $ct^{n-1} + o(t^{n-1})$ as $t \to \infty$, is proved.

In the present paper, we prove similar results for a more general case when the right-hand side depends on Caputo fractional derivatives of the solution of orders $\tilde{r} < r$. Finally, we investigate the problem of asymptotic integration for fractional differential equations with right-hand side depending on Caputo derivatives as well as on Riemann–Liouville fractional integrals of the solution. In the proofs of our results, we apply a desingularization method of nonlinear integral inequalities with weakly singular kernels proposed in [18,19]. Note that all our results are stated for global solution assuming they exist. The problem of existence of global solutions for the below-considered initial value problems is beyond the scope of this paper.

Throughout the paper, we denote $\mathbb{R}_+ = [0, \infty)$.

2 Preliminaries

In this section, we recall some definitions (see e.g. [26, 30]) and basic results.

Definition 1. For $z > 0$, the Euler gamma function is defined as

$$\Gamma(z) := \int_0^\infty t^{z-1}\, \mathrm{e}^{-t} \mathrm{d}t.$$

For $u, v > 0$, the Euler beta function is defined as

$$B(u,v) := \int_0^1 t^{u-1}(1-t)^{v-1} \mathrm{d}t.$$

Definition 2. Let $r > 0$. The Riemann–Liouville integral of a function $h\colon [a,\infty) \to \mathbb{R}$ of order r is defined as

$$I_a^r h(t) = \frac{1}{\Gamma(r)} \int_a^t (t-s)^{r-1} h(s) \mathrm{d}s.$$

Definition 3. Let $r > 0$ and $n \in \mathbb{N}$ be such that $n-1 < r < n$. The Caputo derivative of a C^n function $x(t)$ of order r on the interval $[a,\infty)$, $a \geq 0$ is defined as

$${}^C D_a^r x(t) := I_a^{n-r} x^{(n)}(t) = \frac{1}{\Gamma(n-r)} \int_a^t (t-s)^{n-r-1} x^{(n)}(s) \mathrm{d}s.$$

Definition 4. Let $r > 0$, $n \in \mathbb{N}$ be such that $n-1 < r < n$, $a \geq 0$, $f \in C([a,\infty),\mathbb{R})$, $c_0, c_1, \dots, c_{n-1} \in \mathbb{R}$. A function $x\colon [a,T) \to \mathbb{R}$, $a < T \leq \infty$ is called a solution of the initial value problem

$${}^C D_a^r x(t) = f(t), \quad t \geq a, \tag{2.1}$$

$$x^{(i)}(a) = c_i, \quad i = 0, 1, \dots, n-1 \tag{2.2}$$

if $x \in C^n([a,T),\mathbb{R})$, x satisfies equation (2.1) and initial condition (2.2). This solution is called global if it exists for all $t \in [a,\infty)$.

Lemma 1. *Let $r > 0$, $n \in \mathbb{N}$ be such that $n-1 < r < n$, $a \geq 0$, $f \in C([a,\infty),\mathbb{R})$, $c_0, c_1, \dots, c_{n-1} \in \mathbb{R}$. Then the initial value problem* (2.1), (2.2) *has the solution*

$$x(t) = c_0 + c_1(t-a) + \dots + \frac{c_{n-1}}{(n-1)!}(t-a)^{n-1} + \frac{1}{\Gamma(r)} \int_a^t (t-s)^{r-1} f(s) \mathrm{d}s.$$

The next lemma can be found in [27, 2.2.4.8] or [17].

Lemma 2. *Let $a \geq 0$, $t > a$, $p(\alpha-1)+1 > 0$, $p(\gamma-1)+1 > 0$. Then*

$$\int_a^t (t-s)^{p(\alpha-1)} s^{p(\gamma-1)} \mathrm{d}s \leq t^{\Theta} B\big(p(\gamma-1)+1, p(\alpha-1)+1\big),$$

where $\Theta = p(\alpha+\gamma-2)+1$ and $B(u,v)$ is the Euler beta function.

Lemma 3. *For any $z > 0$, it holds*

$$\Gamma(z) > \frac{\mathrm{e}-1}{\mathrm{e}} \doteq 0.63212.$$

Proof. By its definition the Euler gamma function is positive on $(0,\infty)$. So, its derivative, Γ', and its logarithmic derivative [1], $\Psi = \frac{\Gamma'}{\Gamma}$, have the same sign on $(0,\infty)$. Next, by [1, 6.4.10], $\Psi'(z) = \sum_{k=0}^{\infty} \frac{1}{(k+z)^2} > 0$ for $z > 0$, i.e., Ψ is increasing on $(0,\infty)$. Since by [1, 6.3.5]

$$\Psi(1) = -\mathrm{C} \doteq -0.57722 < 0 \quad (\mathrm{C} \text{ is Euler's constant}),$$
$$\Psi(2) = \Psi(1) + 1 \doteq 0.42278 > 0,$$

Ψ is negative on $(0,1]$ and positive on $[2,\infty)$. Therefore, Γ is decreasing on $(0,1]$, increasing on $[2,\infty)$, and it has a minimum in $(1,2)$. For any $z \in (1,2)$, we estimate

$$\Gamma(z) > \int_0^1 t\,\mathrm{e}^{-t}\mathrm{d}t + \int_1^\infty \mathrm{e}^{-t}\mathrm{d}t = \frac{\mathrm{e}-1}{\mathrm{e}},$$

and the proof is complete. □

Due to the latter lemma, $C_\Gamma := \frac{\mathrm{e}}{\mathrm{e}-1} \doteq 1.58198$ satisfies $C_\Gamma > \frac{1}{\Gamma(z)}$ on $(0,\infty)$.

3 Asymptotic Behavior of Fractional Differential Equations with Fractional Derivative on the Right-Hand Side

This section is devoted to the study of asymptotic behavior of the solutions of fractional differential equations with the right-hand side depending also on fractional derivatives of the solution.

Theorem 1. *Suppose that* $0 < \tilde{r} < r < 1$, $p > 1$, $p(r - \tilde{r} - 1) + 1 > 0$, $a > 0$, $q = \frac{p}{p-1}$ *and the function* $f\colon M := [a,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ *satisfy the following conditions:*

1. $f \in C(M,\mathbb{R})$,
2. *there are continuous functions from* $\mathbb{R}_+$ *to* $\mathbb{R}_+$, g_1, g_2, h_0, h_1, h_2, *such that* g_1, g_2 *are nondecreasing,*

$$\big|f(t,u,v)\big| \le t^{\gamma-1}\left(h_0(t) + h_1(t)g_1\left(\frac{|u|}{t^{\tilde{r}}}\right) + h_2(t)g_2\big(|v|\big)\right)$$

for some $\gamma \in (1 - \frac{1}{p}, 2 - r + \tilde{r} - \frac{1}{p}]$, *and*

$$H_i := \int_a^\infty h_i^q(s)\mathrm{d}s < \infty, \quad i = 0,1,2,$$

3. $$\int_a^\infty \frac{\tau^{q-1}\mathrm{d}\tau}{g_1^q(\tau) + g_2^q(\tau)} = \infty.$$

Then for any global solution $x(t)$ *of the initial value problem*

$$ {}^C D_a^r x(t) = f\big(t, x(t), {}^C D_a^{\tilde{r}} x(t)\big), \quad t \ge a, \tag{3.1}$$
$$ x(a) = c_0, \tag{3.2}$$

there exists a constant $c \in \mathbb{R}$ *such that*

$$x(t) = ct^{\tilde{r}} + o\big(t^{\tilde{r}}\big) \quad \textit{as } t \to \infty.$$

Proof. For simplicity, we denote $F(t) := f(t, x(t), {}^C D_a^{\tilde r} x(t))$. By Lemma 1, the solution $x(t)$ has the form

$$x(t) = c_0 + \frac{1}{\Gamma(r)} \int_a^t (t-s)^{r-1} F(s) \mathrm{d}s, \quad t \geq a.$$

Clearly,

$$\begin{aligned} \frac{|x(t)|}{t^{\tilde r}} &\leq \frac{|c_0|}{t^{\tilde r}} + \frac{1}{\Gamma(r)} \int_a^t \left(\frac{t-s}{t}\right)^{\tilde r} (t-s)^{r-\tilde r-1} \big|F(s)\big| \mathrm{d}s \\ &\leq \frac{|c_0|}{a^{\tilde r}} + \frac{1}{\Gamma(r)} \int_a^t (t-s)^{r-\tilde r-1} \big|F(s)\big| \mathrm{d}s \leq z(t), \quad t \geq a \end{aligned} \tag{3.3}$$

for

$$z(t) := C + C_\Gamma \int_a^t (t-s)^{r-\tilde r-1} \big|F(s)\big| \mathrm{d}s, \quad C = \frac{|c_0|}{a^{\tilde r}}.$$

The fractional derivative ${}^C D_a^{\tilde r} x(t)$ can be obtained by applying the operator $I_a^{r-\tilde r}$ to equation (3.1) (see [26, 2.3.2]):

$$I_a^{r-\tilde r}\big({}^C D_a^r x\big)(t) = I_a^{r-\tilde r}\big(I_a^{1-r} x'\big)(t) = I_a^{1-\tilde r} x'(t) = {}^C D_a^{\tilde r} x(t) = I_a^{r-\tilde r} F(t). \tag{3.4}$$

Hence, by Definition 2,

$${}^C D_a^{\tilde r} x(t) = \frac{1}{\Gamma(r-\tilde r)} \int_a^t (t-s)^{r-\tilde r-1} F(s) \mathrm{d}s$$

yielding the estimation $|{}^C D_a^{\tilde r} x(t)| \leq z(t)$ for $t \geq a$. Using the assumptions on f and the nondecreasing properties of g_1, g_2, we estimate

$$z(t) \leq C + C_\Gamma \int_a^t (t-s)^{r-\tilde r-1} s^{\gamma-1} \big(h_0(s) + h_1(s) g_1\big(z(s)\big) + h_2(s) g_2\big(z(s)\big)\big) \mathrm{d}s.$$

Now, by Hölder inequality and Lemma 2 with $\alpha = r - \tilde r$, we get

$$\int_a^t (t-s)^{r-\tilde r-1} s^{\gamma-1} h_i(s) g_i\big(z(s)\big) \mathrm{d}s \leq t^{\frac{\Theta}{p}} B_1 \left(\int_a^t h_i^q(s) g_i^q\big(z(s)\big) \mathrm{d}s\right)^{\frac{1}{q}}$$

for $i = 1, 2$, where $B_1 = B^{\frac{1}{p}}(p(r-\tilde r-1)+1, p(\gamma-1)+1)$ and $\Theta = p(r - \tilde r + \gamma - 2) + 1 \in (p(r-\tilde r-1), 0]$. Thus

$$\int_a^t (t-s)^{r-\tilde r-1} s^{\gamma-1} h_i(s) g_i\big(z(s)\big) \mathrm{d}s \leq a^{\frac{\Theta}{p}} B_1 \left(\int_a^t h_i^q(s) g_i^q\big(z(s)\big) \mathrm{d}s\right)^{\frac{1}{q}}$$

for each $i = 1, 2$, $t \geq a$. Similarly,

$$\int_a^t (t-s)^{r-\tilde r-1} s^{\gamma-1} h_0(s) \mathrm{d}s \leq a^{\frac{\Theta}{p}} B_1 \left(\int_a^t h_0^q(s) \mathrm{d}s\right)^{\frac{1}{q}}, \quad t \geq a.$$

Therefore,

$$z(t) \le C + \widetilde{C}\Big(\Big(\int_a^t h_0^q(s)\mathrm{d}s\Big)^{\frac{1}{q}} + \Big(\int_a^t h_1^q(s)g_1^q\big(z(s)\big)\mathrm{d}s\Big)^{\frac{1}{q}} + \Big(\int_a^t h_2^q(s)g_2^q\big(z(s)\big)\mathrm{d}s\Big)^{\frac{1}{q}}\Big)$$

with $\widetilde{C} = C_\Gamma a^{\frac{\Theta}{p}} B_1$. Now, we apply the inequality $(\sum_{i=1}^4 a_i)^q \le 4^{q-1}\sum_{i=1}^4 a_i$ for any nonnegative a_i, $i = 1, 2, 3, 4$, to get

$$\begin{aligned} z^q(t) &\le 4^{q-1}\Big(C^q + \widetilde{C}^q\Big(\int_a^t h_0^q(s)\mathrm{d}s + \int_a^t h_1^q(s)g_1^q\big(z(s)\big)\mathrm{d}s \\ &\quad + \int_a^t h_2^q(s)g_2^q\big(z(s)\big)\mathrm{d}s\Big)\Big) \\ &\le 4^{q-1}\big(C^q + \widetilde{C}^q H_0\big) + 4^{q-1}\widetilde{C}^q\left(\int_a^t h_1^q(s)g_1^q\big(z(s)\big)\mathrm{d}s + \int_a^t h_2^q(s)g_2^q\big(z(s)\big)\mathrm{d}s\right). \end{aligned}$$

Denoting $u(t) := z^q(t)$, $A := 4^{q-1}(C^q + \widetilde{C}^q H_0)$, $D := 4^{q-1}\widetilde{C}^q$, we rewrite the last inequality as

$$\begin{aligned} u(t) &\le A + D\left(\int_a^t h_1^q(s)g_1^q\big(u^{\frac{1}{q}}(s)\big)\mathrm{d}s + \int_a^t h_2^q(s)g_2^q\big(u^{\frac{1}{q}}(s)\big)\mathrm{d}s\right) \\ &\le A + D\int_a^t \big(h_1^q(s) + h_2^q(s)\big)\omega\big(u(s)\big)\mathrm{d}s \end{aligned}$$

for $\omega(u) = g_1^q(u^{\frac{1}{q}}) + g_2^q(u^{\frac{1}{q}})$. The Bihari inequality implies

$$\begin{aligned} u(t) &\le \Omega^{-1}\left(\Omega(A) + D\int_a^t h_1^q(s) + h_2^q(s)\mathrm{d}s\right) \\ &\le \Omega^{-1}\big(\Omega(A) + D(H_1 + H_2)\big) =: K_0 < \infty \end{aligned}$$

for

$$\Omega(v) := \int_{v_0}^v \frac{\mathrm{d}s}{\omega(s)}, \quad 0 < v_0 \le v.$$

Note that $\Omega(A) + D(H_1 + H_2)$ is always in the range of Ω, as $\Omega(\infty) = \infty$ by the assumption of the theorem. For $z(t)$ it means that $z(t) \le K_0^{\frac{1}{q}} < \infty$. Consequently from (3.3) it follows that

$$0 \le \int_a^t \left(\frac{t-s}{t}\right)^{\tilde{r}} (t-s)^{r-\tilde{r}-1}\big|F(s)\big|\mathrm{d}s \le \Gamma(r)K_0^{\frac{1}{q}} < \infty, \quad t \ge a,$$

i.e., the integral

$$\int_a^\infty \left(\frac{t-s}{t}\right)^{\tilde{r}} (t-s)^{r-\tilde{r}-1}F(s)\mathrm{d}s$$

converges. In conclusion, we obtain the existence of the limit

$$\lim_{t\to\infty} \frac{x(t)}{t^{\tilde r}} =: c,$$

which is what had to be proved. □

Theorem 2. *Suppose that* $0 < \tilde r < 1 < r < 2$, $p > 1$, $p(r-2)+1 > 0$, $a > 0$, $q = \frac{p}{p-1}$ *and the function* $f\colon M := [a,\infty)\times\mathbb{R}^3 \to \mathbb{R}$ *satisfy the following conditions:*

1. $f \in C(M,\mathbb{R})$,
2. *there are continuous functions from* $\mathbb{R}_+$ *to* $\mathbb{R}_+$, g_1, g_2, g_3, h_0, h_1, h_2, h_3, *such that* g_1, g_2, g_3 *are nondecreasing,*

$$\big|f(t,u,v,w)\big| \le t^{\gamma-1}\left(h_0(t) + h_1(t)g_1\left(\frac{|u|}{t}\right) + h_2(t)g_2(|v|) + h_3(t)g_3\left(\frac{|w|}{t^{1-\tilde r}}\right)\right)$$

for some $\gamma \in (1-\frac{1}{p}, 3-r-\frac{1}{p}]$, *and*

$$H_i := \int_a^\infty h_i^q(s)\mathrm{d}s < \infty, \quad i = 0,1,2,3,$$

3. $$\int_a^\infty \frac{\tau^{q-1}\mathrm{d}\tau}{g_1^q(\tau)+g_2^q(\tau)+g_3^q(\tau)} = \infty.$$

Then for any global solution $x(t)$ *of the initial value problem*

$$\begin{aligned} {}^C D_a^r x(t) &= f\big(t, x(t), x'(t), {}^C D_a^{\tilde r} x(t)\big), \quad t \ge a,\\ x(a) = c_0, &\qquad x'(a) = c_1, \end{aligned}$$

there exists a constant $c \in \mathbb{R}$ *such that*

$$x(t) = ct + o(t) \quad \text{as } t \to \infty.$$

Proof. For simplicity, we denote $F(t) := f(t, x(t), x'(t), {}^C D_a^{\tilde r} x(t))$. Then by Lemma 1, the solution $x(t)$ has the form

$$x(t) = c_0 + c_1(t-a) + \frac{1}{\Gamma(r)}\int_a^t (t-s)^{r-1}F(s)\mathrm{d}s, \quad t \ge a.$$

By differentiation, one gets

$$x'(t) = c_1 + \frac{1}{\Gamma(r-1)}\int_a^t (t-s)^{r-2}F(s)\mathrm{d}s, \quad t \ge a.$$

Consequently,

$$\begin{aligned} \frac{|x(t)|}{t} &\le \frac{|c_0|}{t} + \frac{|c_1|(t-a)}{t} + \frac{1}{\Gamma(r)}\int_a^t \left(\frac{t-s}{t}\right)(t-s)^{r-2}\big|F(s)\big|\mathrm{d}s\\ &\le \frac{|c_0|}{a} + |c_1| + \frac{1}{\Gamma(r)}\int_a^t (t-s)^{r-2}\big|F(s)\big|\mathrm{d}s \le z(t), \quad t \ge a, \end{aligned}$$

and

$$\left|x'(t)\right| \leq |c_1| + \frac{1}{\Gamma(r-1)} \int_a^t (t-s)^{r-2} \left|F(s)\right| \mathrm{d}s \leq z(t), \quad t \geq a \tag{3.5}$$

for

$$z(t) := C + C_\Gamma \int_a^t (t-s)^{r-2} \left|F(s)\right| \mathrm{d}s, \quad C = \frac{|c_0|}{a} + C_\Gamma |c_1|.$$

By Definition 3, ${}^C D_a^{\tilde{r}} x(t)$ is computed as

$$\begin{aligned} {}^C D_a^{\tilde{r}} x(t) &= \frac{1}{\Gamma(1-\tilde{r})} \int_a^t (t-s)^{-\tilde{r}} x'(s) \mathrm{d}s = \frac{c_1}{\Gamma(1-\tilde{r})} \int_a^t (t-s)^{-\tilde{r}} \mathrm{d}s \\ &\quad + \frac{1}{\Gamma(1-\tilde{r})\Gamma(r-1)} \int_a^t (t-s)^{-\tilde{r}} \int_a^s (s-w)^{r-2} F(w) \mathrm{d}w \mathrm{d}s \\ &= \frac{c_1 (t-a)^{1-\tilde{r}}}{\Gamma(2-\tilde{r})} + \frac{1}{\Gamma(1-\tilde{r})\Gamma(r-1)} \int_a^t F(w) \int_w^t (t-s)^{-\tilde{r}} (s-w)^{r-2} \mathrm{d}s \mathrm{d}w. \end{aligned}$$

Then, taking the substitution $s = w + \zeta(t-w)$ and using $B(1-\tilde{r}, r-1) = \frac{\Gamma(1-\tilde{r})\Gamma(r-1)}{\Gamma(r-\tilde{r})}$,

$${}^C D_a^{\tilde{r}} x(t) = \frac{c_1 (t-a)^{1-\tilde{r}}}{\Gamma(2-\tilde{r})} + \frac{1}{\Gamma(r-\tilde{r})} \int_a^t (t-s)^{r-\tilde{r}-1} F(s) \mathrm{d}s.$$

Hence,

$$\frac{|{}^C D_a^{\tilde{r}} x(t)|}{t^{1-\tilde{r}}} \leq \frac{|c_1|}{\Gamma(2-\tilde{r})} + \frac{1}{\Gamma(r-\tilde{r})} \int_a^t (t-s)^{r-2} \left|F(s)\right| \mathrm{d}s \leq z(t), \quad t \geq a.$$

Now, we apply the assumptions on f and the nondecreasing properties of functions g_1, g_2, g_3 to estimate $z(t)$:

$$z(t) \leq C + C_\Gamma \int_a^t (t-s)^{r-2} s^{\gamma-1} \left(h_0(s) + \sum_{i=1}^{3} h_i(s) g_i\big(z(s)\big) \right) \mathrm{d}s.$$

Hölder inequality and Lemma 2 with $\alpha = r - 1$ yield

$$\int_a^t (t-s)^{r-2} s^{\gamma-1} h_i(s) g_i\big(z(s)\big) \mathrm{d}s \leq a^{\frac{\Theta}{p}} B_1 \left(\int_a^t h_i^q(s) g_i^q\big(z(s)\big) \mathrm{d}s \right)^{\frac{1}{q}}, \quad t \geq a$$

for $i = 1, 2, 3$, where $B_1 = B^{\frac{1}{p}}(p(r-2)+1, p(\gamma-1)+1)$ and $\Theta = p(r+\gamma-3)+1 \in (p(r-2), 0]$. Similarly,

$$\int_a^t (t-s)^{r-2} s^{\gamma-1} h_0(s) \mathrm{d}s \leq a^{\frac{\Theta}{p}} B_1 \left(\int_a^t h_0^q(s) \mathrm{d}s \right)^{\frac{1}{q}}, \quad t \geq a.$$

Summarizing the above,

$$z(t) \leq C + C_\Gamma a^{\frac{\Theta}{p}} B_1 \left(\left(\int_a^t h_0^q(s) \mathrm{d}s \right)^{\frac{1}{q}} + \sum_{i=1}^{3} \left(\int_a^t h_i^q(s) g_i^q\big(z(s)\big) \mathrm{d}s \right)^{\frac{1}{q}} \right)$$

for any $t \geq a$. Taking the q-th power and using the inequality $(\sum_{i=1}^{5} a_i)^q \leq 5^{q-1} \sum_{i=1}^{5} a_i^q$ for any $a_i \geq 0$, $i = 1, 2, \ldots, 5$, we obtain

$$\begin{aligned} u(t) := z^q(t) &\leq A + D \sum_{i=1}^{3} \int_a^t h_i^q(s) g_i^q\big(z(s)\big) \mathrm{d}s \\ &\leq A + D \int_a^t \big(h_1^q(s) + h_2^q(s) + h_3^q(s)\big) \omega\big(u(s)\big) \mathrm{d}s, \quad t \geq a \end{aligned}$$

for $A = 5^{q-1}(C^q + C_\Gamma^q a^{\Theta(q-1)} B_1^q H_0)$, $D = 5^{q-1} C_\Gamma^q a^{\Theta(q-1)} B_1^q$, $\omega(u) = \sum_{i=1}^{3} g_i^q(u^{\frac{1}{q}})$. Finally, Bihari inequality implies

$$\begin{aligned} u(t) &\leq \Omega^{-1} \left(\Omega(A) + D \int_a^t h_1^q(s) + h_2^q(s) + h_3^q(s) \mathrm{d}s \right) \\ &\leq \Omega^{-1}\big(\Omega(A) + D(H_1 + H_2 + H_3)\big) =: K_0 < \infty, \end{aligned}$$

where $\Omega(v) = \int_{v_0}^{v} \frac{\mathrm{d}s}{\omega(s)}$, $0 < v_0 \leq v$. Thus $z(t) \leq K_0^{\frac{1}{q}}$ for $t \geq a$, and by (3.5),

$$0 \leq \int_a^t (t-s)^{r-2} \big|F(s)\big| \mathrm{d}s \leq \Gamma(r-1)\big(K_0^{\frac{1}{q}} - |c_1|\big) < \infty, \quad t \geq a,$$

i.e., the integral $\int_a^\infty (t-s)^{r-2} F(s) \mathrm{d}s$ converges. So, there exists a constant c such that $\lim_{t\to\infty} x'(t) = c$, and by applying l'Hôpital's rule,

$$\lim_{t\to\infty} \frac{x(t)}{t} = \lim_{t\to\infty} x'(t) = c.$$

This concludes the proof. □

The following theorem considers a general case when the order r is a positive real non-integer number.

Theorem 3. *Suppose that $r > 0$ and $n \in \mathbb{N}$ be such that $n-1 < r < n$, $m \in \mathbb{N}$, $\tilde{r}_1, \ldots, \tilde{r}_m \in \mathbb{R} \backslash \mathbb{N}$ satisfy $0 < \tilde{r}_1 < \cdots < \tilde{r}_m < r$, $R := \max\{n-1, \tilde{r}_m\}$, $p > 1$, $p(r-R-1)+1 > 0$, $a > 0$, $q = \frac{p}{p-1}$ and the function $f\colon M := [a,\infty) \times \mathbb{R}^{n+m} \to \mathbb{R}$ satisfy the following conditions:*

1. *$f \in C(M, \mathbb{R})$,*
2. *there are continuous functions from $\mathbb{R}_+$ to $\mathbb{R}_+$, $g_1, g_2, \ldots, g_{n+m}$, $h_0, h_1, \ldots, h_{n+m}$, such that $g_1, g_2, \ldots, g_{n+m}$ are nondecreasing,*

$$\begin{aligned} &\big|f(t, u_0, \ldots, u_{n-1}, v_1, \ldots, v_m)\big| \\ &\quad \leq t^{\gamma-1} \left(h_0(t) + \sum_{i=1}^{n} h_i(t) g_i \left(\frac{|u_{i-1}|}{t^{R+1-i}} \right) + \sum_{j=1}^{m} h_{n+j}(t) g_{n+j} \left(\frac{|v_j|}{t^{R-\tilde{r}_j}} \right) \right) \end{aligned}$$

for some $\gamma \in (1 - \frac{1}{p}, 2 - r + R - \frac{1}{p}]$, and

$$H_i := \int_a^\infty h_i^q(s) \mathrm{d}s < \infty, \quad i = 0, 1, \ldots, n+m,$$

3.
$$\int_a^\infty \frac{\tau^{q-1}\mathrm{d}\tau}{\sum_{i=1}^{n+m} g_i^q(\tau)} = \infty.$$

Then for any global solution $x(t)$ of the initial value problem

$$\begin{cases} {}^C D_a^r x(t) = f\big(t, x(t), x'(t), \dots, x^{(n-1)}(t), {}^C D_a^{\tilde r_1} x(t), \dots, {}^C D_a^{\tilde r_m} x(t)\big), \quad t \ge a, \\ x^{(i)}(a) = c_i, \quad i = 0, 1, \dots, n-1, \end{cases} \tag{3.6}$$

there exists a constant $c \in \mathbb{R}$ such that

$$x(t) = ct^R + o\big(t^R\big) \quad \text{as } t \to \infty.$$

Proof. In the whole proof,

$$F(t) := f\big(t, x(t), x'(t), \dots, x^{(n-1)}(t), {}^C D_a^{\tilde r_1} x(t), \dots, {}^C D_a^{\tilde r_m} x(t)\big).$$

By Lemma 1,

$$x(t) = c_0 + c_1(t-a) + \dots + \frac{c_{n-1}}{(n-1)!}(t-a)^{n-1} + \frac{1}{\Gamma(r)} \int_a^t (t-s)^{r-1} F(s)\mathrm{d}s. \tag{3.7}$$

We define

$$z(t) := C + C_\Gamma \int_a^t (t-s)^{r-R-1} \big|F(s)\big|\mathrm{d}s, \quad C = \frac{|c_0|}{a^R} + C_\Gamma \sum_{i=1}^{n-1} \frac{|c_i|}{a^{R-i}}.$$

Differentiating (3.7), we get

$$x^{(i)}(t) = c_i + c_{i+1}(t-a) + \dots + \frac{c_{n-1}(t-a)^{n-1-i}}{(n-1-i)!} + \frac{1}{\Gamma(r-i)} \int_a^t (t-s)^{r-1-i} F(s)\mathrm{d}s \tag{3.8}$$

for $i = 1, 2, \dots, n-1$. It is easy to see, that

$$\begin{aligned} \frac{|x^{(i)}(t)|}{t^{R-i}} &\le \frac{|c_i|}{t^{R-i}} + \frac{|c_{i+1}|(t-a)}{t^{R-i}} + \dots + \frac{|c_{n-1}|(t-a)^{n-1-i}}{(n-1-i)!\, t^{R-i}} \\ &\quad + \frac{1}{\Gamma(r-i)} \int_a^t \left(\frac{t-s}{t}\right)^{R-i} (t-s)^{r-R-1} \big|F(s)\big|\mathrm{d}s \\ &\le \frac{|c_i|}{a^{R-i}} + \frac{|c_{i+1}|}{a^{R-i-1}} + \dots + \frac{|c_{n-1}|}{(n-1-i)!\, a^{R-n+1}} \\ &\quad + \frac{1}{\Gamma(r-i)} \int_a^t (t-s)^{r-R-1} \big|F(s)\big|\mathrm{d}s \le z(t), \quad t \ge a \end{aligned} \tag{3.9}$$

for each $i = 0, 1, \dots, n-1$. Now for each $j \in \{1, 2, \dots, m\}$ there exists $i_j \in \{1, 2, \dots, n\}$ such that $i_j - 1 < \tilde r_j < i_j$.

If $i_j < n$, then ${}^C D_a^{\tilde r_j} x(t) = I_a^{i_j-\tilde r_j} x^{(i_j)}(t)$, and we apply the formula (3.8) to get

$$\begin{aligned}
{}^C D_a^{\tilde r_j} x(t) &= \frac{c_{i_j}}{\Gamma(i_j-\tilde r_j)} \int_a^t (t-s)^{i_j-\tilde r_j-1} \mathrm{d}s \\
&\quad + \frac{c_{i_j+1}}{\Gamma(i_j-\tilde r_j)} \int_a^t (t-s)^{i_j-\tilde r_j-1}(s-a)\mathrm{d}s \\
&\quad + \cdots + \frac{c_{n-1}}{(n-1-i_j)!\,\Gamma(i_j-\tilde r_j)} \int_a^t (t-s)^{i_j-\tilde r_j-1}(s-a)^{n-1-i_j}\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(r-i_j)\Gamma(i_j-\tilde r_j)} \int_a^t (t-s)^{i_j-\tilde r_j-1} \int_a^s (s-w)^{r-1-i_j} F(w)\mathrm{d}w\mathrm{d}s.
\end{aligned}$$

Substituting $s = a + \zeta(t-a)$ and using the beta function give

$$\frac{c_k}{(k-i_j)!\,\Gamma(i_j-\tilde r_j)} \int_a^t (t-s)^{i_j-\tilde r_j-1}(s-a)^{k-i_j}\mathrm{d}s = \frac{c_k (t-a)^{k-\tilde r_j}}{\Gamma(k+1-\tilde r_j)}$$

for $k = i_j, i_j+1, \dots, n-1$, and changing the order of integration and substitution of $s = w + \zeta(t-w)$ yield

$$\begin{aligned}
&\frac{1}{\Gamma(r-i_j)\Gamma(i_j-\tilde r_j)} \int_a^t (t-s)^{i_j-\tilde r_j-1} \int_a^s (s-w)^{r-1-i_j} F(w)\mathrm{d}w\mathrm{d}s \\
&\quad = \frac{1}{\Gamma(r-\tilde r_j)} \int_a^t (t-w)^{r-1-\tilde r_j} F(w)\mathrm{d}w.
\end{aligned}$$

Therefore,

$$\begin{aligned}
{}^C D_a^{\tilde r_j} x(t) &= \frac{c_{i_j}(t-a)^{i_j-\tilde r_j}}{\Gamma(i_j-\tilde r_j+1)} + \frac{c_{i_j+1}(t-a)^{i_j-\tilde r_j+1}}{\Gamma(i_j-\tilde r_j+2)} \\
&\quad + \cdots + \frac{c_{n-1}(t-a)^{n-1-\tilde r_j}}{\Gamma(n-\tilde r_j)} + \frac{1}{\Gamma(r-\tilde r_j)} \int_a^t (t-s)^{r-1-\tilde r_j} F(s)\mathrm{d}s.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\frac{|{}^C D_a^{\tilde r_j} x(t)|}{t^{R-\tilde r_j}} &\le \frac{|c_{i_j}|}{\Gamma(i_j-\tilde r_j+1)t^{R-i_j}} \left(\frac{t-a}{t}\right)^{i_j-\tilde r_j} \\
&\quad + \frac{|c_{i_j+1}|}{\Gamma(i_j-\tilde r_j+2)t^{R-i_j-1}} \left(\frac{t-a}{t}\right)^{i_j-\tilde r_j+1} \\
&\quad + \cdots + \frac{|c_{n-1}|}{\Gamma(n-\tilde r_j)t^{R-n+1}} \left(\frac{t-a}{t}\right)^{n-1-\tilde r_j} \\
&\quad + \frac{1}{\Gamma(r-\tilde r_j)} \int_a^t \left(\frac{t-s}{t}\right)^{R-\tilde r_j} (t-s)^{r-R-1} \big|F(s)\big|\mathrm{d}s \\
&\le \frac{|c_{i_j}|}{\Gamma(i_j-\tilde r_j+1)a^{R-i_j}} + \frac{|c_{i_j+1}|}{\Gamma(i_j-\tilde r_j+2)a^{R-i_j-1}} \\
&\quad + \cdots + \frac{|c_{n-1}|}{\Gamma(n-\tilde r_j)a^{R-n+1}} + \frac{1}{\Gamma(r-\tilde r_j)} \int_a^t (t-s)^{r-R-1} \big|F(s)\big|\mathrm{d}s \\
&\le z(t)
\end{aligned}$$

for any $t \geq a$.

In the other case, when $i_j = n$, the fractional derivative ${}^C D_a^{\tilde{r}_j} x(t)$ is obtained by applying the integral operator $I_a^{r-\tilde{r}_j}$ on equation (3.6) (as in (3.4)). So we get

$$ {}^C D_a^{\tilde{r}_j} x(t) = \frac{1}{\Gamma(r-\tilde{r}_j)} \int_a^t (t-s)^{r-\tilde{r}_j-1} F(s) \mathrm{d}s, $$

hence

$$ \frac{|{}^C D_a^{\tilde{r}_j} x(t)|}{t^{R-\tilde{r}_j}} = \frac{1}{\Gamma(r-\tilde{r}_j)} \int_a^t \left(\frac{t-s}{t} \right)^{R-\tilde{r}_j} (t-s)^{r-R-1} \big| F(s) \big| \mathrm{d}s \leq z(t) \quad (3.10) $$

for any $t \geq a$.

Now, we use the assumptions on f, the above estimates (3.9) and (3.10), and the nondecreasing properties of functions $g_1, g_2, \ldots, g_{n+m}$ to estimate

$$ \begin{aligned} z(t) \leq{} & C + C_\Gamma \int_a^t (t-s)^{r-R-1} s^{\gamma-1} \\ & \times \left(h_0(s) + \sum_{i=1}^n h_i(s) g_i \left(\frac{|x^{(i-1)}(s)|}{t^{R+1-i}} \right) + \sum_{j=1}^m h_{n+j}(s) g_{n+j} \left(\frac{|{}^C D_a^{\tilde{r}_j} x(s)|}{t^{R-\tilde{r}_j}} \right) \right) \mathrm{d}s \\ & \leq C + C_\Gamma \int_a^t (t-s)^{r-R-1} s^{\gamma-1} \left(h_0(s) + \sum_{i=1}^{n+m} h_i(s) g_i \big(z(s) \big) \right) \mathrm{d}s, \quad t \geq a. \end{aligned} $$

Hölder inequality and Lemma 2 with $\alpha = r - R$ imply

$$ \int_a^t (t-s)^{r-R-1} s^{\gamma-1} h_i(s) g_i \big(z(s) \big) \mathrm{d}s \leq a^{\frac{\Theta}{p}} B_1 \left(\int_a^t h_i^q(s) g_i^q \big(z(s) \big) \mathrm{d}s \right)^{\frac{1}{q}}, \quad t \geq a $$

for $i = 1, 2, \ldots, n+m$, where $B_1 = B^{\frac{1}{p}}(p(r-R-1)+1, p(\gamma-1)+1)$ and $\Theta = p(r-R+\gamma-2)+1 \in (p(r-R-1), 0]$. Similarly,

$$ \int_a^t (t-s)^{r-R-1} s^{\gamma-1} h_0(s) \mathrm{d}s \leq a^{\frac{\Theta}{p}} B_1 \left(\int_a^t h_0^q(s) \mathrm{d}s \right)^{\frac{1}{q}}, \quad t \geq a. $$

Thus

$$ z(t) \leq C + C_\Gamma a^{\frac{\Theta}{p}} B_1 \left(\left(\int_a^t h_0^q(s) \mathrm{d}s \right)^{\frac{1}{q}} + \sum_{i=1}^{n+m} \left(\int_a^t h_i^q(s) g_i^q \big(z(s) \big) \mathrm{d}s \right)^{\frac{1}{q}} \right), $$

and after taking the q-th power and using the inequality

$$ \left(\sum_{i=1}^{n+m+2} a_i \right)^q \leq (n+m+2)^{q-1} \sum_{i=1}^{n+m+2} a_i^q $$

for any $a_i \geq 0$, $i = 1, 2, \ldots, n+m+2$, one arrives at

$$\begin{aligned} u(t) := z^q(t) &\leq A + D \sum_{i=1}^{n+m} \int_a^t h_i^q(s) g_i^q\big(z(s)\big) \mathrm{d}s \\ &\leq A + D \int_a^t \bigg(\sum_{i=1}^{n+m} h_i^q(s) \bigg) \omega\big(u(s)\big) \mathrm{d}s \end{aligned}$$

with $A = (n+m+2)^{q-1}(C^q + C_\Gamma^q a^{\Theta(q-1)} B_1^q H_0)$, $D = (n+m+2)^{q-1} C_\Gamma^q a^{\Theta(q-1)} B_1^q$, $\omega(u) = \sum_{i=1}^{n+m} g_i^q(u^{\frac{1}{q}})$. Finally, by Bihari inequality

$$\begin{aligned} u(t) &\leq \Omega^{-1}\bigg(\Omega(A) + D \int_a^t \sum_{i=1}^{n+m} h_i^q(s) \mathrm{d}s \bigg) \\ &\leq \Omega^{-1}\bigg(\Omega(A) + D \sum_{i=1}^{n+m} H_i \bigg) =: K_0 < \infty, \quad t \geq a, \end{aligned}$$

where

$$\Omega(v) = \int_{v_0}^v \frac{\mathrm{d}s}{\omega(s)}, \quad 0 < v_0 \leq v,$$

i.e., $z(t) \leq K_0^{\frac{1}{q}}$ for any $t \geq a$. Note that for (3.9) with $i = n-1$, this means that

$$\begin{aligned} \frac{|x^{(n-1)}(t)|}{t^{R-n+1}} &\leq \frac{|c_{n-1}|}{t^{R-n+1}} + \frac{1}{\Gamma(r-n+1)} \int_a^t \left(\frac{t-s}{t} \right)^{R-n+1} (t-s)^{r-R-1} \big|F(s)\big| \mathrm{d}s \\ &\leq z(t) \leq K_0^{\frac{1}{q}} < \infty, \quad t \geq a. \end{aligned}$$

In other words,

$$\int_a^t \left(\frac{t-s}{t} \right)^{R-n+1} (t-s)^{r-R-1} \big|F(s)\big| \mathrm{d}s \leq \Gamma(r-n+1) K_0^{\frac{1}{q}}, \quad t \geq a,$$

and so there exists the limit

$$\lim_{t\to\infty} \int_a^t \left(\frac{t-s}{t} \right)^{R-n+1} (t-s)^{r-R-1} F(s) \mathrm{d}s =: \tilde{c} \in [0, \infty).$$

The statement follows by applying the l'Hôpital rule

$$\begin{aligned} \lim_{t\to\infty} \frac{x(t)}{t^R} &= \frac{1}{\prod_{i=0}^{n-2}(R-i)} \lim_{t\to\infty} \frac{x^{(n-1)}(t)}{t^{R-n+1}} \\ &= \frac{1}{\prod_{i=0}^{n-2}(R-i)} \left(\lim_{t\to\infty} \frac{c_{n-1}}{t^{R-n+1}} + \frac{\tilde{c}}{\Gamma(r-n+1)} \right) =: c, \end{aligned}$$

where the value of c depends on R. □

At the end, we consider the case when the right-hand side depends also on Riemann–Liouville integrals of the solution.

Theorem 4. *Suppose that $r > 0$ and $n \in \mathbb{N}$ be such that $n-1 < r < n$, $m \in \mathbb{N}$, $\tilde{r}_1, \dots, \tilde{r}_m \in \mathbb{R}\backslash\mathbb{N}$ satisfy $0 < \tilde{r}_1 < \dots < \tilde{r}_m < r$, $R := \max\{n-1, \tilde{r}_m\}$, $\widetilde{m} \in \mathbb{N}$, $q_1, \dots, q_{\widetilde{m}} > 0$, $p > 1$, $p(r-R-1)+1 > 0$, $a > 0$, $q = \frac{p}{p-1}$ and the function $f\colon M := [a,\infty) \times \mathbb{R}^{n+m+\widetilde{m}} \to \mathbb{R}$ satisfy the following conditions:*

1. $f \in C(M, \mathbb{R})$,
2. *there are continuous functions from $\mathbb{R}_+$ to $\mathbb{R}_+$, $g_1, g_2, \dots, g_{n+m+\widetilde{m}}$, h_0, $h_1, \dots, h_{n+m+\widetilde{m}}$, such that $g_1, g_2, \dots, g_{n+m+\widetilde{m}}$ are nondecreasing,*

$$\begin{aligned}
&\bigl|f(t, u_0, \dots, u_{n-1}, v_1, \dots, v_m, w_1, \dots, w_{\widetilde{m}})\bigr| \\
&\quad\le t^{\gamma-1}\Biggl(h_0(t) + \sum_{i=1}^{n} h_i(t) g_i\left(\frac{|u_{i-1}|}{t^{R+1-i}}\right) + \sum_{j=1}^{m} h_{n+j}(t) g_{n+j}\left(\frac{|v_j|}{t^{R-\tilde{r}_j}}\right) \\
&\qquad + \sum_{j=1}^{\widetilde{m}} h_{n+m+j}(t) g_{n+m+j}\left(\frac{|w_j|}{t^{R+q_j}}\right)\Biggr)
\end{aligned}$$

for some $\gamma \in (1 - \frac{1}{p}, 2 - r + R - \frac{1}{p}]$, and

$$H_i := \int_a^\infty h_i^q(s)\mathrm{d}s < \infty, \quad i = 0, 1, \dots, n+m+\widetilde{m},$$

3. $$\int_a^\infty \frac{\tau^{q-1}\mathrm{d}\tau}{\sum_{i=1}^{n+m+\widetilde{m}} g_i^q(\tau)} = \infty.$$

Then for any global solution $x(t)$ of the initial value problem

$$\begin{cases} {}^C D_a^r x(t) = f\bigl(t, x(t), x'(t), \dots, x^{(n-1)}(t), {}^C D_a^{\tilde{r}_1} x(t), \dots, {}^C D_a^{\tilde{r}_m} x(t), \\ \qquad\qquad I_a^{q_1} x(t), \dots, I_a^{q_{\widetilde{m}}} x(t)\bigr), \quad t \ge a, \\ x^{(i)}(a) = c_i, \quad i = 0, 1, \dots, n-1, \end{cases} \tag{3.11}$$

there exists a constant $c \in \mathbb{R}$ such that

$$x(t) = ct^R + o(t^R) \quad \text{as } t \to \infty.$$

Proof. For simplicity we denote,

$$\begin{aligned} F(t) := f\bigl(&t, x(t), x'(t), \dots, x^{(n-1)}(t), {}^C D_a^{\tilde{r}_1} x(t), \dots, {}^C D_a^{\tilde{r}_m} x(t), \\ &I_a^{q_1} x(t), \dots, I_a^{q_{\widetilde{m}}} x(t)\bigr). \end{aligned}$$

By Lemma 1, the solution $x(t)$ of (3.11) has the form (3.7). Note that the estimations (3.9) and (3.10) remain valid for

$$z(t) := C + C_\Gamma \int_a^t (t-s)^{r-R-1}\bigl|F(s)\bigr|\mathrm{d}s, \quad C = C_\Gamma \sum_{i=0}^{n-1} \frac{|c_i|}{a^{R-i}}.$$

For each $j = 1, 2, \dots, \widetilde{m}$, the fractional integral $I_a^{q_j} x(t)$ is obtained by integrating formula (3.7) to get

$$\begin{aligned}
I_a^{q_j} x(t) = {} & \frac{c_0}{\Gamma(q_j)} \int_a^t (t-s)^{q_j - 1} \mathrm{d}s + \frac{c_1}{\Gamma(q_j)} \int_a^t (t-s)^{q_j-1}(s-a)\mathrm{d}s \\
& + \dots + \frac{c_{n-1}}{\Gamma(q_j)(n-1)!} \int_a^t (t-s)^{q_j-1}(s-a)^{n-1}\mathrm{d}s \\
& + \frac{1}{\Gamma(r)\Gamma(q_j)} \int_a^t (t-s)^{q_j-1} \int_a^s (s-w)^{r-1} F(w)\mathrm{d}w\mathrm{d}s \\
= {} & \frac{c_0(t-a)^{q_j}}{\Gamma(q_j+1)} + \frac{c_1(t-a)^{q_j+1}}{\Gamma(q_j+2)} \\
& + \dots + \frac{c_{n-1}(t-a)^{q_j+n-1}}{\Gamma(q_j+n)} + \frac{1}{\Gamma(q_j+r)} \int_a^t (t-s)^{q_j+r-1} F(s)\mathrm{d}s.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{|I_a^{q_j} x(t)|}{t^{R+q_j}} \le {} & \frac{|c_0|}{\Gamma(q_j+1)t^R} \left(\frac{t-a}{t}\right)^{q_j} + \frac{|c_1|}{\Gamma(q_j+2)t^{R-1}} \left(\frac{t-a}{t}\right)^{q_j+1} \\
& + \dots + \frac{|c_{n-1}|}{\Gamma(q_j+n)t^{R-n+1}} \left(\frac{t-a}{t}\right)^{q_j+n-1} \\
& + \frac{1}{\Gamma(q_j+r)} \int_a^t \left(\frac{t-s}{t}\right)^{R+q_j} (t-s)^{r-R-1} \big|F(s)\big|\mathrm{d}s \\
\le {} & \frac{|c_0|}{\Gamma(q_j+1)a^R} + \frac{|c_1|}{\Gamma(q_j+2)a^{R-1}} + \dots + \frac{|c_{n-1}|}{\Gamma(q_j+n)a^{R-n+1}} \\
& + \frac{1}{\Gamma(q_j+r)} \int_a^t (t-s)^{r-R-1} \big|F(s)\big|\mathrm{d}s \le z(t)
\end{aligned}$$

for any $t \ge a$, $j = 1, 2, \dots, \widetilde{m}$. So, after applying the assumption on f, one arrives at

$$\begin{aligned}
z(t) \le {} & C + C_\Gamma \int_a^t (t-s)^{r-R-1} s^{\gamma-1} \Bigg(h_0(s) + \sum_{i=1}^n h_i(s) g_i\left(\frac{|x^{(i-1)}(s)|}{t^{R+1-i}}\right) \\
& + \sum_{j=1}^m h_{n+j}(s) g_{n+j}\left(\frac{|{}^C D_a^{\tilde{r}_j} x(s)|}{t^{R-\tilde{r}_j}}\right) + \sum_{j=1}^{\widetilde{m}} h_{n+m+j}(s) g_{n+m+j}\left(\frac{|I_a^{q_j} x(s)|}{t^{R+q_j}}\right)\Bigg)\mathrm{d}s \\
\le {} & C + C_\Gamma \int_a^t (t-s)^{r-R-1} s^{\gamma-1} \left(h_0(s) + \sum_{i=1}^{n+m+\widetilde{m}} h_i(s) g_i\big(z(s)\big)\right)\mathrm{d}s, \quad t \ge a.
\end{aligned}$$

The rest of the proof can be carried out as the proof of Theorem 3. □

4 Conclusion

In this paper, we considered fractional differential equations with Caputo derivative of any positive non-integer order of a solution on the left-hand side and a general right-hand side depending on a solution, its integer and fractional derivatives, and its Riemann–Liouville integrals of arbitrary order. We stated sufficient conditions for any global solution to behave like $ct^R + o(t^R)$ for a convenient $R > 0$ as $t \to \infty$. The existence of the global solution for these equations was not investigated, and it remains to be proved in another paper.

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