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ON UNIFORM IN TIME ERROR ESTIMATES FOR INVESTIGATION OF NONLINEAR DIFFERENCE SCHEMES

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1. INTRODUCTION

One general methodology for investigation of difference schemes, which approximate nonstationary nonlinear differential equations, is given in [1,2]. That methodology is based on the nonlinear stability definition and enables us to use the investigation formula "aproximation + stability= convergence" in the nonlinear cases.

In this note we modify this scheme of investigation by using the definition of asymptotical stability of nonlinear difference schemes. It enables us to prove uniform in time error estimates. The efficiency of such methodology is demonstrated for implicit and explicit finite - difference schemes which approximate the semilinear diffusion-reaction problems

$$\frac{\partial u}{\partial t} = \sum_{j=1}^{d} \frac{\partial}{\partial x_j}\Big(a_j(x)\frac{\partial u}{\partial x_j}\Big) + f(x,t,u) \quad \text{in} \quad \Omega \times (0,\infty), \tag{1.1}$$

$$u = 0 \quad \text{on} \quad \partial\Omega \times [0,\infty) \tag{1.2}$$

$$u(\cdot,0) = u_0 \quad \text{in} \quad \Omega. \tag{1.3}$$

Here Ω is a rectangular domain in $R^d, d \geq 1, a_j(x)$ are given functions satisfying conditions

$$0 < A_L \leq a_j(x) \leq A_R \quad \text{in} \quad \Omega. \tag{1.4}$$

In this paper we use the following norm convention

$$\|v(\cdot,t)\|_{L_\infty} = \sup_{x\in\Omega} |v(x,t)|.$$

Then we define the neighbourhood of the solution

$$B(u,R) = \{v : \|u(\cdot,t) - v(\cdot,t)\|_{L_\infty} \le R\}.$$

We assume that the differential equation has some structure which forces a solution to approach an equilibrium. More precisely, we assume the following:

(H1) $f(x,t,v) \underset{t\to\infty}{\longrightarrow} F(x,v)$.

(H2) The problem (1.1) – (1.3) defines asymptotically stable nonlinear operator in $B(u,R)$, i.e.:

$$\lambda_{\max}(t) = \max_{\Phi\in H_0^1} \frac{\int\limits_\Omega \big[- \sum\limits_{j=1}^{d} a_j(x)\Phi_{x_j}^2 + f_u'(\cdot,t,U)\Phi^2\big] dx}{\int\limits_\Omega \Phi^2 dx} < 0$$

for any $U \in B(u,R)$, here H_0^1 is the subspace of the standard Sobolev space $H^1(\Omega)$ satisfying the homogeneous Dirichlet boundary conditions.

Then $||u(\cdot,t) - \bar U||_{L_\infty} \to 0$, where $\bar U$ satisfies the stationary problem

$$\begin{aligned} -\sum_{j=1}^{d} \frac{\partial}{\partial x_j}\big(a_j(x)\frac{\partial \bar U}{\partial x_j}\big) &= F(x,\bar U) \qquad \text{in} \quad \Omega, \\ \bar U &= 0 \qquad \text{on} \quad \partial\Omega. \end{aligned} \tag{1.5}$$

Following [2] (see, also [3]) in the difference scheme to be described below we replace function f by a smooth function f_R, which coincides with f in $B(u,R)$. Outside of $B(u,R)$ f_R is bounded and Lipschitz continuous, where all bounds depend only on u and R. This replacement does not affect the exact solution $u(x,t)$.

Finally we mention some related work. Larsson [3] analyses the long-time behavior of the dissipative backward Euler method. He proves error estimates in the L_2 norm. The approximation of "contracting" trajectories near asymptotically stable equilibra by an explicit Euler finite-difference scheme is considered by Sanz-Serna and Stuart [4]. The qualitative behaviour of spatially semidiscrete finite element solutions of a semilinear parabolic problem near an unstable hyperbolic equilibrium is studied by Larsson and Sanz-Serna [5]. We mention the important work of Heywood and Rannacher [6].

2. FINITE-DIFFERENCE SCHEME

In this section we describe the finite-difference approximation of (1.1) – (1.3).

Let Ω_τ be the uniform time mesh with the time step τ. Let $\Omega_h = \Omega_{1h} \times \Omega_{2h} \times \cdots \times \Omega_{dh}$ be a discretization of Ω, where Ω_{ih} are space meshes obtained by dividing space intervals $]0,1[$ into mesh intervals by a sequence of points $x_{ij} = jh, j = 1, 2, \dots, N-1$, where $Nh = 1$ and h denotes the space step. Then for any vector $j = (j_1, j_2, \dots, j_d)$ with $0 < j_k < N$ we get a discrete point $X_j = (x_{1j_1}, \dots, x_{dj_\alpha}) \in \Omega_h$.

Let U_j^n denotes the discrete approximation of $u(X_j, t_n)$. We also use the notation

$$V(x+\alpha; k) = V(x_1, \dots, x_k + \alpha, \dots, x_d).$$

The inner product between mesh functions U and V and the discrete L_2 and L_∞ norms are defined by

$$\begin{aligned} &\Big(U, V\Big) = h^d \sum_{X \in \Omega_h} U(X)V(X), \quad \|V\| = \sqrt{(V,V)}, \\ &\|V\|_\infty = \max_{X \in \Omega_h} |V(X)|. \end{aligned}$$

The finite-difference scheme is defined as follows

$$\frac{U^{n+1} - U^n}{\tau} = \sum_{k=1}^{d} D_k U^\sigma + f_R(X, t_{n+\sigma}, U^\sigma) \quad \text{in} \quad \Omega_h, \tag{2.1}$$

$$U^{n+1} = 0 \qquad \text{on} \quad \partial\Omega_h, \tag{2.2}$$

$$U(X,0) = u_0(X) \qquad \text{in} \quad \Omega_h \cup \partial\Omega_h, \tag{2.3}$$

where D_k is an approximation of the differential operator by central differences

$$\begin{aligned} D_k U \;=\; & \frac{1}{h^2}\Big[a_k\big(X + \frac{h}{2}; k\big)\Big(U(X+h;k) - U(X)\Big) \\ & - a_k\big(X - \frac{h}{2}; k\big)\Big(U(X) - U(X-h;k)\Big)\Big], \end{aligned}$$

and we use the following notations

$$U^\sigma = \sigma U^{n+1} + (1-\sigma)U^n, \quad t_{n+\sigma} = t_n + \sigma\tau, \quad 0 \le \sigma \le 1.$$

It is well-known that D_k satisfies the estimates (see, e.g. [6])

$$\frac{4A_L}{h^2}\sin^2\Big(\frac{\pi h}{2}\Big)\|U\|^2 \le \Big(-D_k U, U\Big) \le \frac{4}{h^2} A_R \|U\|^2. \tag{2.4}$$

3. THE METHOD FOR INVESTIGATION OF NONLINEAR DIFFERENCE SCHEMES

In this section we consider a modification of a general investigation method which was proposed in [1,2].

First the existence of the unique solution of finite-difference scheme (2.1) – (2.3) is proved. We find a solution by using the iterative method

$$\frac{\overset{s+1}{U} - U^n}{\tau} = \sum_{k=1}^{d} D_k \overset{s+1}{U}{}^{\sigma} + f_R(X, t_{n+\sigma}, \overset{s}{U}{}^{\sigma}) \quad \text{in } \Omega_h, \tag{3.1}$$

$$\overset{s+1}{U} = 0 \qquad \text{on } \partial\Omega_h, \tag{3.2}$$

$$\overset{\circ}{U} = U^n,$$

where $\overset{s}{U}$ denotes the s-th iterative approximation and $\overset{s}{U}{}^{\sigma} = \sigma \overset{s}{U} + (1-\sigma)U^n$.

As it follows from [2] the convergence of the iterative method (3.1) – (3.2) and the uniqueness of the solution depends on the following stability property.

Let us consider the auxiliary discrete problem for the difference $V^{n+1} - W^{n+1}$

$$\begin{aligned} \frac{V^{n+1} - W^{n+1}}{\tau} &= \sigma \sum_{k=1}^{d} D_k \big(V^{n+1} - W^{n+1}\big) \\ &\quad + f_R\Big(X, t_{n+\sigma}, \sigma P^{n+1} + (1-\sigma)U^n\Big) \\ &\quad - f_R\big(X, t_{n+\sigma}, \sigma Q^{n+1} + (1-\sigma)U^n\big) \quad \text{in } \Omega_h, \end{aligned} \tag{3.3}$$

$$V^{n+1} - W^{n+1} = 0 \qquad \text{on } \partial\Omega_h, \tag{3.4}$$

where $U^n \in B\Big(u(t_n), R\Big)$ and P^{n+1}, Q^{n+1} are any functions satisfying boundary conditions (3.4).

Definition 1. *The finite-difference scheme is said to be stable, if for sufficiently small $\tau \le \tau_0, h \le h_0$ the following estimate*

$$\|V^{n+1} - W^{n+1}\|_{(1)} \le \tau C_D \|P^{n+1} - Q^{n+1}\|_{(1)} \tag{3.5}$$

holds for problem (3.3) – (3.4), *where C_D may depend on constants which are used for the definition of f_R.*

Then the following theorem is proved in [2].

Theorem 3.1. *If the difference scheme* (2.1) – (2.3) *is stable then for sufficiently small $\tau \le \tau_1$ the iterative sequence defined by problem* (3.1) – (3.2) *converges to the solution of the difference scheme* (2.1) – (2.3), *the following estimate*

$$\|U^{n+1} - \overset{s}{U}\|_{(1)} \le \frac{q^s}{1-q} \|\overset{1}{U} - U^n\|_{(1)}, \quad q < 1$$

holds and this solution is unique.

Now we will give one important remark about the realization details of the iterative method (3.1) – (3.2).

REMARK 1. The main norm $\|\cdot\|_{(1)}$ can be weaker than the maximum norm $\|\cdot\|_\infty$. We will prove below that for sufficiently small $\tau \le \tau_0$ the solution $U^{n+1} \in B\Big(u(t_{n+1}), R\Big)$. Hence we have that $f_R\Big(X, t_{n+\sigma}, U^\sigma\Big) = f\Big(X, t_{n+\sigma}, U^\sigma\Big)$. In the formulation of the iterative method we can use a ball $B(U^n, 2R)$. If after convergence of the iterative sequence $U^{n+1} \notin B(U^n, 2R)$, we decrease the time step τ. It is important to note that some $\overset{s}{U}$ may not belong to $B(U^n, 2R)$ $\Big($nor to $B\Big(u(t_{n+1}), R\Big)\Big)$.

Now we will investigate the convergence of the discrete solution. The global error $Z_j^n = U_j^n - u(X_j, t_n)$ satisfies the problem

$$\frac{Z^{n+1} - Z^n}{\tau} = \sum_{k=1}^{d} D_k Z^\sigma + f_R\Big(X, t_{n+\sigma}, U^\sigma\Big) \qquad (3.6)$$

$$-f_R\big(X, t_{n+\sigma}, u^\sigma\big) + \Psi^n \quad \text{in} \quad \Omega_h,$$

$$Z^n = 0 \qquad \text{on} \quad \partial\Omega_h, \qquad (3.7)$$

$$Z(X, 0) = 0 \qquad \text{in} \quad \Omega_h \cup \partial\Omega_h,$$

where the function

$$\Psi_j^n = -\frac{u^{n+1} - u^n}{\tau} + \sum_{k=1}^{d} D_k u^\sigma + f\big(x, t_{n+\sigma}, u^\sigma\big)$$

is called the truncation error.

DEFINITION 2. *Finite-difference scheme* (2.1) − (2.3) *is said to be asymptotically stable, if for sufficiently small $\tau \le \tau_0, h \le h_0$ the following estimate*

$$\|Z^{n+1}\|_{(1)} \le e^{-C_S \tau} \|Z^n\|_{(1)} + \tau C_T \|\Psi^n\|_{(2)}, \qquad (3.8)$$

holds for problem (3.6) − (3.7), *where C_S, C_T are nonnegative constants that may depend on constants used in the definition of function f_R.*

Let assume that the following estimate

$$\|\Psi^n\|_{(2)} \le C_A(\tau^\alpha + h^\beta), \quad \alpha, \beta > 0, n \ge 0 \qquad (3.9)$$

is valid for the truncation error.

THEOREM 3.2. *Let finite-difference scheme* (2.1) − (2.3) *be asymptotically stable. Then for sufficiently small $\tau \le \tau_3$ the global error of the solution of difference scheme* (2.1) − (2.3) *satisfies the uniform in time estimate*

$$\|U^n - u(t_n)\|_{(1)} \le C(\tau^\alpha + h^\beta). \qquad (3.10)$$

P r o o f. By iterating the stability estimate (3.8) we have

$$\|Z^n\|_{(1)} \leq e^{-C_S(t_n-t_m)}\|Z^{n-m}\|_{(1)} + \tau C_T \max_{n-m\leq j\leq n-1} \|\Psi^j\|_{(2)} \big(1 + e^{-C_S\tau} + \cdots + e^{-C_S(m-1)\tau}\big).$$

With $\tau \leq \tau_0$ we thus get

$$\|Z^n\|_{(1)} \leq e^{-C_S(t_n-t_m)}\|Z^{n-m}\|_{(1)} + \frac{C_T}{C_S} e^{C_S\tau_0} \max_{n-m\leq j\leq n-1} \|\Psi^j\|_{(2)}. \quad (3.11)$$

Taking $t_m = t_0$ and using (3.8) we prove the required uniform in time global error estimate (3.10). This completes the proof. □

There we will make one important remark (see also [2,3]). Although the replacement of $f(x,t,u)$ by $f_R(x,t,u)$ does not affect the exact solution of the differential problem, it may change the solution of the difference scheme. Hence it is necessary additionally to prove the convergence estimate in the maximum norm L_∞. Examples of such analysis will be given below.

Next we consider the corresponding discrete stationary problem

$$\begin{aligned} -\sum_{k=1}^{d} D_k V &= F_R(x,V) \quad \text{in} \quad \Omega_h, \\ V &= 0 \quad \text{on} \quad \partial\Omega_h. \end{aligned} \quad (3.12)$$

The existence of the solution V is guaranted for small h if F_R satisfies the assumption (H2) and since V is a finite-dimensional vector.

Then finite-difference scheme (2.1)–(2.3) can be used as an iterative method for finding the solution of stationary problem (3.12).

THEOREM 3.3. *Let finite-difference scheme* (2.1) − (2.3) *is asymptotically stable. Then* U^n *converges to the stationary solution* V *of* (3.12) *and the following error estimate is valid*

$$\|U^n - V\|_{(1)} \leq e^{-C_S t_n}\|U^0 - V\|_{(1)}. \quad (3.13)$$

Moreover, V *converges to a stationary solution* $\bar{U}$ *of* (1.5) *and* $V - \bar{U}$ *satisfies the estimates*

$$\|V - \bar{U}\|_{(1)} \leq Ch^\beta. \quad (3.14)$$

P r o o f. The difference $Z^n = U^n - V$ satisfies the problem

$$\frac{Z^{n+1} - Z^n}{\tau} = \sum_{k=1}^{d} D_k Z^\sigma.$$

Then it follows from (3.11) that

$$\|Z^n\| \le e^{-Ct_n}\|Z^0\|.$$

Hence an initial error $\|Z^0\|$ is reduced $1/\varepsilon$ times if $n \ge \ln(1/\varepsilon)/(C_S\tau)$.

We now turn to the proof of (3.14). Substituting $\bar{U}$ into (2.1) we get that the truncation error satisfies

$$\|\Psi\|_{(2)} \le C_A h^\beta, \quad \beta > 0.$$

Then by (3.11) we obtain the uniform in time error estimate

$$\|U^n - \bar{U}\| \le e^{-C_S t_n}\|U^0 - \bar{U}\| + \frac{C_T}{C_S}h^\beta.$$

Using this inequality and the established fact that $U^n \to V$ as $n \to \infty$ we prove (3.14). The theorem is proved . □

4. THE IMPLICIT FINITE-DIFFERENCE SCHEME

In this section we apply general results of Section 3 to the implicit scheme (2.1) – (2.3) with $\sigma = 1$

$$\frac{U^{n+1} - U^n}{\tau} = \sum_{k=1}^{d} D_k U^{n+1} + f_R(X, t_{n+1}, U^{n+1}) \qquad \text{in} \quad \Omega_h, \tag{4.1}$$

$$U^{n+1} = 0 \qquad \text{on} \quad \partial\Omega_h. \tag{4.2}$$

We assume the following hypotheses.

(H3) The function f_R is globally Lipschitz function, i.e.

$$|f_R(x,t,U_1) - f_R(x,t,U_2)| \le L|U_1 - U_2|.$$

(H4) The smooth function f_R satisfies the estimate

$$\frac{\partial f_R}{\partial u}(x,t,v) \le -S_F, \quad S_F > 0.$$

It is well-known that for a sufficiently smooth solution $u(x,t)$ of (1.1) – (1.3) the truncation error satisfies

$$|\Psi| \le C_A(\tau + h^2) \quad \text{in} \quad \Omega_h. \tag{4.3}$$

It is sufficient to prove that finite-difference scheme (4.1)–(4.2) is stable and asymptotically stable.

LEMMA 4.1. *Let assume that (H3) and (H4) are satisfied. Then finite-difference scheme (4.1) is stable and asymptotically stable in* L_∞, *i.e.* $\|\cdot\|_{(j)} = \|\cdot\|_\infty, j = 1, 2$.

P r o o f. In order to prove that scheme (4.1) is stable we consider the auxiliary discrete problem (see (3.5))

$$\begin{aligned} \frac{V^{n+1} - W^{n+1}}{\tau} &= \sum_{k=1}^{\infty} D_k(V^{n+1} - W^{n+1}) + f_R(X, t_{n+1}, P^{n+1}) \\ &\quad - f_R(X, t_{n+1}, Q^{n+1}) \end{aligned}$$

It follows from the maximum principle that

$$\|V^{n+1} - W^{n+1}\|_\infty \le \tau \|f_R(\cdot, t_{n+1}, P^{n+1}) - f_R(\cdot, t_{n+1}, Q^{n+1})\|_\infty.$$

By the estimate (H3) we now have that for $n \ge 0$

$$\|V^{n+1} - W^{n+1}\|_\infty \le \tau L \|P^{n+1} - Q^{n+1}\|_\infty.$$

We conclude that finite-difference scheme (4.1) – (4.2) is stable.

We now turn to the proof of asymptotical stability. The global error $Z = U - u$ satisfies the discrete problem

$$\begin{aligned} \frac{Z^{n+1} - Z^n}{\tau} &= \sum_{k=1}^{d} D_k Z^{n+1} + f_R(X, t_{n+1}, U^{n+1}) \\ &\quad - f_R(X, t_{n+1}, u(t_{n+1})) + \Psi^n. \end{aligned}$$

Then it follows from (H4) and from the maximum principle that

$$\|Z^{n+1}\|_\infty \le \frac{1}{1 + \tau S_F} \big(\|Z^n\|_\infty + \tau \|\Psi^n\|_\infty \big).$$

We note that

$$\frac{1}{1 + \tau S_F} = e^{-\tau C_F} \quad \text{for} \quad C_F > 0.$$

This completes the proof of the lemma. □

Hence the conclusions of Theorem 3.1 and Theorem 3.2 hold for the implicit difference scheme (4.1) – (4.2) if the function f_R satisfies assumptions (H3) and (H4).

We also remark that, since the asymptotical stability is proved in the maximum norm L_∞, we have

$$\|U^n - u(t_n)\|_\infty \le R, \quad 0 \le t_n < \infty$$

for τ and h sufficiently small, so that, in fact,

$$f_R(X, t_n, U^n) = f(X, t_n, U^n).$$

Now we replace the assumptions (H3), (H4) with the following more general assumption:

(H5) The smooth function f_R satisfies globally the estimates

$$-S_L \leq \frac{\partial f_R(x,t,v)}{\partial u} \leq S_R \quad \text{for} \quad S_L, S_R > 0.$$

In order to use the results of Section 3 we will prove the stability estimates in Lemma 4.2 and Lemma 4.3.

LEMMA 4.2. *Let f_R be a smooth function satisfying (H5). Then finite-difference scheme (4.1) is stable in L_∞.*

P r o o f. It is sufficient to note that the assumption (H3) follows from the assumption (H5) with $L = \max(S_L, S_R)$. Then the stability inequality (3.5) follows from Lemma 4.1. This completes the proof of Lemma 4.2. □

LEMMA 4.3. *Let f_R be a smooth function satisfying (H5) and the following inequality*

$$dA_L\lambda_1 - S_R > C_f, \quad \lambda_1 = \frac{4}{h^2}\sin^2\left(\frac{\pi h}{2}\right), \ C_f > 0 \tag{4.4}$$

is valid. Then finite-difference scheme (4.1) is asymptotically stable in L_2, i.e. the stability inequality

$$\|Z^{n+1}\| \leq e^{-\tau C_F}\Big(\|Z^n\| + \tau\|\Psi^n\|\Big) \quad \text{for} \quad C_F > 0 \tag{4.5}$$

holds for the global error.

P r o o f. We first split f_R as $f_R = f_R^+ - f_R^-$, where

$$0 \leq \frac{\partial f_R^+}{\partial u} \leq S_R, \quad 0 \leq \frac{\partial f_R^-}{\partial u} \leq S_L. \tag{4.6}$$

Then we have the discrete problem for the global error

$$\frac{Z^{n+1} - Z^n}{\tau} + \frac{\partial f_R^-}{\partial u} Z^{n+1} = \sum_{j=1}^{d} D_j Z^{n+1} + \frac{\partial f_R^+}{\partial u} Z^{n+1} + \Psi^n. \tag{4.7}$$

Taking the inner product of (4.7) with Z^{n+1} and using (4.6), we get

$$\|Z^{n+1}\|^2 \quad \leq \quad \|Z^n\|\,\|Z^{n+1}\| + \tau\sum_{j=1}^{d}\Big(D_j Z^{n+1}, Z^{n+1}\Big)$$

$$+\tau S_R \|Z^{n+1}\|^2 + \tau \|\Psi^n\| \, \|Z^{n+1}\|.$$

Using (2.4) and (4.6), we have

$$\Big(1 + \tau\big(dA_L\lambda_1 - S_R\big)\Big)\|Z^{n+1}\| \le \|Z^n\| + \tau\|\Psi^n\|. \tag{4.8}$$

We note that (4.4) implies

$$1 + \tau\big(dA_L\lambda_1 - S_R\big) = e^{\tau C_F} \quad \text{for} \quad C_F > 0,$$

hence (4.5) follows from (4.8). This completes the proof of Lemma 4.3. □

Since the asymptotical stability is proved only in the L_2 norm, we must prove that $U^n \in B(u(t_n), R)$. It follows from (4.3) and the well-known inverse inequality

$$\|Z^n\|_\infty \le h^{-d/2}\|Z^n\|$$

that the global error satisfies the estimate

$$\|Z^n\|_\infty \le C h^{-d/2}(\tau + h^2).$$

Hence, for $d \le 3$ and $\tau = C_0 h^{d/2+\varepsilon}$ with $\varepsilon > 0$, we have

$$\|Z^n\|_\infty \le C(h^\varepsilon + h^{2-d/2}) \le R$$

for sufficiently small h and τ.

In the one dimensional case $d = 1$ we can prove that $U^n \in B(u(t_n), R)$ without imposing the restrictive relation between sizes of parameters τ and h.

LEMMA 4.4. *Let $d = 1$ and f_R be a smooth function satisfying $(H5)$ and the following inequalities*

$$\|\frac{\partial f_R}{\partial x}\|_\infty \le S_F, \quad A_L\lambda_1 - S_R = C_F > 0. \tag{4.9}$$

Then for sufficiently small τ and h we have that $U^n \in B(u(t_n), R)$.

P r o o f. We will use the well-known multiplicative inequality (see, e.g. [7])

$$\|Z^n\|_\infty \le C\|Z^n_{\bar{x}}]|^{0.5}\|Z^n\|^{0.5},$$

where we denote

$$\|Z^n_{\bar{x}}]|^2 = \sum_{j=1}^{N} ha(X_{j-0.5})\big(\frac{Z_j - Z_{j-1}}{h}\big)^2.$$

It is easy to see that

$$\|Z_{\bar{x}}^n]| \le \|U_{\bar{x}}^n]| + \|u_{\bar{x}}^n]|.$$

Since $u(x,t)$ is a smooth function, we have $\|u_{\bar{x}}^n]| \le C$. It remains to prove that $\|U_{\bar{x}}^n]|$ is also globally bounded.

Taking the inner product of (4.1) with $D_1 U^{n+1}$ and using the Green's first formula we get

$$\begin{aligned} \|U_{\bar{x}}^{n+1}]|^2 + \tau \|D_1 U^{n+1}\|^2 \quad &\le \quad \|U_{\bar{x}}^n]|\,\|U_{\bar{x}}^{n+1}]| \\ &\quad + \tau \Big(A_R \|\frac{\partial f_R}{\partial x}\|_\infty + S_R \|U_{\bar{x}}^{n+1}]| \Big) \|U_{\bar{x}}^{n+1}]|. \end{aligned}$$

Using the inequality (see, e.g. [6])

$$\|D_1 U^{n+1}\|^2 \ge \lambda_1 A_L \|U_{\bar{x}}^{n+1}]|^2$$

and the assumption (4.9), we get

$$\Big(1 + \tau\big(\lambda_1 A_L - S_R\big)\Big)\|U_{\bar{x}}^{n+1}]| \le \|U_{\bar{x}}^n]| + \tau A_R S_F.$$

By iterating this inequality, we have the uniform estimate

$$\|U_{\bar{x}}^n]| \le \|U_{\bar{x}}^0]| + \big(1 + \tau C_F\big) A_R S_F / C_F.$$

Hence the global error Z^n can be estimated in the L_∞ norm by

$$\|Z^n\|_\infty \le C(\tau + h^2)^{0.5}$$

and $U^n \in B\big(u(t_n), R\big)$ for sufficiently small τ and h. This completes the proof of Lemma 4.4. □

5. THE EXPLICIT FINITE-DIFFERENCE SCHEME

In this section we apply general results of Section 3 to the explicit scheme (2.1) – (2.3) with $\sigma = 0$

$$\frac{U^{n+1} - U^n}{\tau} = \sum_{k=1}^{d} D_k U^n + f_R(X, t_n, U^n) \quad \text{in} \quad \Omega_h, \qquad (5.1)$$

$$U^{n+1} = 0 \qquad \text{on} \quad \partial\Omega_h. \qquad (5.2)$$

Since we are using an explicit scheme, only asymptotical stability must be investigated. We again note that for a sufficiently smooth solution $u(x,t)$ of (1.1) – (1.3) the truncation error satisfies (4.3).

LEMMA 5.1. *Let f_R be a smooth function satisfying $(H5)$ and the inequality (4.4) is valid. Then for sufficiently small $\tau \le \tau_0$, where*

$$\tau_0 = \frac{2}{A_R 4d/h^2 + S_L + C_f} \tag{5.3}$$

the finite-difference scheme $(5.1) - (5.2)$ *is asymptotically stable.*

P r o o f. The global error $Z = U - u$ satisfies the discrete problem

$$\begin{aligned} \frac{Z^{n+1} - Z^n}{\tau} &= \sum_{k=1}^{d} D_k Z^n + f_R(X, t_n, U^n) \\ &\quad - f_R(X, t_n, u/(t_n)) + \Psi^n. \end{aligned} \tag{5.4}$$

In our stability analysis we will use the method similar to one presented in [4]. Let define the following matrices

$$AZ = \sum_{k=1}^{d} D_k Z, \quad T^{\pm} Z = \frac{\partial f_R^{\pm}}{\partial u} Z.$$

Taking L_2 norms of (5.4), we obtain

$$\|Z^{n+1}\| \le \|I + \tau\Big(A + T^+ - T^-\Big)\| \, \|Z^n\| + \tau \|\Psi^n\|. \tag{5.5}$$

Since A is symmetric and $T^{\pm}$ are diagonal the norm $\|I + \tau\big(A + T^+ - T^-\big)\|$ coincides with the spectral radius ρ of $I + \tau\big(A + T^+ - T^-\big)$. Using (H5) and standard results on eigenvalues of the matrix A, we get

$$\rho = \max\Big(|1 - \tau\big(A_R \frac{4d}{h^2} + S_L\big)|, |1 - \tau\big(A_L \lambda_1 - S_R\big)|\Big).$$

Taking $\tau \le \tau_0$, where τ_0 is given by (5.3), we get that $\rho = 1 - \tau C_f < \exp(-\tau C_f)$. Hence it follows from (5.5) that

$$\|Z^{n+1}\| \le e^{-\tau C_f} \|Z^n\| + \tau \|\Psi^n\|.$$

This completes the proof of Lemma 5.1. $\square$

Again we must prove that $U^n \in B\big(u(t_n), R\big)$. Since $\tau \le Ch^2$ by (5.3), the required global error estimate follows for $d \le 3$ and small h from the inverse inequality

$$\|Z^n\|_\infty \le h^{-d/2} \|Z^n\| \le Ch^{2-d/2}.$$

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