Generating Pareto Optimal Solutions of Multi-Objective LFPP with Interval Coefficients Using ϵ-Constraint Method

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Abstract. This paper illustrates a procedure to generate pareto optimal solutions of multi-objective linear fractional programming problem (MOLFPP) with closed interval coefficients of decision variables both in objective and constraint functions. ϵ-constraint method is applied to produce pareto optimal solutions comprising most preferred solution to satisfy all objectives. A numerical example is solved using our proposed method and the result so obtained is compared with that of fuzzy programming which justifies the efficiency and authenticity of the proposed method.

Keywords: multi-objective LFPP, ϵ-constraint method, closed interval coefficients, Fuzzy programming.

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1 Introduction

Mathematical optimization is described as the science of determining the best solution to the mathematically modelled real world problems. Generally, it is the process of formulation and solution of constrained or unconstrained optimization problems occurring in every branch of science and engineering designed problems. Various real world problems concerned with decision making situations can be modelled as constrained or unconstrained multi-objective optimization (i.e., vector optimization or pareto optimization) problems. In multi-objective optimization problems, usually there does not exist single solution which satisfies all the objectives, however a set of pareto optimal(non-inferior, non-dominated) solutions can be generated from which the decision maker (DM) has a choice to decide the most preferred pareto optimal solution. In the recent past, researchers have given more emphasis on multi-objective techniques to handel fractional optimization problems. In fractional programming problems, the numerators and denominators of the objectives or

the constraints are considered as linear or nonlinear functions. In order to optimize a multi-objective fractional programming problem, several ratios of functions need to be optimized simultaneously. It has wide range of application in numerous important fields like finance, engineering, business, economics, management, and so forth. Various mathematical optimization problems comprising fractional objectives are frequently encountered [1, 20] in many real life situations like optimization of profit/cost, output/employee, cost/time, inventory/sale, etc. Linear fractional programming problem (LFPP) contains some linear constraints and a fractional objective function i.e., fraction of two linear functions was developed by Hungarian mathematician B.Martos [14] in 1960. Several methods have been proposed in literature for obtaining the optimal solution of LFPP. Charnes and Cooper [5] developed a variable transformation technique in 1960 which converts the fractional objective function to a non-fractional linear form with an additional variable and a constraint. Costa [7] proposed weighted sum technique to solve linear fractional programming problem. Jain and Arya [12] proposed an inverse optimization model to find out the solutions of a linear fractional programming problem. Borza et al. [3] derived a method to solve LFPP with interval coefficients in the objective function. Effati and Pakdaman [8] converted the objective function with interval coefficients to an interval valued objective function. Some important concepts on interval coefficients in the constraints and the objective functions are explained by H.C. Wu. [22]. Multi-objective LFPP comprises many fractional objective functions to be optimized subject to a certain set of linear constraints. Many methods have been proposed for solution in this regard. Pal and Sen [19] proposed goal programming method to solve interval valued MOLFPP. Milan [11] discussed linear fractional programming problem under interval uncertainty to obtain a range for optimal values of the objective functions. Nykowski and Zolkiewski [17] proposed a compromise procedure to determine the non-dominated solution whereas Caballero and Hernandez [4] developed a method to find a set of weakly efficient solutions of MOLFPP. Suprajitno [21] proposed a method to solve multiobjective linear programming with interval coefficients. Bellman and Zadeh [2] presented some basic concepts regarding fuzzy programming which is widely followed. Luhandjula [13] proposed fuzzy approach to find the efficient solution of MOLFPP.

Many real world problems when are mathematically modelled, do not carry certain values as coefficients. In such situations, intervals of fixed values are better to be assumed as the coefficients of the decision variables. In this paper, closed intervals of certain values are assumed as the coefficients of the decision variables both in objective and constraint functions of a multi-objective LFPP. Relative minimum and maximum values of each objective function with respect to others are determined then ϵ-constraint method due to Haimes et al. [10] is used to derive a set of non-inferior solutions from which the decision maker(DM) decides the most preferred optimal solution to satisfy all the objectives with best compromising objective values on preference basis. Efficiency of our proposed method is ensured by comparing the results with the existing fuzzy programming(Zimmermann's max-min operator) method through an example. The organisation of the paper is as follows: Following introduction, some useful

results regarding interval arithmetic have been presented in Section 2. Some brief ideas of multi-objective optimization problem and existence of it's optimal solution have been concisely interpreted in Section 3 whereas ϵ-constraint method has been incorporated in Section 4. Fuzzy programming method to solve a multi-objective minimization problem has been explained in Section 5. Our proposed method for solving multi-objective linear fractional programming problem with interval coefficients has been discussed in Section 6. An illustrative example has been solved using both the proposed method and existing fuzzy programming method separately then some remarks regarding the solutions obtained are added in Section 7. Finally, some conclusions are drawn from the discussion have been incorporated in Section 8.

2 Preliminaries of Interval Arithmetic

Assume that I denotes the colletion of all closed intervals in real line $\mathbb{R}$. Let $b, c \in$ I such that $b = [b^L, b^U], c = [c^L, c^U]$, where $b^L, c^L, b^U, c^U \in \mathbb{R}$ are lower and upper bounds of b and c. The results of interval arithmetic given by [16] are interpreted as follows.

- $b + c = [b^L + c^L, b^U + c^U]$.

- for $\alpha \in \mathbb{R}$, $\alpha b = \begin{cases} [\alpha b^L, \alpha b^U], & \alpha \geq 0, \\ [\alpha b^U, \alpha b^L], & \alpha < 0. \end{cases}$

- $b - c = [b^L - c^U, b^U - c^L]$.

- $bc = [b^L, b^U][c^L, c^U] = [Min D_1, Max D_1]$,
  where $D_1 = \{b^L c^L, b^L c^U, b^U c^L, b^U c^U\}$.

- $b/c = [b^L, b^U]/[c^L, c^U] = [Min D_2, Max D_2]$, where $D_2 = [\frac{b^L}{c^L}, \frac{b^L}{c^U}, \frac{b^U}{c^L}, \frac{b^U}{c^U}]$ and $0 \notin c$.

3 Multi-Objective Optimization Problem (MOOP)

Optimization is an important activity in many fields of science and engineering. The classical frame work for optimization is to find optimal value of objective functions with respect to given constraints. A multi-objective optimization problem [9, 15] can be stated as:

$$\max/\min f(x) = \big(f_1(x), f_2(x), \ldots, f_k(x)\big) \qquad (3.1)$$
$$\text{subject to } x \in \Omega,$$

where Ω is a nonempty compact feasible region. Usually, there does not exist a single solution which optimizes all the objective funtions simultaneously with cent percent satisfaction. Therefore, an appropriate method is used to generate a set of non-inferior or pareto optimal solutions from which the most preferred optimal solution(that satisfies all the objective functions with best possibility) can be determined by the decision maker (DM).

DEFINITION [See [15]]. $x^* \in \Omega$ is a pareto optimal solution of MOOP (3.1) if there does not exist another feasible solution $x^f \in \Omega$ such that $f_i(x^f) \le f_i(x^*)$ $\forall i$ and $f_j(x^f) < f_j(x^*)$ for at least one j.

4 ϵ-Constraint Method

The ϵ-constraint method was proposed by Haimes et al. [10] for generating pareto optimal solutions of a MOOP . It is a posteriori preference based method that generates the non-inferior solutions [6, 18] by considering one objective function at a time as primary one and converting the remaining objective functions as constraints. In other words, it minimizes one objective function and simultaneously maintains the maximum acceptability level for other objective functions. The ϵ-constraint method is defined as:

$$\text{Min } f_s(x), \ s \in \{1, 2, \ldots, k\} \tag{4.1}$$
subject to
$$f_i(x) \le \epsilon_i, \ i = 1, 2, \ldots, s-1, s+1, \ldots, k, \quad x \in \Omega,$$

where, $\epsilon_i^L \le \epsilon_i \le \epsilon_i^U$, $\epsilon_i^L = Min \ f_i(x) \ \forall x \in \Omega$, $\epsilon_i^U = Max \ f_i(x) \ \forall x \in \Omega$, i.e., the nonempty compact feasible region of the problem. Taking the value of ϵ_i in $[\epsilon_i^L, \epsilon_i^U]$, a sequence of optimal solutions for the objective function of high priority can be generated from which the most preferred optimal solution is determined.

5   Fuzzy Programming

The concept of fuzzy set theory was first proposed by Zadeh in 1965 which represents the imprecise expressions of real world problems in precise mathematical forms. Zimmermann proposed max-min operator in fuzzy approach [23] to solve multi-objective linear optimization problem. Each fuzzy set is associated with a membership function that determines the degree of membership of it's elements in the range $[0, 1]$. Several types of membership functions are available in fuzzy programming to solve a multi-objective optimization problem but a suitable one is to be chosen in order to determine the optimal solution. The following steps are followed by fuzzy programming(max-min operator technique) to determine the optimal solution of an MOOP as defined in (3.1).

Step 1. Obtain the values of relative minimum(L_i) and relative maximum(U_i) of each objective $f_i(x)$ with respect to other objective functions such that $L_i \le f_i \le U_i, i = 1, 2, \ldots, k$.

Step 2. We consider the following membership function in our proposed method to determine the optimal solution of an MOOP. Define the fuzzy membership function $\mu(x)$ for each objective f_i of the problem (3.1) as:

$$\mu_i(x) = \begin{cases} 1, & f_i(x) \le L_i, \\ \dfrac{U_i - f_i(x)}{U_i - L_i}, & L_i \le f_i(x) \le U_i, \\ 0, & f_i(x) \ge U_i. \end{cases} \tag{5.1}$$

Step 3. Construct the following crisp model using Zimmermann's max–min operator and solve it to obtain the optimal compromise solution of MOOP (3.1).

$$\text{Max } \left\{ \min_{1 \le i \le k} \mu_i(x) \right\} \tag{5.2}$$

$$\text{subject to } x \in \Omega.$$

This multi-objective optimization problem (5.2) is equivalent to the following problem (5.3) defined as:

$$\text{Max } \lambda \tag{5.3}$$
$$\text{subject to}$$
$$\mu_i(x) = \frac{U_i - f_i(x)}{U_i - L_i} \ge \lambda, \ i = 1, 2...k, \quad x \in \Omega,$$

where λ is an auxiliary variable and the constraints $\mu_i(x) \ge \lambda$ can be replaced by $f_i(x) + \lambda(U_i - L_i) \le U_i, i = 1, 2, \ldots, k$ for simplicity.

Step 4. Use an appropriate method to solve the above single objective optimization problem (5.3) to obtain the optimal solution x^* and evaluate the functional values of the objectives $f_i(x)$ $(i = 1, 2, \ldots, k)$ at this solution to be considered as their optimal objective values.

6 Multi-Objective LFPP with Interval Coefficients

Multi-objective LFPP with interval coefficients of decision variables both in objective and constraint functions can be generally formulated as follows.

$$\text{Min } f(x) = \big(f_1(x), f_2(x), \ldots, f_k(x)\big)$$
$$\text{subject to}$$
$$\Omega = \big\{ x = (x_j) \in \mathbb{R}^n \mid Ax \le b, \ x \ge 0 \big\},$$

where

$$f_i(x) = \frac{\sum_{j=1}^{n} c_{ij} x_j + \alpha_i}{\sum_{j=1}^{n} d_{ij} x_j + \beta_i}, i = 1, 2, \ldots, k,$$
$$A = (a_{tj})_{m \times n} \text{ and } b = (b_t) \in \mathbb{R}^m, \ t = 1, 2, \ldots, m.$$

Assume that

$$\sum_{j=1}^{n} d_{ij} x_j + \beta_i > 0 \ \forall \ i, j \text{ and } c_{ij}, d_{ij}, \alpha_i, \beta_i, a_{tj} \in I,$$

$$c_{ij} = [c_{ij}^L, c_{ij}^U], d_{ij} = [d_{ij}^L, d_{ij}^U], \alpha_i = [\alpha_i^L, \alpha_i^U], \beta_i = [\beta_i^L, \beta_i^U], a_{tj} = [a_{tj}^L, a_{tj}^U]$$

$\forall i, j, t$. Let $d_{ij}^L, \beta_i^L > 0$ for satisfying the positivity assumption of the denominator objective functions.

The above multi-objective LFPP can be stated as follows.

Problem P.

$$\text{Min } f_i(x) = \frac{\sum_{j=1}^{n} [c_{ij}^L, c_{ij}^U] x_j + [\alpha_i^L, \alpha_i^U]}{\sum_{j=1}^{n} [d_{ij}^L, d_{ij}^U] x_j + [\beta_i^L, \beta_i^U]}, i = 1, 2, \ldots, k$$

subject to

$$\sum_{j=1}^{n} [a_{tj}^L, a_{tj}^U] x_j \le b_t, t = 1, 2, \ldots, m \text{ and } x_j \ge 0.$$

As the values of b_t mostly remains in knowledge of DM, we assume here fixed values for it but can be considered in interval form regarding the necessity of the real world problem.

The objective functions and the constraints of the above Problem P are formulated into the following forms.

6.1   Objective functions

The objective functions of Problem P can be simplified into the following forms:

$$f_i(x) = \frac{\left[\sum_{j=1}^{n} c_{ij}^L x_j + \alpha_i^L, \sum_{j=1}^{n} c_{ij}^U x_j + \alpha_i^U\right]}{\left[\sum_{j=1}^{n} d_{ij}^L x_j + \beta_i^L, \sum_{j=1}^{n} d_{ij}^U x_j + \beta_i^U\right]}, \ i = 1, 2, \ldots, k.$$

We use the following notations for convenience.

$$f_i(x) = \frac{f_i^N(x)}{f_i^D(x)} = \frac{[f_i^{NL}(x), f_i^{NU}(x)]}{[f_i^{DL}(x), f_i^{DU}(x)]}, \ i = 1, 2, \ldots, k.$$

Since $f_i^D(x) > 0 \ \forall x \in \Omega$ (assumed), the following are the possible cases:

- **Case-I:** If $0 \le f_i^{NL}(x) \le f_i^{NU}(x)$ then $f_i(x) = \left[\frac{f_i^{NL}(x)}{f_i^{DU}(x)}, \frac{f_i^{NU}(x)}{f_i^{DL}(x)}\right]$.

- **Case-II:** If $f_i^{NL}(x) \le f_i^{NU}(x) \le 0$ then $f_i(x) = \left[\frac{f_i^{NL}(x)}{f_i^{DL}(x)}, \frac{f_i^{NU}(x)}{f_i^{DU}(x)}\right]$.

- **Case-III:** If $f_i^{NL}(x) \le 0 \le f_i^{NU}(x)$ then $f_i(x) = \left[\frac{f_i^{NL}(x)}{f_i^{DL}(x)}, \frac{f_i^{NU}(x)}{f_i^{DL}(x)}\right]$.

Selection criterion of the cases:

The affine functions $f_i^{NL}(x)$ and $f_i^{NU}(x)$ can be both positive and negative for different values of $x \in \Omega$. Since $f_i^D(x) > 0$ (assumed), depending on the values of $f_i^{NL}(x)$ and $f_i^{NU}(x)$, the following criteria are undertaken for the problem to decide the cases that occur. As it is a minimization problem, the numerators of the objectives need to be minimized. So the signs of $f_i^{NL}(x)$, $f_i^{NU}(x)$ are considered about their individual minimal points to obtain the left and right end points of the interval form of $f_i(x)$. The minimum values of $f_i^{NL}(x)$ and $f_i^{NU}(x)$ are evaluated with the given constraints treating them as individual objectives and depending on the signs of their individual optimal objective values i.e., f_i^{NL*} and f_i^{NU*}, the following observations are made.

Observations:

1. If $f_i^{NL*} \geq 0$ and $f_i^{NU*} \geq 0$ then **Case-I** occurs.

2. If $f_i^{NL*} \leq 0$ and $f_i^{NU*} \leq 0$ then **Case-II** occurs.

3. If $f_i^{NL*} \leq 0$ and $f_i^{NU*} \geq 0$ then **Case-III** occurs.

4. If $f_i^{NL*} \geq 0$ and $f_i^{NU*} \leq 0$ then **Not Possible**.

Special observations:

1. If $c_{ij}^L, \alpha_i^L \geq 0 \ \forall j = 1, 2, \ldots, n$ then **Case-I** occurs.

2. If $c_{ij}^U, \alpha_i^U \leq 0 \ \forall j = 1, 2, \ldots, n$ then **Case-II** occurs.

We consider **Case-I** to illustrate the procedure for computing pareto optimal solutions of the interval-valued objective functions of the MOLFPP and for the **Cases-(II, III)**, it can be done in a similar manner.

For **Case-I** i.e., $\min_{x \in \Omega} \left[\frac{f_i^{NL}(x)}{f_i^{DU}(x)}, \frac{f_i^{NU}(x)}{f_i^{DL}(x)} \right]$ the optimal objective value must remain in interval form as the objective function $f_i(x)$ is interval-valued. However the following real-valued fractional programming problems can be considered to obtain the pareto optimal solutions for **Case-I** problem as we have a minimization problem.

- $\min_{x \in \Omega} \frac{f_i^{NL}(x)}{f_i^{DU}(x)}$ (Best-case problem).

- $\min_{x \in \Omega} \frac{f_i^{NU}(x)}{f_i^{DL}(x)}$ (Worst-case problem).

- $\min_{x \in \Omega} f_i^{ND}(x)$ where $\frac{f_i^{NL}(x)}{f_i^{DU}(x)} \leq f_i^{ND}(x) \leq \frac{f_i^{NU}(x)}{f_i^{DL}(x)}$ (Other-case problem).

Particularly, $\min_{x \in \Omega} \frac{f_i^{NL}(x)}{f_i^{DL}(x)}$ and $\min_{x \in \Omega} \frac{f_i^{NU}(x)}{f_i^{DU}(x)}$ belong to the class of Other-case problem, solving which the optimal objective values can be obtained in between the range of its best and worst values. Each case problem generates pareto optimal solutions but the theoretical explanation to compute the pareto optimal solutions by the proposed method is presented for the Worst-case problem.

6.2 Constraints

Proposition. *Let* $(x_1, x_2, \ldots, x_n)$ *be a feasible point of Problem P and* $\lambda_j \in [a_{mj}^L, a_{mj}^U]$, $j = 1, 2, \ldots, n$ *then the following results hold true.*

- *If* $\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n \leq b_m$ *then* $a_{m1}^L x_1 + a_{m2}^L x_2 + \cdots + a_{mn}^L x_n \leq b_m$.

- *If* $\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n \geq b_m$ *then* $a_{m1}^U x_1 + a_{m2}^U x_2 + \cdots + a_{mn}^U x_n \geq b_m$.

Using the above proposition, the constraints of Problem P can be stated as: $\sum_{j=1}^n a_{tj}^L x_j \leq b_t$, $t = 1, 2, \ldots, m$. This reduction of constraints can be used for computing each (Best, Worst and Other)-case problm of the objective functions. If b_t is considered as an interval $[b_t^L, b_t^U]$ then both sides of each constraint of Problem P exist in form of intervals. As $[b_t^L, b_t^U]$ is considered

the range for existence of the interval $\sum_{j=1}^{n}[a_{tj}^L, a_{tj}^U]x_j$, the constraints can be stated as $\sum_{j=1}^{n}[a_{tj}^L, a_{tj}^U]x_j \subseteq [b_t^L, b_t^U]$ which can be further splitted into the following two constraints [22]:

$$\sum_{j=1}^{n} a_{tj}^L x_j \geq b_t^L \quad \text{and} \quad \sum_{j=1}^{n} a_{tj}^U x_j \leq b_t^U.$$

6.3   Solution procedure by the proposed method

Thus, the Problem P is transformed into Problem P* using aforementioned objective assumptions and constraint rules.

Problem P*.

$$\text{Min } f_i(x) = \frac{\sum_{j=1}^{n} c_{ij}^U x_j + \alpha_i^U}{\sum_{j=1}^{n} d_{ij}^L x_j + \beta_i^L}, \ i = 1, 2, \ldots, k$$

subject to

$$\sum_{j=1}^{n} a_{tj}^L x_j \leq b_t, \ t = 1, 2, \ldots, m \text{ and } x_j \geq 0.$$

A sequence of pareto optimal solutions of Problem P* can be generated using relative minimum and maximum of the functional values of each objective function which can be evaluated through constructing the pay-off tables by following the procedure described below.

Step 1.

$$\text{Min } f_i^N(x) = \sum_{j=1}^{n} c_{ij}^U x_j + \alpha_i^U$$

subject to

$$\sum_{j=1}^{n} a_{tj}^L x_j \leq b_t, \ t = 1, 2, \ldots, m, \quad x_j \geq 0.$$

Solve the above problem individually for each objective numerator function $f_i^N(x)$ with $i = 1, 2 \ldots k$. Let X_i^{N*} be the optimal solutions of the above problem.

Step 2.

$$\text{Max } f_i^D(x) = \sum_{j=1}^{n} d_{ij}^L x_j + \beta_i^L$$

subject to

$$\sum_{j=1}^{n} a_{tj}^L x_j \leq b_t, \ t = 1, 2, \ldots, m, \quad x_j \geq 0.$$

Solve the above problem individually for each objective denominator function $f_i^D(x)$ with $i = 1, 2, \ldots, k$. Let X_i^{D*} be the optimal solutions of the above problem.

Step 3. Construct the pay-off table for objective numerator functions. Find $f_j^{NL} = f_j^N(X_j^{N*})$ and $f_j^{NU} = \max\{f_j^N(X_i^{N*}) \mid i = 1, 2, \ldots, k\}$ for each objective numerator $f_j^N(x)$, $j = 1, 2 \ldots k$ from the minimum and maximum values of each column of the following pay-off Table 1.

Table 1. Pay-off table for numerator functions.

X_i^{N*}	$f_1^N(X_i^{N*})$	$f_2^N(X_i^{N*})$	$\cdots$	$f_k^N(X_i^{N*})$
X_1^{N*}	$f_1^N(X_1^{N*})$	$f_2^N(X_1^{N*})$	$\cdots$	$f_k^N(X_1^{N*})$
X_2^{N*}	$f_1^N(X_2^{N*})$	$f_2^N(X_2^{N*})$	$\cdots$	$f_k^N(X_2^{N*})$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
X_k^{N*}	$f_1^N(X_k^{N*})$	$f_2^N(X_k^{N*})$	$\cdots$	$f_k^N(X_k^{N*})$

Step 4. Similarly construct another pay-off table for the denominator functions to determine the relative minimum and maximum of each objective denominator $f_j^D(x)$, $j = 1, 2, \ldots, k$ as:

- $f_j^{DL} = \min\{f_j^D(X_i^{D*}) \mid i = 1, 2, \ldots, k\}$.

- $f_j^{DU} = f_j^D(X_j^{D*})$.

Step 5. Since $f_i^{NL} \leq f_i^N(x) \leq f_i^{NU}$ where $f_i^N(x) \leq f_i^{NU}$ holds true for $x = X_j^{N*}$, $j = 1, 2, \ldots, k$ and $f_i^{DL} \leq f_i^D(x) \leq f_i^{DU}$ where $f_i^{DL} \leq f_i^D(x)$ holds true for $x = X_j^{D*}$, $j = 1, 2, \ldots, k$, it can be considered as $f_i(x) = \frac{f_i^N(x)}{f_i^D(x)} \in \frac{[f_i^{NL}, f_i^{NU}]}{[f_i^{DL}, f_i^{DU}]} = [\epsilon_i^L, \epsilon_i^U]$. The values of ϵ_i^L and ϵ_i^U can be determined using interval arithmetic (defined in Section 2).

For each objective function $f_i(x)$, the value of ϵ_i is restricted as $\epsilon_i^L \leq \epsilon_i \leq \epsilon_i^U$. If $f_s(x)$ is highest prioritized by the DM, using ϵ-constraint method, Problem P* is transformed into the following Problem P$_s^*$.

Problem P$_s^*$.

$$\text{Min } f_s(x) = \frac{\sum_{j=1}^n c_{sj}^U x_j + \alpha_s^U}{\sum_{j=1}^n d_{sj}^L x_j + \beta_s^L}, \ s \in \{1, 2, \ldots, k\}$$

subject to

$$f_l(x) \leq \epsilon_l, \ l = 1, 2, \ldots, s - 1, s + 1, \ldots, k \text{ and } \Omega.$$

For $f_l(x) \leq \epsilon_l$, $l = 1, 2, \ldots, s - 1, s + 1, \ldots, k$

$$\frac{\sum_{j=1}^n c_{lj}^U x_j + \alpha_l^U}{\sum_{j=1}^n d_{lj}^L x_j + \beta_l^L} \leq \epsilon_l,$$

i.e., $\sum_{j=1}^n (c_{lj}^U - \epsilon_l d_{lj}^L) x_j \leq (\epsilon_l \beta_l^L - \alpha_l^U)$.

Using the above results, Problem P_s^* can be reformulated in generalised form as,

$$\text{Min } f_s(x) = \frac{c_{s1}^U x_1 + c_{s2}^U x_2 + \cdots + c_{sn}^U x_n + \alpha_s^U}{d_{s1}^L x_1 + d_{s2}^L x_2 + \cdots + d_{sn}^L x_n + \beta_s^L}$$

subject to Ω,

$$\left(c_{l1}^U - \epsilon_l d_{l1}^L\right)x_1 + \left(c_{l2}^U - \epsilon_l d_{l2}^L\right)x_2 + \cdots + \left(c_{ln}^U - \epsilon_l d_{ln}^L\right)x_n \leq \left(\epsilon_l \beta_l^L - \alpha_l^U\right).$$

Charnes and Cooper transformation technique

Consider the following optimization problem with fractional objective:

$$\text{Optimize } g(x_i) = \frac{\sum_{i=1}^n d_i' x_i + \gamma'}{\sum_{i=1}^n d_i'' x_i + \gamma''} \text{ subject to } \Omega. \qquad (*_1)$$

Assuming $\frac{1}{\sum_{i=1}^n d_i'' x_i + \gamma''} = z$ and $x_i z = y_i$, the fractional objective is transformed into non-fractional linear form with an additional variable i.e.,

$$\text{optimize } g(y_i, z) = \sum_{i=1}^n d_i' y_i + \gamma' z \qquad (*_2)$$

$$\text{subject to } \sum_{j=i}^n d_i'' y_i + \gamma'' z = 1 \text{ and } \Omega.$$

If (y_i^*, z^*) is the optimal solution of problem $(*_2)$ then $x^* = (x_i^*) = (\frac{y_i^*}{z^*})$ is considered as the optimal solution of problem $(*_1)$.

Using the aforesaid transformation technique, reformulated Problem P_s^* can be stated as:

Problem P_s^{**}.

$$\text{Min } f_s(y, z) = \sum_{j=1}^n c_{sj}^U y_j + \alpha_s^U z_s$$

subject to

$$\sum_{j=1}^n d_{sj}^L y_j + \beta_s^L z_s = 1,$$

$$\sum_{j=1}^n (c_{lj}^U - \epsilon_l d_{lj}^L) y_j \leq \left(\epsilon_l \beta_l^L - \alpha_l^U\right) z_s, \ l = 1, 2, \ldots, s-1, s+1, \ldots, k,$$

$$a_{tj}^L y_j \leq b_t z_s, \ t = 1, 2, \ldots, m, \quad y_j \geq 0, \ z_s > 0.$$

Changing the values of each ϵ_l in the interval $[\epsilon_l^L, \epsilon_l^U]$ and substituting in the constraints of the above Problem P_s^{**}, we can generate a sequence of solutions by using any suitable method. If $(y_1^*, y_2^*, \ldots, y_n^*, z_s^*)$ be the optimal solutions of the above problem then $(\frac{y_1^*}{z_s^*}, \frac{y_2^*}{z_s^*}, \ldots, \frac{y_n^*}{z_s^*}) = (x_1^*, x_2^*, \ldots, x_n^*)$ are considered as the optimal solutions of the Problem P_s^* and pareto optimal(non-inferior) solutions of the Problem P. Here ϵ-constraint method generates k number of single objective LFPP i.e., Problem P_s^* as 's' varies in $\{1, 2, \ldots, k\}$. From the sequence of pareto optimal solutions the DM can choose the most preferred optimal solution for the given multi-objective LFPP.

6.4 Algorithm for solving MOLFPP

We adopt the following steps for solving MOLFPP with interval coefficients.

Step 1. Convert the given Problem P into Problem P*.

Step 2. Determine $[\epsilon_i^L, \epsilon_i^U]$ for each objective function $f_i(x)$ using the approach described in our proposed technique.

Step 3. Choose the objective function $f_s(x)$ of highest priority and solve Problem P_s^{**} by changing the value of ϵ_i in $[\epsilon_i^L, \epsilon_i^U]$.

Step 4. Obtain a sequence of pareto optimal solutions and determine the most preferred optimal solution.

Step 5. The objective function of highest priority is altered to obtain another single objective LFPP and a new set of pareto optimal solutions can be generated using ϵ-constraint method.

7 Numerical Example

Let us consider the following bi-objective LFPP with interval coefficients of the decision variables both in the objective and constraint functions.

Problem P.

$$\text{Min } f_1(x) = \frac{[-3, -2]x_1 + [1, 3]x_2 + [-1, 1]}{[7, 8]x_1 + [5, 7]x_2 + [8, 9]},$$

$$f_2(x) = \frac{[1, 3]x_1 + [-3, -2]x_2 + [1, 2]}{[5, 6]x_1 + [4, 7]x_2 + [7, 9]}$$

subject to

$$[2, 4]x_1 + [3, 5]x_2 \le 6, \ [1, 3]x_1 + [-4, 1]x_2 \le 3, \ x_1, x_2 \ge 0.$$

Using the selection criteria of the proposed technique (as described in Subsection 6.1), it is observed that the objective functions $f_1(x)$ and $f_2(x)$ both belong to Case-II category i.e.,

$$f_i(x) = \left[\frac{f_i^{NL}(x)}{f_i^{DL}(x)}, \frac{f_i^{NU}(x)}{f_i^{DU}(x)} \right], \ i = 1, 2.$$

For Problem P,

$$f_1(x) = \left[\frac{-3x_1 + x_2 - 1}{7x_1 + 5x_2 + 8}, \frac{-2x_1 + 3x_2 + 1}{8x_1 + 7x_2 + 9} \right],$$

$$f_2(x) = \left[\frac{x_1 - 3x_2 + 1}{5x_1 + 4x_2 + 7}, \frac{3x_1 - 2x_2 + 2}{6x_1 + 7x_2 + 9} \right].$$

So Problem P can be formulated into the following real model problems:

Best-case problem: $\min_{x \in \Omega} \left\{ \dfrac{-3x_1 + x_2 - 1}{7x_1 + 5x_2 + 8}, \dfrac{x_1 - 3x_2 + 1}{5x_1 + 4x_2 + 7} \right\}.$

Worst-case problem: $\min\limits_{x \in \Omega} \left\{ \dfrac{-2x_1 + 3x_2 + 1}{8x_1 + 7x_2 + 9}, \dfrac{3x_1 - 2x_2 + 2}{6x_1 + 7x_2 + 9} \right\}.$

Each case can be individually solved, however we solve a case

$$\min\limits_{x \in \Omega} \left\{ \frac{f_1^{NU}(x)}{f_1^{DL}(x)}, \frac{f_2^{NU}(x)}{f_2^{DL}(x)} \right\}$$

in between them to generate the pareto optimal solutions i.e., Problem P* which belongs to the category of Other-case problem as it is interpreted for Case-I category in Subsection 6.1.

Problem P.*

$$\text{Min } f_1(x) = \frac{f_1^N(x)}{f_1^D(x)} = \frac{-2x_1 + 3x_2 + 1}{7x_1 + 5x_2 + 8},$$

$$f_2(x) = \frac{f_2^N(x)}{f_2^D(x)} = \frac{3x_1 - 2x_2 + 2}{5x_1 + 4x_2 + 7}$$

subject to

$$\Omega = \begin{cases} 2x_1 + 3x_2 \le 6, \\ x_1 - 4x_2 \le 3, \\ x_1, x_2 \ge 0. \end{cases}$$

Optimizing $f_1^N(x)$, $f_1^D(x)$, $f_2^N(x)$ and $f_1^D(x)$ individually subject to the constraints set Ω, the following results are obtained:

$$\min f_1^N(x) = -5 \text{ at } X_1^{N*} = (3, 0),$$
$$\min f_2^N(x) = -2 \text{ at } X_2^{N*} = (0, 2),$$
$$\max f_1^D(x) = 29 \text{ at } X_1^{D*} = (3, 0),$$
$$\max f_2^D(x) = 22 \text{ at } X_2^{D*} = (3, 0).$$

Constructing the pay-off table for both objective numerator and denominator functions as defined in our proposed technique, the relative minimum and maximum values of the objectives are obtained, as

$$\frac{-5}{29} = \frac{f_1^{NL}}{f_1^{DU}} \le f_1(x) \le \frac{f_1^{NU}}{f_1^{DL}} = \frac{7}{29},$$

$$\frac{-2}{22} = \frac{f_2^{NL}}{f_2^{DU}} \le f_2(x) \le \frac{f_2^{NU}}{f_2^{DL}} = \frac{11}{22}.$$

Thus, $\epsilon_1 \in [\epsilon_1^L, \epsilon_1^U] = [-0.1724, 0.2414]$ and $\epsilon_2 \in [\epsilon_2^L, \epsilon_2^U] = [-0.0909, 0.5]$.

The DM prioritizes best to either $f_1(x)$ or $f_2(x)$. Pareto optimal solutions are generated for both the cases using ϵ-constraint method by mathematically modelling them as Problem P_1^* and Problem P_2^*.

Problem P_1^*. (The objective function $f_1(x)$ is prioritized best.)

$$\text{Min } f_1(x) = \frac{-2x_1 + 3x_2 + 1}{7x_1 + 5x_2 + 8}$$

subject to

Ω and $f_2(x) \le \epsilon_2$.

Using transformation technique, it can be reformulated as:

$$\text{Min } f_1(y) = -2y_1 + 3y_2 + z_1$$

subject to

$$2y_1 + 3y_2 - 6z_1 \le 0, \quad y_1 - 4y_2 - 3z_1 \le 0,$$
$$(3 - 5\epsilon_2)y_1 + (-2 - 4\epsilon_2)y_2 - (7\epsilon_2 - 2)z_1 \le 0,$$
$$7y_1 + 5y_2 + 8z_1 = 1, \quad y_1, y_2, z_1 \ge 0.$$

If (y_1^*, y_2^*, z_1^*) be the optimal solution of the above reformulated problem then Problem P_1^* has optimal solution, $x_1^* = \frac{y_1^*}{z_1^*}$ and $x_2^* = \frac{y_2^*}{z_1^*}$. Now substituting different values of ϵ_2 in reformulated form of Problem P_1^* such that $-0.0909 \le \epsilon_2 \le 0.5$, a sequence of solutions (x_1^*, x_2^*) can be obtained in Table 2 which are considered as the pareto optimal solutions of Problem P.

Table 2. Solution of problem P_1^*.

ϵ_2	y_1^*	y_2^*	z_1^*	x_1^*	x_2^*	$f_1^* = f_1(x_1^*, x_2^*)$
-0.0812	0.0093	0.1011	0.0537	0.1732	1.8827	0.3385
-0.0636	0.0133	0.0968	0.0528	0.2519	1.8334	0.3168
0.0069	0.0246	0.0847	0.0505	0.4871	1.6772	0.2555
0.0773	0.0366	0.0718	0.0481	0.7609	1.4927	0.1903
0.1830	0.0541	0.0530	0.0445	**1.2157**	**1.1910**	**0.0953**
0.2182	0.0598	0.0469	0.0434	1.3779	1.0806	0.0645
0.2887	0.0711	0.0348	0.0411	1.7299	0.8467	0.0033
0.3591	0.0821	0.0230	0.0388	2.1160	0.5928	-0.0564
0.3943	0.0875	0.0171	0.0377	2.3210	0.4536	-0.0860
0.4824	0.1008	0.0028	0.0350	2.8800	0.0800	-0.1583

Prioritizing best to $f_1(x)$ and comparing the obtained Table 2 values of ϵ_2 and f_1^*, it is observed that $|\epsilon_2 - f_1^*|$ i.e., $|f_2^* - f_1^*|$ is minimum at $x^* = (1.2157, 1.1910)$ as compared to the other solutions. Since at this solution the optimal objective values f_1^* and f_2^* remain in most compromised state i.e., at a shortest distance from each other, the most preferred optimal solution is considered as $x^* = (1.2157, 1.1910)$ where the optimal objective values are $f_1(x^*) = 0.0953$ and $f_2(x^*) = 0.1830$.

Problem P_2^*. (The objective function f_2 is prioritized best.)

$$\text{Min } f_2(x) = \frac{3x_1 - 2x_2 + 2}{5x_1 + 4x_2 + 7}$$

subject to

Ω and $f_1(x) \leq \epsilon_1$.

Using transformation technique, Problem P_2^* can be reformulated as:

$$\text{Min } f_2(y) = 3y_1 - 2y_2 + 2z_2$$

subject to

$$2y_1 + 3y_2 - 6z_2 \leq 0, \quad y_1 - 4y_2 - 3z_2 \leq 0,$$
$$(-2 - 7\epsilon_1)y_1 + (3 - 5\epsilon_1)y_2 - (8\epsilon_1 - 1)z_2 \leq 0,$$
$$5y_1 + 4y_2 + 7z_2 = 1, \quad y_1, y_2 \geq 0, \ z_2 > 0.$$

If (y_1^*, y_2^*, z_2^*) be the optimal solution of the above reformulated Problem P_2^* then $(x_1^*, x_2^*) = \left(\frac{y_1^*}{z_2^*}, \frac{y_2^*}{z_2^*}\right)$ is considered as the optimal solution of Problem P_2^*. Now substituting different values of ϵ_1 in the reformulated Problem P_2^* such that $-0.1724 \leq \epsilon_1 \leq 0.2414$, the following pareto optimal solutions of Problem P can be obtained in Table-3.

Prioritizing best to $f_2(x)$ and comparing the values of ϵ_1 and f_2^* from the set of non-inferior solutions(in Table-3), the most preferred optimal solution can be considered as $x^* = (0.9948, 1.3380)$ where the optimal objective values are $f_1(x^*) = 0.1397$ and $f_2(x^*) = 0.1332$.

Table 3. Solution of Problem P_2^*.

ϵ_1	y_1^*	y_2^*	z_2^*	x_1^*	x_2^*	$f_2^* = f_2(x_1^*, x_2^*)$
-0.1408	0.1280	0.0082	0.0468	2.7350	0.1752	0.4610
-0.1097	0.1198	0.0162	0.0480	2.4958	0.3375	0.4231
-0.0473	0.1037	0.0319	0.0505	2.0535	0.6317	0.3484
0.0150	0.0879	0.0473	0.0530	1.6585	0.8925	0.2752
0.0773	0.0725	0.0624	0.0554	1.3087	1.1264	0.2035
0.1397	0.0574	0.0772	0.0577	**0.9948**	**1.3380**	**0.1332**
0.2020	0.0426	0.0917	0.0600	0.7100	1.5283	0.0644
0.2130	0.0400	0.0942	0.0604	0.6623	1.5510	0.0526
0.2332	0.0353	0.0988	0.0612	0.5768	1.6144	0.0307
0.2401	0.0337	0.1004	0.0614	0.5488	1.6352	0.0232

Table-2 and Table-3 represent the set of non-inferior or pareto optimal solutions of the given multi-objective LFPP from which the most preferred optimal solution is ascertained by the DM assigning highest priority to only one objective function and accordingly optimal objective values are determined.

7.1 Result by fuzzy programming

Our example stated as Problem P is formulated to Problem P* implementing some propositions. Using the pay-off tables (both for numerator and denominator objective functions) of our proposed technique, the values of relative minimum and maximum of each objective function with respect to others are ascertained as $-0.1724 = L_1 \leq f_1(x) \leq U_1 = 0.2414$ and $-0.0909 = L_2 \leq f_2(x) \leq U_2 = 0.5$. According to the algorithm of max-min operator in fuzzy programming (defined in Section 5) the following crisp model obtained from Problem P* is to be solved.

$$\text{Max } \lambda$$
$$\text{subject to}$$
$$f_i(x) + (U_i - L_i)\lambda \leq U_i, \ (i = 1, 2) \text{ and } \Omega.$$

Substituting the values and on simplifying, it can be reformulated as:

$$\text{Max } \lambda$$
$$\text{subject to}$$
$$-3.6898x_1 + 1.793x_2 + 0.438\lambda(7x_1 + 5x_2 + 8) \leq 0.9312,$$
$$0.5x_1 - 4x_2 + 0.5909\lambda(5x_1 + 4x_2 + 7) \leq 1.5,$$
$$2x_1 + 3x_2 \leq 6, \quad x_1 - 4x_2 \leq 3,$$
$$\lambda > 0, \ x_1, x_2 \geq 0.$$

The fuzzy optimal solution is obtained as $\lambda = 0.4435$, $x_1 = 1.4740$ and $x_2 = 1.0170$. Thus considering the example (converted to problem P*), the optimal solution obtained by fuzzy programming is $x^* = (x_1^*, x_2^*) = (1.4740, 1.0170)$ where the optimal objective values are $f_1(x^*) = 0.0471$ and $f_2(x^*) = 0.2380$.

7.2 Remarks

In the example defined as Problem P, our method provides the most preferred optimal solution (Table-2) by changing the values of ϵ_2 in the constraints as $x^* = (1.2157, 1.1910)$ and the corresponding optimal objective values are $(f_1^*, f_2^*) = (0.0953, 0.1830)$. Changing the values of ϵ_1 in the constraints, the most preferred optimal solution is considered as $x^* = (0.9948, 1.3380)$ (Table-3) and the corresponding optimal objective values are $(f_1^*, f_2^*) = (0.1397, 0.1332)$. But using fuzzy programming(max-min operator) method the optimal solution is obtained as $x^* = (1.4740, 1.0170)$ and the corresponding optimal objective values are $(f_1^*, f_2^*) = (0.0471, 0.2380)$.

Comparing the optimal objective values, it is clear that our method has several better options to choose the optimal solution and also the optimal objective values exist in more compromising state that reduces the feasibility of conflict among the objectives whereas fuzzy programming gives single optimal solution so that the DM has no choice to choose. So our method may be considered as helpful to determine the most preferred optimal solution of a MOLFPP.

8   Conclusions

In this paper, ϵ-constraint method is applied to generate a number of pareto optimal solutions of the multi-objective LFPP with interval coefficients both in the obective and constraint functions. As a result, several choices are available to select the most preferred optimal solution in order to satisfy all the objectives with own desired level of satisfation as all the objectives can not be fully satisfied at single solution. MATLAB programming language has been used to obtain the computational results presented in the aforementioned example. The proposed method generates set of optimal solutions which have been compared with that of fuzzy technique in the illustrated example. The pareto optimal solutions obtained by our method can be approximated with the solution due to fuzzy by adjusting the value of ϵ_i in the constraints. Our method gives some other better solutions than fuzzy approach so that the DM has several options to choose the best one according to own desire on preference basis. As a few methods are available in literature to solve such type of problems, our proposed method is expected to assist in this regard.

References

[1] E.B. Bajalinov. *Linear-Fractional Programming: Theory, Methods, Applications and Software*, volume 84 of *Appl. Optim.* Springer, 2003.

[2] R.E. Bellman and L.A. Zadeh. Decision-making in a fuzzy environment. *Management Sci.*, **17**(4):B–141, 1970. http://dx.doi.org/10.1287/mnsc.17.4.B141.

[3] M. Borza, A.S. Rambely and M. Saraj. Solving linear fractional programming problems with interval coefficients in the objective function. A new approach. *Appl. Math. Sci.*, **6**(69):3443–3452, 2012.

[4] R. Caballero and M. Hernandez. The controlled estimation method in the multiobjective linear fractional problem. *Comput. Oper. Res.*, **31**(11):1821–1832, 2004. http://dx.doi.org/10.1016/S0305-0548(03)00141-2.

[5] A. Charnes and W.W. Cooper. Programming with linear fractional functionals. *Naval Res. Logist. Q.*, **9**(3–4):181–186, 1962.
http://dx.doi.org/10.1002/nav.3800090303.

[6] Y. Collette and P. Siarry. *Multiobjective Optimization: Principles and Case Studies.* Springer, 2003.

[7] J.P. Costa. Computing non-dominated solutions in MOLFP. *European J. Oper. Res.*, **181**(3):1464–1475, 2007. http://dx.doi.org/10.1016/j.ejor.2005.11.051.

[8] S. Effati and M. Pakdaman. Solving the interval-valued linear fractional programming problem. *Amer. J. Comput. Math.*, **2**:51, 2012.
http://dx.doi.org/10.4236/ajcm.2012.21006.

[9] M. Ehrgott. *Multicriteria Optimization*. Springer, 2005.

[10] Y.Y. Haimes, L.S. Ladson and D.A. Wismer. On a bicriterion formulation of problems of integrated system identification and system optimization. *IEEE Trans. Syst., Man, Cybern., Syst.*, **1**(3):296–297, 1971. http://dx.doi.org/10.1109/TSMC.1971.4308298.

[11] M. Hladík. Generalized linear fractional programming under interval uncertainty. *European J. Oper. Res.*, **205**(1):42–46, 2010. http://dx.doi.org/10.1016/j.ejor.2010.01.018.

[12] S. Jain and N. Arya. Inverse optimization for linear fractional programming. *Int. J. Phys. Math. Sci.*, **4**(1), 2013.

[13] M.K. Luhandjula. Fuzzy approaches for multiple objective linear fractional optimization. *Fuzzy Sets and Systems*, **13**(1):11–23, 1984. http://dx.doi.org/10.1016/0165-0114(84)90023-X.

[14] B. Martos. Hyperbolic programming. *Publ. Res. Inst. Math. Sci.*, **5**:386–407, 1960.

[15] K.M. Miettinen. *Nonlinear Multiobjective Optimization*. Kluwer Academic Publisher, 2004.

[16] R.E. Moore. *Interval Analysis*. Prentice–Hall Englewood Cliffs, 1966.

[17] I. Nykowski and Z. Zolkiewski. A compromise procedure for the multiple objective linear fractional programming problem. *European J. Oper. Res.*, **19**(1):91–97, 1985. http://dx.doi.org/10.1016/0377-2217(85)90312-1.

[18] A.K. Ojha and K.K. Biswal. Multi-objective geometric programming problem with ϵ-constraint method. *Appl. Math. Model.*, **38**(2):747–758, 2014. http://dx.doi.org/10.1016/j.apm.2013.07.003.

[19] B.B. Pal and S. Sen. A goal programming procedure for solving interval valued multiobjective fractional programming problems. In *Advanced Computing and Communications, 2008. ADCOM 2008. 16th International Conference on*, pp. 297–302. IEEE, 2008.

[20] I.M. Stancu-Minasian. *Fractional Programming*. Springer, 1997.

[21] H. Suprajitno. Solving multiobjective linear programming problem using interval arithmetic. *Appl. Math. Sci.*, **6**(80):3959–3968, 2012.

[22] H.C. Wu. On interval-valued nonlinear programming problems. *J. Math. Anal. Appl.*, **338**(1):299–316, 2008. http://dx.doi.org/10.1016/j.jmaa.2007.05.023.

[23] H.-J. Zimmermann. Fuzzy programming and linear programming with several objective functions. *Fuzzy Sets and Systems*, **1**(1):45–55, 1978. http://dx.doi.org/10.1016/0165-0114(78)90031-3.