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# On the Muth Distribution*

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**Abstract.** The Muth distribution is a continuous random variable introduced in the context of reliability theory. In this paper, some mathematical properties of the model are derived, including analytical expressions for the moment generating function, moments, mode, quantile function and moments of the order statistics. In this regard, the generalized integro-exponential function, the Lambert $W$ function and the golden ratio arise in a natural way. The parameter estimation of the model is performed by the methods of maximum likelihood, least squares, weighted least squares and moments, which are compared via a Monte Carlo simulation study. A natural extension of the model is considered as well as an application to a real data set.

**Keywords:** Muth distribution, moment generating function, order statistics, parameter estimation, simulation.

**AMS Subject Classification:** 60E05; 33B30; 33E20.

## 1 Introduction

Muth [18] introduced a continuous probability distribution with application in reliability theory. A random variable $X$ is said to have a Muth distribution with parameter $\alpha$ if the probability density function is given by

$$f(x;\alpha) := \left(e^{\alpha x} - \alpha\right) \exp\left(\alpha x - \frac{1}{\alpha}\left(e^{\alpha x} - 1\right)\right), \quad x > 0, \tag{1.1}$$

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where $\alpha \in (0,1]$. Figure 1 represents the density function of $X$ for several values of $\alpha$.

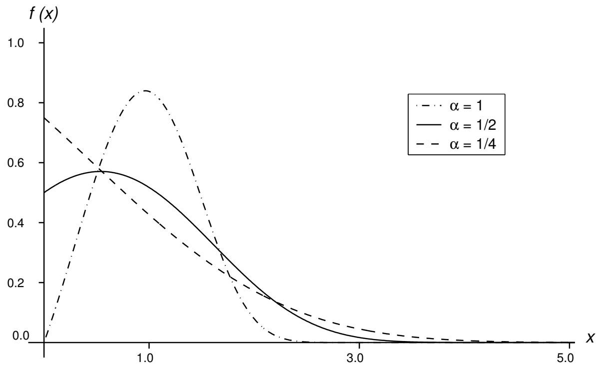


**Figure 1.** Density functions of the Muth distribution with parameters $\alpha = 1, 1/2, 1/4$.

The cumulative distribution function of $X$, $F(x;\alpha) := P(X \leq x)$, is the following

$$F(x;\alpha) = 1 - \exp\left(\alpha x - \frac{1}{\alpha}\left(e^{\alpha x} - 1\right)\right), \quad x > 0. \tag{1.2}$$

Leemis and McQueston [15] summarized in a schematic figure the most important univariate probability distributions including many of the relationships among these distributions together with the main statistical properties that each distribution possesses. In particular, in [15] we can find the probability distribution defined by Eq. (1.1) which the authors called the Muth distribution. However, the unique statistical property of $X$ highlighted in [15] is that its limit distribution as the parameter $\alpha$ decreases to zero is the standard exponential distribution. In fact, only a few properties of $X$ have been given in [18], specifically that it is a model with strictly positive memory, that the mean residual life function corresponds to the exponential function and that it has considerably less probability mass in the tail than commonly used unimodal distributions, such as the gamma, lognormal and Weibull distributions. The last property is outlined in [18] and can be easily checked as follows. Denoting by $S_\alpha(x) := P(X > x)$ the survival function of the Muth distribution with parameter $\alpha$ and by $S(x)$ the survival function of the gamma, lognormal or Weibull distribution, routine calculations show that $\lim_{x\to\infty} S_\alpha(x)/S(x) = 0$, which implies the result.

As far as the Muth distribution is concerned, with the exception of [15], it has been overlooked in the literature. As it will be seen in this paper, the Muth distribution has interesting mathematical properties which have not been previously considered in [15, 18]. In Section 2, we provide an explicit expression of the moment generating function in terms of the exponential integral function. This is used to derive the moments of $X$ as a function of the generalized integro-exponential function. In Section 3, we show that the Muth distribution has the variate generation property. In this regard, the Lambert $W$ function plays a central role. Section 4 establishes a connection between the mode of $X$ and the golden ratio. In Section 5, the closed-form expressions obtained for the mode, median and expected value of $X$ are used to show that the Muth distribution satisfies the so-called mode-median-mean inequality. Section 6 provides analytical expressions for the moments of the order statistics. Section 7 deals with the parameter estimation problem. The methods of maximum likelihood, least squares, weighted least squares and moments are described and compared via a Monte Carlo simulation study. Section 8 describes a scaling transformation of the Muth distribution as well as the properties inherited from the original distribution. An application to a real data set is presented in Section 9. Finally, Section 10 summarizes the results and concludes the paper.

## 2 The Moment Generating Function

In this section, we initially provide an analytical expression for the moment generating function of the Muth distribution in terms of the exponential integral function. Recall that the moment generating function of $X$ is defined by $M(t;\alpha) := E\left[e^{tX}\right]$, $t \in \mathbb{R}$. Using this function, we shall derive the moments of $X$ in terms of the generalized integro-exponential function.

First, we introduce some notation. Denote by $\Gamma(a,z)$ the upper incomplete gamma function (cf. Olver et al. [19, p. 174]), that is,

$$\Gamma(a,z) := \int_z^\infty t^{a-1}e^{-t}dt, \quad a \in \mathbb{C},\ z \in \mathbb{C} \setminus \mathbb{R}^-. \tag{2.1}$$

In addition, the usual exponential integral function $E_s(z)$ can be defined in terms of the upper incomplete gamma function as follows (cf. Olver et al. [19, p. 185])

$$E_s(z) := z^{s-1}\Gamma(1-s,z), \quad s,z \in \mathbb{C}. \tag{2.2}$$

With the preceding notations, we state the following.

**Theorem 1.** *Let $X$ be a random variable having a Muth distribution with parameter $\alpha \in (0,1]$. The moment generating function of $X$ is*

$$M(t;\alpha) = \frac{e^{1/\alpha}}{\alpha}\, tE_{-t/\alpha}(1/\alpha) + 1, \quad -\infty < t < \infty. \tag{2.3}$$

*Proof.* From the definition of $M$ together with Eq. (1.2), for any $t \in \mathbb{R}$, we have

$$\begin{aligned} M(t;\alpha) = E\left[e^{tX}\right] &= \int_0^\infty e^{tx}dF(x;\alpha) \\ &= \int_0^\infty e^{tx}\left(e^{\alpha x} - \alpha\right)\exp\left\{\alpha x - \frac{1}{\alpha}\left(e^{\alpha x} - 1\right)\right\}dx. \end{aligned}$$

By making the change of variable $u = e^{\alpha x}/\alpha$, we obtain

$$\begin{aligned} M(t;\alpha) &= e^{1/\alpha}\alpha^{1+t/\alpha}\left(\int_{1/\alpha}^\infty u^{1+t/\alpha}e^{-u}du - \int_{1/\alpha}^\infty u^{t/\alpha}e^{-u}du\right) \\ &= e^{1/\alpha}\alpha^{1+t/\alpha}\left\{\Gamma\left(2 + \frac{t}{\alpha}, \frac{1}{\alpha}\right) - \Gamma\left(1 + \frac{t}{\alpha}, \frac{1}{\alpha}\right)\right\}. \end{aligned} \tag{2.4}$$

Now, taking into account in Eq. (2.4) that the upper incomplete gamma function satisfies the recurrence relation $\Gamma(a+1,z) = a\Gamma(a,z) + z^a e^{-z}$ (cf. Olver et al. [19, p. 178]), we get

$$M(t;\alpha) = e^{1/\alpha}\alpha^{t/\alpha} t\Gamma\left(1+\frac{t}{\alpha},\frac{1}{\alpha}\right) + 1, \quad -\infty < t < \infty. \tag{2.5}$$

Finally, from the above equation and Eq. (2.2) we obtain the desired result. □

*Remark 1.* We highlight that the moment generating function of the Muth distribution can also be expressed alternatively in terms of the upper incomplete gamma function by virtue of Eq. (2.5).

In order to compute the $k$th moment of $X$

$$E[X^k] := \int_0^\infty x^k dF(x;\alpha), \quad k = 1,2,\ldots,$$

we shall use the well-known property that

$$E[X^k] = \frac{\partial^k}{\partial t^k} M(t;\alpha)|_{t=0}, \quad k = 1,2,\ldots$$

see, Bartoszyński and Niewiadomska-Bugaj [7, pp. 230–231]. With this aim, first we show in the next lemma that the derivatives of $M(t;\alpha)$ can be expressed in terms of the generalized integro-exponential function. The generalized integro-exponential function is defined by the following integral representation (cf. Milgram [17] for further details)

$$E_s^m(z) := \frac{1}{\Gamma(m+1)} \int_1^\infty (\log u)^m u^{-s} e^{-zu} du, \quad s, z \in \mathbb{C},\ m = 0,1,\ldots, \tag{2.6}$$

where log stands for the natural logarithm. An efficient and accurate computation algorithm for the above integrals can be found in Ozalp and Bairamov [20]. We also note that these integrals are related to the exponential integral distribution (cf. Meijer and Baken [16]).

**Lemma 1.** *Let $X$ be a random variable having a Muth distribution with parameter $\alpha \in (0,1]$. The derivatives of the moment generating function $M$ are given by*

$$\frac{\partial^k}{\partial t^k} M(t;\alpha) = \frac{e^{1/\alpha}\Gamma(k+1)}{\alpha^k}\left(E_{-t/\alpha}^{k-1}(1/\alpha) + \frac{t}{\alpha} E_{-t/\alpha}^{k}(1/\alpha)\right), \quad k = 1,2,\ldots.$$

*Proof.* From Eq. (2.3), after some calculations we get the following

$$\frac{\partial^k}{\partial t^k} M(t;\alpha) = \frac{e^{1/\alpha}}{\alpha}\left(k\frac{\partial^{k-1}}{\partial t^{k-1}} E_{-t/\alpha}(1/\alpha) + t\frac{\partial^k}{\partial t^k} E_{-t/\alpha}(1/\alpha)\right), \quad k = 1,2,\ldots.$$

Then, the result follows from the above equation by taking into account that (cf. Milgram [17])

$$(-1)^k \frac{\partial^k}{\partial s^k} E_s(z) = \Gamma(k+1) E_s^k(z), \quad k = 1, 2, \dots,$$

and assuming that $\frac{\partial^0}{\partial s^0} E_s(z) := E_s(z)$. This completes the proof. □

**Proposition 1.** *Let $X$ be a random variable having a Muth distribution with parameter $\alpha \in (0, 1]$. The moments of $X$ are given by*

$$E[X^k] = \frac{e^{1/\alpha} \Gamma(k+1)}{\alpha^k} E_0^{k-1}(1/\alpha), \quad k = 1, 2, \dots. \tag{2.7}$$

*Proof.* The result follows from Lemma 1 because $E[X^k] = \frac{\partial^k}{\partial t^k} M(t; \alpha)\big|_{t=0}$. □

As a consequence of Proposition 1, it is clear that it is not possible to obtain expressions for the moments of the Muth distribution in terms of elementary functions, with the exception of the special case $k = 1$ as we shall see below. Moreover, in the next result, we also see that $E[X^2]$ can be alternatively expressed in terms of the best-known upper incomplete gamma function.

*Corollary 1.* Let $X$ be a random variable having a Muth distribution with parameter $\alpha \in (0, 1]$. Then,

$$(i) \quad E[X] = 1, \qquad (ii) \quad E[X^2] = \frac{2\, e^{1/\alpha}}{\alpha} \Gamma(0, 1/\alpha).$$

*Proof.* (i) The result follows from Eq. (2.7) since $E_0^0(1/\alpha) = \alpha e^{-1/\alpha}$. (ii) From Eqs. (2.6) and (2.7), we have

$$E\big[X^2\big] = \frac{2e^{1/\alpha}}{\alpha^2} E_0^1(1/\alpha) = \frac{2e^{1/\alpha}}{\alpha^2} \int_1^\infty \log(u) e^{-u/\alpha} du = \frac{2e^{1/\alpha}}{\alpha} \int_1^\infty \frac{e^{-u/\alpha}}{u} du,$$

where the last equality is obtained by integration by parts. Now, part (ii) follows by virtue of Eq. (2.1). □

For several values of $\alpha$, Table 1 displays some numerical results concerning the variance of $X$ defined by $\sigma^2 := E[X^2] - E^2[X]$, skewness of $X$ defined by $\gamma_1 := E[(X - E[X])^3]/\sigma^3$ and kurtosis of $X$ defined by $\gamma_2 := E[(X - E[X])^4]/\sigma^4 - 3$. Table 1 shows how rapidly the variance, skewness and kurtosis decrease as $\alpha$ increases. Recalling that the limit distribution of $X$ as $\alpha$ decreases to zero is the standard exponential distribution, we also know that $\lim_{\alpha \to 0^+} \sigma^2 = 1$, $\lim_{\alpha \to 0^+} \gamma_1 = 2$ and $\lim_{\alpha \to 0^+} \gamma_2 = 6$. It is also clear that the Muth distribution is a right-skewed distribution, that is, $\gamma_1 > 0$.

**Table 1.** Values of $\sigma^2$, $\gamma_1$ and $\gamma_2$ for different values of $\alpha$.

| $\alpha$ | $\sigma^2$ | $\gamma_1$ | $\gamma_2$ |
|---|---|---|---|
| 0.01 | 0.980388458 | 1.941997413 | 5.548021863 |
| 0.1 | 0.831266679 | 1.541622673 | 2.989014642 |
| 0.2 | 0.704221762 | 1.235599336 | 1.579050074 |
| 0.3 | 0.602372560 | 1.002240929 | 0.764615221 |
| 0.4 | 0.517629182 | 0.812204500 | 0.255908197 |
| 0.5 | 0.445314468 | 0.651973092 | -0.069242338 |
| 0.6 | 0.382451878 | 0.514927916 | -0.270994230 |
| 0.7 | 0.327020548 | 0.398595883 | -0.383652943 |
| 0.8 | 0.277582209 | 0.304031421 | -0.430372680 |
| 0.9 | 0.233075574 | 0.236683935 | -0.432614204 |
| 1.0 | 0.192694724 | 0.209347826 | -0.421933390 |

## 3 Quantile Function via the Lambert $W$ Function

In this section, we show that the Muth distribution has the variate generation property, that is, its quantile function can be given in closed form. This is a useful property since, by using the inverse transform method (cf., for example, Fishman [13, pp. 149–156]), it allows us to generate by computer pseudo-random data from that probability distribution.

For the sake of completeness, we recall that the quantile function of an arbitrary random variable $T$ is defined as the function

$$Q_T(u) := \inf\{t \in \mathbb{R} : \ F_T(t) \geq u\}, \quad 0 < u < 1,$$

where $F_T(t)$ is the cumulative distribution function of $T$. In particular, the above definition implies that if $F_T$ is a continuous and strictly increasing function then $F_T$ has a unique inverse and $Q_T(u) = F_T^{-1}(u)$, $0 < u < 1$. See Parzen [21] for a short review on the quantile function and its applications.

More specifically, the quantile function of the Muth distribution can be expressed in closed form in terms of the Lambert $W$ function. We briefly remind that the Lambert $W$ function is defined as the solution of the equation

$$W(z)\exp\big(W(z)\big) = z, \quad z \in \mathbb{C}. \tag{3.1}$$

The multivalued complex function $W$ has two real branches if $z$ is a real number such that $z \geq -1/e$. The real branch taking on values in $(-\infty, -1]$ is called the negative branch and denoted by $W_{-1}(z)$, where $-1/e \leq z < 0$. The real branch taking on values in $[-1, \infty)$ is called the principal branch and denoted by $W_0(z)$, where $z \geq -1/e$. For our purpose, we use the negative branch which has the following elementary properties: $W_{-1}(-1/e) = -1$, $W_{-1}(z)$ is decreasing as $z$ increases and $W_{-1}(z) \to -\infty$ as $z \to 0$ (cf. Corless et al. [10] for more details).

**Proposition 2.** *Let $X$ be a random variable having a Muth distribution with parameter $\alpha \in (0, 1]$. The quantile function of $X$, $Q(u; \alpha)$, is*

$$Q(u;\alpha) = \frac{1}{\alpha}\log(1-u) - \frac{1}{\alpha}W_{-1}\left(\frac{u-1}{\alpha e^{1/\alpha}}\right) - \frac{1}{\alpha^2}, \quad 0 < u < 1. \tag{3.2}$$

*Proof.* For any $\alpha \in (0,1]$ and $u \in (0,1)$, we have to solve with respect to $x$ the equation $F(x;\alpha) = u$, $x > 0$, that is,

$$\exp\left(\alpha x - \frac{1}{\alpha}\left(e^{\alpha x} - 1\right)\right) = 1 - u.$$

This equation can be rewritten as follows

$$\alpha x - \frac{1}{\alpha}\, e^{\alpha x} = \log\left(1 - u\right) - \frac{1}{\alpha}.$$

Now, from Jodrá [14, Lemma 1] we have

$$x = \frac{1}{\alpha}\,\log\left(1 - u\right) - \frac{1}{\alpha}\, W\left(\frac{u-1}{\alpha e^{1/\alpha}}\right) - \frac{1}{\alpha^2}. \tag{3.3}$$

Moreover, for any $\alpha \in (0,1]$, $u \in (0,1)$ and $x > 0$, it can be checked that $(u-1)/(\alpha e^{1/\alpha}) \in (-1/e, 0)$, since $\lim_{\alpha \to 0^+} (u-1)/(\alpha e^{1/\alpha}) = 0$, and also that $\log\left(1-u\right) - 1/\alpha - \alpha x < -1$, which imply that the Lambert $W$ function in Eq. (3.3) corresponds to the negative branch $W_{-1}$. The proof is completed. □

*Remark 2.* As the Lambert $W$ function is implemented in computer algebra systems, pseudo-random data from the Muth distribution can be computer-generated in a straightforward manner by virtue of Proposition 2.

Similarly, as a consequence of Proposition 2, we can also generate by computer pseudo-random data from the extreme order statistics of $X$. To be more precise, let $X_1, \ldots, X_n$ be $n$ independent random variables having a Muth distribution with parameter $\alpha$. The extreme order statistics of $X$ are defined by $X_{n:n} := \max\{X_1, \ldots, X_n\}$ and $X_{1:n} := \min\{X_1, \ldots, X_n\}$. It is well-known that the cumulative distribution functions of $X_{n:n}$ and $X_{1:n}$ can be expressed in terms of $F$, namely, $F_{X_{n:n}}(x;\alpha) = F^n(x;\alpha)$ and $F_{X_{1:n}}(x;\alpha) = 1 - (1 - F(x;\alpha))^n$, $x > 0$, where $F$ is given by Eq. (1.2). Let us denote by $Q_{X_{n:n}}$ and $Q_{X_{1:n}}$ the quantile functions of $X_{n:n}$ and $X_{1:n}$, respectively. Then the following result is valid:

(i) $Q_{X_{n:n}}(u;\alpha) = Q(u^{1/n};\alpha)$, $0 < u < 1$,

(ii) $Q_{X_{1:n}}(u;\alpha) = Q(1 - (1-u)^{1/n};\alpha)$, $0 < u < 1$,

where $Q$ is given by Eq. (3.2). Therefore, $X_{n:n}$ and $X_{1:n}$ also have the variate generation property.

To end this section, the following result provides a more compact expression for the quantile function of $X$, which will be used in Section 5.

*Corollary 2.* Let $X$ be a random variable having a Muth distribution with parameter $\alpha \in (0,1]$. The quantile function of $X$ is

$$Q(u;\alpha) = \frac{1}{\alpha}\,\log\left(-\alpha W_{-1}\left(\frac{u-1}{\alpha e^{1/\alpha}}\right)\right), \quad 0 < u < 1. \tag{3.4}$$

*Proof.* It must be noted that multiplying by $(-1)$ both sides of Eq. (3.1) and taking logarithms we have the equivalent expression

$$W(z) = \log(-z) - \log\big(-W(z)\big). \tag{3.5}$$

Then, from Eqs. (3.2) and (3.5), the quantile function of $X$ can be rewritten as follows

$$Q(u;\alpha) = \frac{1}{\alpha}\left\{\log(1-u) - \log\left(\frac{1-u}{\alpha e^{1/\alpha}}\right) + \log\left(-W_{-1}\left(\frac{u-1}{\alpha e^{1/\alpha}}\right)\right)\right\} - \frac{1}{\alpha^2},$$

which leads to Eq. (3.4). □

## 4 The Mode and the Golden Ratio

The mode of a continuous probability distribution is the value at which the probability density function has its maximum. This section shows that the mode of the Muth distribution, denoted as mode$(X)$, can be expressed in closed form in terms of the golden ratio. This is an interesting mathematical property since, as far as we know, the Muth distribution is the only probability distribution whose mode involves the golden ratio.

Recall that the golden ratio $\varphi$ can be defined as the positive solution of the equation $x^2 - x - 1 = 0$, that is, it is the algebraic number

$$\varphi := \frac{1+\sqrt{5}}{2} \approx 1.618033988749.$$

We state the following proposition.

**Proposition 3.** *Let $X$ be a random variable having a Muth distribution with parameter $\alpha \in (0,1]$. Then,*

$$\text{mode}(X) = \begin{cases} 0, & 0 < \alpha \le 1/\varphi^2, \\ \frac{\log(\alpha\varphi^2)}{\alpha}, & 1/\varphi^2 < \alpha \le 1. \end{cases} \tag{4.1}$$

*Proof.* The first derivative of Eq. (1.1) can be written as follows

$$\frac{\partial}{\partial x} f(x;\alpha) = \left(\alpha e^{\alpha x} - \left(e^{\alpha x} - \alpha\right)^2\right) \exp\left(\alpha x - \frac{1}{\alpha}\left(e^{\alpha x} - 1\right)\right).$$

In order to obtain the mode of $X$ we have to solve with respect to $x$ the equation $(\partial/\partial x) f(x;\alpha) = 0$, which is equivalent to solve the equation

$$\alpha e^{\alpha x} - \left(e^{\alpha x} - \alpha\right)^2 = 0. \tag{4.2}$$

Now, by taking into account that $\varphi^2 = 1 + \varphi$, it is easy to see that $x = \log(\alpha\varphi^2)/\alpha$ is the unique positive solution of Eq. (4.2) if $\alpha\varphi^2 > 1$, otherwise, the left-hand side of Eq. (4.2) is less than zero. Moreover, after some calculations it can be checked that

$$\left.\frac{\partial^2}{\partial x^2} f(x;\alpha)\right|_{x=\log(\alpha\varphi^2)/\alpha} = e^{(1-\alpha\varphi^2)/\alpha}\alpha^4\varphi^2\left(\varphi^6 - 6\varphi^4 + 7\varphi^2 - 1\right)$$

and also that $(\varphi^6 - 6\varphi^4 + 7\varphi^2 - 1) < 0$. This implies that the mode of the Muth distribution is located at $x = \log(\alpha\varphi^2)/\alpha$ for $\alpha \in (1/\varphi^2, 1]$, whereas the mode is zero for $\alpha \in (0, 1/\varphi^2]$. The proof is completed. □

## 5 The Mode, Median and Mean Inequality

The closed-form expressions obtained for the mode, the quantile function and the expected value of the Muth distribution can be used to show an inequality commonly known as the mode-median-mean inequality, which in general holds for unimodal right-skewed distributions (cf. Abadir [1] for counterexamples). Denote by median$(X)$ the median of $X$, that is, median$(X) = Q(1/2; \alpha)$ where $Q$ is given in Proposition 2. We state the following proposition.

**Proposition 4.** *Let $X$ be a random variable having a Muth distribution with parameter $\alpha \in (0, 1]$. Then,*

$$\operatorname{mode}(X) < \operatorname{median}(X) < E[X] = 1.$$

*Proof.* Before proceeding with the proof, recall that the derivative of the Lambert $W$ function is (cf. Corless et al. [10])

$$\frac{\partial}{\partial z} W(z) = \frac{W(z)}{z(1 + W(z))}, \quad z \neq 0. \tag{5.1}$$

Additionally, from Eq. (3.4) the median of $X$ is

$$\operatorname{median}(X) = \frac{1}{\alpha} \log\left(-\alpha W_{-1}\left(\frac{-1}{2\alpha e^{1/\alpha}}\right)\right). \tag{5.2}$$

It is interesting to note that the above expression implies that the argument of the logarithm function is strictly greater than one, that is

$$-\alpha W_{-1}\left(\frac{-1}{2\alpha e^{1/\alpha}}\right) > 1 \quad \forall \alpha \in (0, 1],$$

since the median of a strictly positive random variable is a positive number.

Now we are in a position to proceed with the proof. We start by showing that median$(X) < E[X]$. To prove this result, we will show that the argument of the logarithmic function in Eq. (5.2) is a strictly increasing function in $\alpha$. Using Eq. (5.1), for any $\alpha \in (0, 1]$ we get

$$\frac{\partial}{\partial \alpha}\left\{-\alpha W_{-1}\left(\frac{-1}{2\alpha e^{1/\alpha}}\right)\right\} = -\frac{W_{-1}\left(\frac{-1}{2\alpha e^{1/\alpha}}\right)\left(1 + \alpha W_{-1}\left(\frac{-1}{2\alpha e^{1/\alpha}}\right)\right)}{\alpha\left(1 + W_{-1}\left(\frac{-1}{2\alpha e^{1/\alpha}}\right)\right)} > 0,$$

where the last inequality follows from the fact that $\alpha W_{-1}\left(\frac{-1}{2\alpha e^{1/\alpha}}\right) < -1$ for any $\alpha \in (0, 1]$. Therefore, the argument of the logarithmic function in Eq. (5.2) is a strictly increasing function in $\alpha$ and, as a consequence, the median of $X$ is a strictly increasing function in $\alpha$. In particular, this implies that median$(X) < E[X] = 1$ for any $\alpha \in (0, 1]$ since $Q(1/2; 1) \approx 0.985199809$.

Next, we see that $\mathrm{mode}(X) < \mathrm{median}(X)$. For $\alpha \in (0, 1/\varphi^2]$, the inequality holds since the mode is zero whereas the median of $X$ is a strictly increasing function in $\alpha$ and we also know that $\lim_{\alpha \to 0^+} Q(1/2; \alpha) = \log(2)$, the latter because the limit distribution of $X$ as $\alpha$ decreases to zero is the standard exponential distribution. On the other hand, for $\alpha \in (1/\varphi^2, 1]$, from Eqs. (4.1) and (5.2) we have

$$\mathrm{mode}(X) - \mathrm{median}(X) = \frac{1}{\alpha}\left\{2\log(\varphi) - \log\left(-W_{-1}\left(\frac{-1}{2\alpha e^{1/\alpha}}\right)\right)\right\}. \tag{5.3}$$

Below, we see that the argument of the logarithm function on the right-hand side of Eq. (5.3) is a strictly decreasing function in $\alpha$. Using Eq. (5.1), for any $\alpha \in (0, 1)$ we obtain

$$\frac{\partial}{\partial \alpha}\left\{-W_{-1}\left(\frac{-1}{2\alpha e^{1/\alpha}}\right)\right\} = \frac{(\alpha - 1)W_{-1}\left(\frac{-1}{2\alpha e^{1/\alpha}}\right)}{\alpha^2\left(1 + W_{-1}\left(\frac{-1}{2\alpha e^{1/\alpha}}\right)\right)} < 0,$$

where the last inequality holds since $W_{-1}\left(\frac{-1}{2\alpha e^{1/\alpha}}\right) < -1$. As a consequence, we have

$$2\log(\varphi) < \log\left(-W_{-1}\left(\frac{-1}{2\alpha e^{1/\alpha}}\right)\right), \quad \alpha \in \left(1/\varphi^2, 1\right), \tag{5.4}$$

because $2\log(\varphi) \approx 0.962423$ whereas the logarithmic function on the right-hand side of Eq. (5.4) takes values in the interval $(0.985199, 1.292264)$ for any $\alpha \in (1/\varphi^2, 1)$. Thus, from Eqs. (5.3) and (5.4) we have $\mathrm{mode}(X) < \mathrm{median}(X)$ for $\alpha \in (1/\varphi^2, 1)$; this inequality can be directly checked in the particular case $\alpha = 1$. This completes the proof. □

## 6 Moments of Order Statistics

This section provides analytical expressions for the moments of the order statistics of the Muth distribution. For reasons that will be given later, we pay special attention to the minimum order statistic.

First, we introduce some notation. Let $X_1, \ldots, X_n$ be $n$ independent random variables having a Muth distribution with parameter $\alpha \in (0, 1]$, that is, a random sample of size $n$ from $X$. Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the order statistics obtained by arranging $X_i$, $i = 1, \ldots, n$, in non-decreasing order of magnitude. For any $n = 1, 2, \ldots$ and $k = 1, 2, \ldots$, it is known that the $k$th moment of $X_{r:n}$, $r = 1, \ldots, n$, can be computed using the following formula (cf. Balakrishnan and Rao [5, p. 7])

$$E\left[X_{r:n}^k\right] = r\binom{n}{r}\int_0^\infty x^k \left(F(x;\alpha)\right)^{r-1}\left(1 - F(x;\alpha)\right)^{n-r} dF(x;\alpha). \tag{6.1}$$

The next result gives an analytical expression for the moments of the minimum order statistic $E[X_{1:n}^k]$, which can be used to compute $E[X_{r:n}^k]$ for $r = 2, \ldots, n$ and $k = 1, 2, \ldots$, as we shall see at the end of this section.

**Theorem 2.** *Let $X_1, \dots, X_n$ be $n$ independent random variables having a Muth distribution with parameter $\alpha \in (0,1]$. The moments of the minimum order statistic $X_{1:n}$ are given by*

$$E\big[X_{1:n}^k\big] = \frac{\Gamma(k+1)e^{n/\alpha}}{\alpha^k}\, E_{-(n-1)}^{k-1}(n/\alpha), \quad k = 1, 2, \dots.$$

*Proof.* From Eqs. (6.1) and (1.2), for $k = 1, 2, \dots$ we have

$$\begin{aligned} E\big[X_{1:n}^k\big] &= \int_0^\infty x^k \big(1 - F(x;\alpha)\big)^{n-1} dF(x;\alpha) \\ &= n\int_0^\infty x^k \big(e^{\alpha x} - \alpha\big) \exp\bigg(n\Big(\alpha x - \frac{1}{\alpha}\big(e^{\alpha x} - 1\big)\Big)\bigg)\, dx. \end{aligned}$$

Now, by making the change of variable $u = e^{\alpha x}$ we get

$$\begin{aligned} E\big[X_{1:n}^k\big] &= \frac{n e^{n/\alpha}}{\alpha^k}\left\{\frac{1}{\alpha}\int_1^\infty (\log u)^k u^n e^{-nu/\alpha} du - \int_1^\infty (\log u)^k u^{n-1} e^{-nu/\alpha} du\right\} \\ &= \frac{n e^{n/\alpha}}{\alpha^k}\Gamma(k+1)\left\{\frac{1}{\alpha}\, E_{-n}^k(n/\alpha) - E_{-(n-1)}^k(n/\alpha)\right\}, \end{aligned}$$

where in the last equality we have used Eq. (2.6). Finally, by taking into account in the above equation the following recurrence formula (cf. Milgram [17])

$$(1-s)E_s^m(z) = zE_{s-1}^m(z) - E_s^{m-1}(z), \quad z > 0,\ s \neq 1,\ m = 0, 1, \dots,$$

where it is assumed $E_s^{-1}(z) := e^{-z}$, we obtain the desired result. □

As a consequence of Theorem 2, in the next result we see that the expected value of the minimum order statistic $X_{1:n}$ can be easily computed as a finite sum.

*Corollary 3.* Let $X_1, \dots, X_n$ be $n$ independent random variables having a Muth distribution with parameter $\alpha \in (0,1]$. The expected value of the minimum order statistic $X_{1:n}$ is

$$E[X_{1:n}] = \frac{\alpha^{n-1}\Gamma(n)}{n^n}\sum_{i=0}^{n-1}\frac{n^i}{i!\alpha^i}, \quad n = 1, 2, \dots.$$

*Proof.* From Theorem 2, we get

$$E[X_{1:n}] = \frac{e^{n/\alpha}}{\alpha}\, E_{-(n-1)}^0(n/\alpha) = \frac{\alpha^{n-1}e^{n/\alpha}}{n^n}\,\Gamma\Big(n, \frac{n}{\alpha}\Big),$$

where in the last equality we use the fact that $E_s^0(z) = E_s(z)$ together with Eq. (2.2). The result follows directly by taking into account that $\Gamma(n,z) = \Gamma(n-1)e^{-z}\sum_{i=0}^{n-1}(z^i/i!)$ for $n = 1, 2, \dots$ (cf. Olver et al. [19, p. 177]). □

*Remark 3.* We highlight the importance of Corollary 3 since the expected value of the minimum order statistic can be used to determine if two distributions with finite expected values are identical (cf. Chan [9]), that is, to characterize probability distributions.

*Remark 4.* The expression of $E[X_{1:n}^k]$ in Theorem 2 can be used to compute $E[X_{r:n}^k]$, for $r = 2, \ldots, n$ and $k = 1, 2, \ldots$, avoiding the use of Eq. (6.1). To do so, we can employ the following well-known formula (see, for example, Balakrishnan and Rao [5, p. 156] and David and Nagaraja [12, Chapter 3])

$$E\big[X_{r:n}^k\big] = \sum_{j=n-r+1}^{n} (-1)^{j-(n-r+1)} \binom{n}{j} \binom{j-1}{n-r} E\big[X_{1:j}^k\big], \quad r = 2, \ldots, n.$$

## 7 Parameter Estimation

In this section, we describe the following methods to estimate the parameter $\alpha$: maximum likelihood (ML), least squares (ULS), weighted least squares (WLS) and moments (MM), which are presented in Subsections 7.1, 7.2, 7.3 and 7.4, respectively. As it will be seen, these estimators cannot be obtained in closed form so their performance must be assessed via a Monte Carlo simulation study, which is given in Subsection 7.5.

### 7.1 Maximum likelihood estimate

Let $X_1, \ldots, X_n$ be a random sample of size $n$ from a Muth distribution with unknown parameter $\alpha$. Let us denote by $x_1$, $x_2$, $\ldots$, $x_n$ the observed values. From the likelihood function, $L(\alpha) := \prod_{i=1}^{n} f(x_i; \alpha)$, the log-likelihood function can be written as follows

$$\log L(\alpha) = \sum_{i=1}^{n} \log\big(e^{\alpha x_i} - \alpha\big) - \frac{1}{\alpha} \sum_{i=1}^{n} \big(e^{\alpha x_i} - 1\big) + \alpha \sum_{i=1}^{n} x_i. \tag{7.1}$$

The ML estimate of $\alpha$ is the value, say $\hat{\alpha}$, that maximizes Eq. (7.1). Then, to get $\hat{\alpha}$ we must numerically solve

$$\frac{\partial}{\partial \alpha} \log L(\alpha) = \sum_{i=1}^{n} \frac{x_i e^{\alpha x_i} - 1}{e^{\alpha x_i} - \alpha} + \frac{1}{\alpha^2} \sum_{i=1}^{n} e^{\alpha x_i} - \frac{1}{\alpha} \sum_{i=1}^{n} x_i e^{\alpha x_i} + \sum_{i=1}^{n} x_i - \frac{n}{\alpha^2} = 0,$$

and check that the solution $\hat{\alpha}$ satisfies $(\partial^2 \log L(\alpha)/\partial \alpha^2)|_{\alpha=\hat{\alpha}} < 0$.

### 7.2 Least squares estimate

Let $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ be the order statistics of a random sample of size $n$ from $X$ and denote by $x_{(1)} < x_{(2)} < \ldots < x_{(n)}$ the ordered data. As an estimator of $F(x_{(i)})$, we consider the empirical distribution function given by

$$F_n(x_{(i)}; d) := \frac{i - d}{n - 2d + 1}, \quad i = 1, \ldots, n \tag{7.2}$$

for some real number $d$, $0 \le d < 1$. In particular, in Subsection 7.5 we shall use the commonly used values $d = 0, 0.3, 0.375, 0.5$. For a justification of choosing Eq. (7.2) and these values of $d$ we refer the reader to Barnett [6] and D'Agostino and Stephens [11, Chapter 2].

In order to obtain the ULS estimate of $\alpha$, say $\hat{\alpha}$, we apply the approach proposed by Bain [4], so the parameter $\alpha$ is estimated by minimizing

$$UL_d(\alpha) := \sum_{i=1}^{n} \big(\log\big(1 - F_n(x_{(i)}; d)\big) - \log\big(1 - F(x_{(i)}; \alpha)\big)\big)^2. \tag{7.3}$$

Hence, to get $\hat{\alpha}$ we must numerically solve

$$\frac{\partial}{\partial\alpha} UL_d(\alpha) = \sum_{i=1}^{n} \left\{ \left( \log\left(1 - \frac{i-d}{n-2d+1}\right) + \frac{1}{\alpha}\left(e^{\alpha x_{(i)}} - 1\right) - \alpha x_{(i)} \right) \times \left( \frac{1}{\alpha} x_{(i)} e^{\alpha x_{(i)}} - \frac{1}{\alpha^2}\left(e^{\alpha x_{(i)}} - 1\right) - x_{(i)} \right) \right\} = 0,$$

and check that the solution $\hat{\alpha}$ satisfies $(\partial^2 UL_d(\alpha)/\partial\alpha^2)|_{\alpha=\hat{\alpha}} > 0$.

## 7.3 Weighted least squares estimate

Next, we estimate $\alpha$ by WLS, using a weight for each term in the sum in Eq. (7.3). Following Bickel and Doksum [8, pp. 316–317]), we shall consider $w_{i,d} := (1 - F_n(x_{(i)}; d))^2$, $i = 1, \ldots, n$. Therefore, the WLS estimate of $\alpha$, say $\hat{\alpha}$, is obtained by minimizing

$$WL_d(\alpha) := \sum_{i=1}^{n} w_{i,d}\big(\log\big(1 - F_n(x_{(i)}; d)\big) - \log\big(1 - F(x_{(i)}; \alpha)\big)\big)^2.$$

Then, to get $\hat{\alpha}$ we must numerically solve

$$\frac{\partial}{\partial\alpha} WL_d(\alpha) = \sum_{i=1}^{n} \left\{ w_{i,d} \left( \log\left(1 - \frac{i-d}{n-2d+1}\right) + \frac{1}{\alpha}\left(e^{\alpha x_{(i)}} - 1\right) - \alpha x_{(i)} \right) \times \left( \frac{1}{\alpha} x_{(i)} e^{\alpha x_{(i)}} - \frac{1}{\alpha^2}\left(e^{\alpha x_{(i)}} - 1\right) - x_{(i)} \right) \right\} = 0,$$

and check that the solution $\hat{\alpha}$ satisfies $(\partial^2 WL_d(\alpha)/\partial\alpha^2)\big|_{\alpha=\hat{\alpha}} > 0$.

## 7.4 Method of moments

Let $\overline{x}_2$ be the second sample moment, $\overline{x}_2 := (1/n)\sum_{i=1}^{n} x_i^2$, where $x_1$, $x_2$, $\ldots$, $x_n$ are the observed values. By virtue of Corollary 1 (ii), the method of moments estimate of $\alpha$ is obtained by numerically solving in $\alpha$ the equation

$$\frac{2\,e^{1/\alpha}}{\alpha}\,\Gamma(0, 1/\alpha) = \overline{x}_2.$$

### 7.5 Simulation study

A simulation study was carried out to compare the estimation methods ML, ULS, WLS and MM. For this purpose, we generated $N = 100$ random samples of different sizes $n$ for selected values of $\alpha$. Pseudo-random data from the Muth distribution were computer-generated by means of Eq. (3.2). For each estimation method the following quantities were calculated:

(i) The mean of the simulated estimates $\hat{\alpha}_j$, $j = 1, \ldots, N$, that is,

$$\bar{\alpha} := (1/N) \sum_{j=1}^{N} \hat{\alpha}_j.$$

(ii) The bias of the simulated estimates

$$\text{Bias}(\hat{\alpha}) := (1/N) \sum_{j=1}^{N} (\hat{\alpha}_j - \alpha).$$

(iii) The mean-square error of the simulated estimates

$$\text{MSE}(\hat{\alpha}) := (1/N) \sum_{j=1}^{N} (\hat{\alpha}_j - \alpha)^2.$$

The estimation methods described in Subsections 7.1–7.4 were implemented in Matlab R2014a. In particular, ULS and WLS were performed using the values $d = 0, 0.3, 0.375, 0.5$. The equations involved were solved numerically using the Matlab functions `solve` and `vpasolve`; more precisely, the value 0.5 was used as the starting point of `vpasolve` since $\alpha \in (0, 1]$. All the computations were performed on an Intel Core i7-4700MQ CPU at 2.40GHz with 16GB RAM.

From the simulation study, we observed that, in general, ML provided better estimates of $\alpha$, with less bias and mean-square error, than those obtained by ULS, WLS and MM. For the sake of saving space, here we present only the numerical results obtained using ML and ULS in the particular case $d = 0$, which are given in Tables 2 and 3, respectively. The latter table is included because the simulation study suggested that ULS with $d = 0$ can produce estimates of $\alpha$ with less bias and mean-square error than those obtained using ML in the particular cases of small values of $\alpha$ and small sample sizes (see the results for $\alpha = 0.10$ in Table 3). In addition, we also observed that ULS and WLS produced quite similar results, whereas MM yielded poor estimates. Overall, from the simulation results, we conclude that ML provides better estimates of $\alpha$ than the other methods.

## 8 A Scaling Transformation of the Muth Distribution

We remark that the Muth distribution is normalized in [18] to have an expected value of 1. This fact is a rather strong restriction if we want to use the model

**Table 2.** ML estimates.

| | $\alpha = 0.10$ | | | $\alpha = 0.20$ | | | $\alpha = 0.30$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\bar{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) | $\bar{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) | $\bar{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) |
| $n = 25$ | 0.219418 | 0.119418 | 0.032232 | 0.262985 | 0.062985 | 0.029696 | 0.324885 | 0.024885 | 0.026077 |
| $n = 50$ | 0.160877 | 0.060877 | 0.015763 | 0.227187 | 0.027187 | 0.014470 | 0.308306 | 0.008306 | 0.012125 |
| $n = 75$ | 0.144202 | 0.044202 | 0.009905 | 0.212892 | 0.012892 | 0.009650 | 0.310618 | 0.010618 | 0.013558 |
| $n = 100$ | 0.131696 | 0.031696 | 0.009668 | 0.224059 | 0.024059 | 0.010286 | 0.303534 | 0.003534 | 0.007449 |
| $n = 150$ | 0.125450 | 0.025450 | 0.005604 | 0.207290 | 0.007290 | 0.004966 | 0.308581 | 0.008581 | 0.004489 |
| $n = 250$ | 0.111412 | 0.011412 | 0.003152 | 0.213609 | 0.013609 | 0.002399 | 0.299468 | -0.000531 | 0.002844 |
| $n = 500$ | 0.105561 | 0.005561 | 0.002023 | 0.206993 | 0.006993 | 0.001270 | 0.304072 | 0.004072 | 0.001408 |
| $n = 1000$ | 0.101412 | 0.001412 | 0.001064 | 0.197219 | -0.002780 | 0.000614 | 0.302025 | 0.002025 | 0.000742 |

| | $\alpha = 0.40$ | | | $\alpha = 0.50$ | | | $\alpha = 0.60$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\bar{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) | $\bar{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) | $\bar{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) |
| $n = 25$ | 0.393594 | -0.006405 | 0.025965 | 0.512817 | 0.012817 | 0.029792 | 0.594649 | -0.005350 | 0.017332 |
| $n = 50$ | 0.442308 | 0.042308 | 0.016519 | 0.511660 | 0.011660 | 0.014444 | 0.635556 | 0.035556 | 0.010812 |
| $n = 75$ | 0.397479 | -0.002520 | 0.009973 | 0.504789 | 0.004789 | 0.008033 | 0.599469 | -0.000530 | 0.008323 |
| $n = 100$ | 0.402401 | 0.002401 | 0.005069 | 0.507494 | 0.007494 | 0.006870 | 0.607463 | 0.007463 | 0.006747 |
| $n = 150$ | 0.412889 | 0.012889 | 0.004289 | 0.497006 | -0.002993 | 0.003922 | 0.606248 | 0.006248 | 0.003886 |
| $n = 250$ | 0.401093 | 0.001093 | 0.002631 | 0.507523 | 0.007523 | 0.003210 | 0.599820 | -0.000179 | 0.002188 |
| $n = 500$ | 0.401904 | 0.001904 | 0.001430 | 0.499173 | -0.000826 | 0.001369 | 0.596405 | -0.003594 | 0.001321 |
| $n = 1000$ | 0.399123 | -0.000876 | 0.000761 | 0.501869 | 0.001869 | 0.000661 | 0.604364 | 0.004364 | 0.000586 |

| | $\alpha = 0.70$ | | | $\alpha = 0.80$ | | | $\alpha = 0.90$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\bar{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) | $\bar{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) | $\bar{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) |
| $n = 25$ | 0.679706 | -0.020293 | 0.019868 | 0.759433 | -0.040566 | 0.018329 | 0.838109 | -0.061890 | 0.013060 |
| $n = 50$ | 0.682240 | -0.017759 | 0.008705 | 0.780940 | -0.019059 | 0.007914 | 0.881449 | -0.018550 | 0.004974 |
| $n = 75$ | 0.688793 | -0.011206 | 0.006290 | 0.805659 | 0.005659 | 0.004915 | 0.866872 | -0.033127 | 0.005921 |
| $n = 100$ | 0.689708 | -0.010291 | 0.005491 | 0.794081 | -0.005918 | 0.005488 | 0.891536 | -0.008463 | 0.003351 |
| $n = 150$ | 0.700634 | 0.000634 | 0.003293 | 0.793459 | -0.006540 | 0.003453 | 0.897719 | -0.002280 | 0.002491 |
| $n = 250$ | 0.708097 | 0.008097 | 0.002177 | 0.801045 | 0.001045 | 0.001816 | 0.898684 | -0.001315 | 0.001556 |
| $n = 500$ | 0.701857 | 0.001857 | 0.000864 | 0.793970 | -0.006029 | 0.000953 | 0.894444 | -0.005555 | 0.000723 |
| $n = 1000$ | 0.697103 | -0.002896 | 0.000563 | 0.796297 | -0.003702 | 0.000485 | 0.898980 | -0.001019 | 0.000473 |

**Table 3.** ULS estimates.

| | $\alpha = 0.10$ | | | $\alpha = 0.20$ | | | $\alpha = 0.30$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\bar{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) | $\bar{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) | $\bar{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) |
| $n = 25$ | 0.174095 | 0.074095 | 0.028127 | 0.212987 | 0.012987 | 0.030726 | 0.220504 | -0.079495 | 0.032534 |
| $n = 50$ | 0.128237 | 0.028237 | 0.011718 | 0.158049 | -0.041950 | 0.014260 | 0.225794 | -0.074205 | 0.024967 |
| $n = 75$ | 0.127940 | 0.027940 | 0.010146 | 0.168815 | -0.031184 | 0.012574 | 0.252180 | -0.047819 | 0.016563 |
| $n = 100$ | 0.112016 | 0.012016 | 0.005783 | 0.150410 | -0.049589 | 0.008663 | 0.237681 | -0.062318 | 0.015039 |
| $n = 150$ | 0.096551 | -0.003448 | 0.004663 | 0.146608 | -0.053391 | 0.008955 | 0.248848 | -0.051151 | 0.012128 |
| $n = 250$ | 0.096939 | -0.003060 | 0.004033 | 0.161360 | -0.038639 | 0.006409 | 0.275772 | -0.024227 | 0.004923 |
| $n = 500$ | 0.084962 | -0.015037 | 0.003167 | 0.181867 | -0.018132 | 0.002594 | 0.291162 | -0.008837 | 0.002389 |
| $n = 1000$ | 0.088875 | -0.011124 | 0.001569 | 0.183846 | -0.016153 | 0.001355 | 0.286931 | -0.013068 | 0.001176 |

| | $\alpha = 0.40$ | | | $\alpha = 0.50$ | | | $\alpha = 0.60$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\bar{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) | $\bar{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) | $\bar{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) |
| $n = 25$ | 0.269170 | -0.130829 | 0.052210 | 0.393805 | -0.106194 | 0.047733 | 0.464956 | -0.135043 | 0.055357 |
| $n = 50$ | 0.301432 | -0.098567 | 0030100 | 0.433146 | -0.066853 | 0.025520 | 0.494654 | -0.105345 | 0.028829 |
| $n = 75$ | 0.330180 | -0.069819 | 0.017874 | 0.438766 | -0.061233 | 0.015044 | 0.532039 | -0.067960 | 0.017121 |
| $n = 100$ | 0.348750 | -0.051249 | 0.015091 | 0.468291 | -0.031708 | 0.011865 | 0.546227 | -0.053772 | 0.013628 |
| $n = 150$ | 0.336158 | -0.063841 | 0.010401 | 0.452330 | -0.047669 | 0.009580 | 0.553814 | -0.046185 | 0.008865 |
| $n = 250$ | 0.362530 | -0.037469 | 0.005741 | 0.461115 | -0.038884 | 0.004553 | 0.566173 | -0.033826 | 0.005419 |
| $n = 500$ | 0.383750 | -0.016249 | 0.002596 | 0.484314 | -0.015685 | 0.002216 | 0.576416 | -0.023583 | 0.002761 |
| $n = 1000$ | 0.386173 | -0.013826 | 0.001137 | 0.485701 | -0.014298 | 0.001262 | 0.593185 | -0.006814 | 0.001385 |

| | $\alpha = 0.70$ | | | $\alpha = 0.80$ | | | $\alpha = 0.90$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\bar{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) | $\bar{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) | $\bar{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) |
| $n = 25$ | 0.561582 | -0.138417 | 0.056624 | 0.653860 | -0.146139 | 0.051370 | 0.720243 | -0.179756 | 0.053009 |
| $n = 50$ | 0.606705 | -0.093294 | 0.024722 | 0.703384 | -0.096615 | 0.027535 | 0.800719 | -0.099280 | 0.021167 |
| $n = 75$ | 0.630258 | -0.069741 | 0.016247 | 0.747176 | -0.052823 | 0.012286 | 0.819144 | -0.080855 | 0.016402 |
| $n = 100$ | 0.668332 | -0.031667 | 0.010667 | 0.745189 | -0.054810 | 0.013088 | 0.815587 | -0.084412 | 0.014422 |
| $n = 150$ | 0.667490 | -0.032509 | 0.008337 | 0.757325 | -0.042674 | 0.008475 | 0.857242 | -0.042757 | 0.006831 |
| $n = 250$ | 0.674489 | -0.025510 | 0.005357 | 0.773928 | -0.026071 | 0.004435 | 0.868235 | -0.031764 | 0.004111 |
| $n = 500$ | 0.688233 | -0.011766 | 0.002378 | 0.789655 | -0.010344 | 0.002108 | 0.877969 | -0.022030 | 0.002280 |
| $n = 1000$ | 0.685223 | -0.014776 | 0.001378 | 0.791106 | -0.008893 | 0.001182 | 0.894334 | -0.005665 | 0.000960 |

with real data. To overcome this limitation, we consider as a natural extension the scaled Muth distribution defined by $Y = \beta X$, with parameters $\alpha \in (0,1]$ and $\beta > 0$, which is a more flexible model with expected value $E[Y] = \beta > 0$. In this case, the cumulative distribution function of $Y$, denoted by $F_Y(y;\alpha,\beta)$, is the following

$$F_Y(y;\alpha,\beta) = F(y/\beta;\alpha) = 1 - \exp\left(\frac{\alpha}{\beta}\,y - \frac{1}{\alpha}\left(e^{\alpha y/\beta} - 1\right)\right), \quad y > 0, \tag{8.1}$$

where $F(\cdot\,;\alpha)$ is given by Eq. (1.2). From Eq. (8.1), clearly we see that $Y$ does not have the scaling property, that is, $Y$ does not come from the same distribution family defined by Eq. (1.2). Fortunately, as $Y$ is obtained by a scaling transformation of $X$, the new probability distribution inherits some properties from $X$, as we summarize below. We use notations similar to those used in the previous sections.

**Table 4.** The scaled Muth distribution: ML estimates.

| | $\alpha = 0.25,\ \beta = 5.0$ | | | | | | $\alpha = 0.25,\ \beta = 15.0$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\bar{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) | $\bar{\beta}$ | Bias($\hat{\beta}$) | MSE($\hat{\beta}$) | $\bar{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) | $\bar{\beta}$ | Bias($\hat{\beta}$) | MSE($\hat{\beta}$) |
| $n = 50$ | 0.28330 | 0.03330 | 0.01398 | 5.12731 | 0.12731 | 0.28259 | 0.29431 | 0.04431 | 0.01807 | 15.09489 | 0.09489 | 3.40721 |
| $n = 75$ | 0.25422 | 0.00422 | 0.00969 | 4.91343 | -0.08656 | 0.18581 | 0.26228 | 0.01228 | 0.00957 | 15.07469 | 0.07469 | 2.14589 |
| $n = 100$ | 0.25444 | 0.00444 | 0.00820 | 5.02966 | 0.02966 | 0.15416 | 0.26872 | 0.01872 | 0.00985 | 15.11893 | 0.11893 | 1.50599 |
| $n = 200$ | 0.26060 | 0.01060 | 0.00433 | 4.99879 | -0.00120 | 0.08426 | 0.26899 | 0.01899 | 0.00472 | 15.07645 | 0.07645 | 0.61633 |
| | $\alpha = 0.25,\ \beta = 30.0$ | | | | | | $\alpha = 0.25,\ \beta = 50.0$ | | | | | |
| | $\bar{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) | $\bar{\beta}$ | Bias($\hat{\beta}$) | MSE($\hat{\beta}$) | $\bar{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) | $\bar{\beta}$ | Bias($\hat{\beta}$) | MSE($\hat{\beta}$) |
| $n = 50$ | 0.28061 | 0.03061 | 0.01824 | 29.74512 | -0.25487 | 10.73001 | 0.26800 | 0.01800 | 0.01810 | 49.19190 | -0.80809 | 40.55703 |
| $n = 75$ | 0.27035 | 0.02035 | 0.01011 | 29.77874 | -0.22125 | 7.83400 | 0.27294 | 0.02294 | 0.01064 | 50.55815 | 0.55815 | 24.07844 |
| $n = 100$ | 0.27484 | 0.02484 | 0.00927 | 30.25639 | 0.25639 | 6.64147 | 0.27165 | 0.02165 | 0.00913 | 49.42345 | -0.57654 | 17.86408 |
| $n = 200$ | 0.26739 | 0.01739 | 0.00374 | 30.00175 | 0.00175 | 2.35839 | 0.25655 | 0.00655 | 0.00445 | 49.68650 | -0.31349 | 8.57021 |
| | $\alpha = 0.50,\ \beta = 5.0$ | | | | | | $\alpha = 0.50,\ \beta = 15.0$ | | | | | |
| | $\bar{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) | $\bar{\beta}$ | Bias($\hat{\beta}$) | MSE($\hat{\beta}$) | $\bar{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) | $\bar{\beta}$ | Bias($\hat{\beta}$) | MSE($\hat{\beta}$) |
| $n = 50$ | 0.53185 | 0.03185 | 0.01943 | 5.14962 | 0.14962 | 0.17187 | 0.52449 | 0.02449 | 0.01997 | 15.10753 | 0.10753 | 1.65353 |
| $n = 75$ | 0.52068 | 0.02068 | 0.01393 | 5.10329 | 0.10329 | 0.12565 | 0.52215 | 0.02215 | 0.01286 | 14.93310 | -0.06689 | 1.60916 |
| $n = 100$ | 0.52824 | 0.02824 | 0.01002 | 5.14817 | 0.14817 | 0.10565 | 0.52490 | 0.02490 | 0.00917 | 15.18816 | 0.18816 | 1.13546 |
| $n = 200$ | 0.50963 | 0.00963 | 0.00427 | 5.07146 | 0.07146 | 0.04963 | 0.50869 | 0.00869 | 0.00517 | 15.01271 | 0.01271 | 0.56019 |
| | $\alpha = 0.50,\ \beta = 30.0$ | | | | | | $\alpha = 0.50,\ \beta = 50.0$ | | | | | |
| | $\bar{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) | $\bar{\beta}$ | Bias($\hat{\beta}$) | MSE($\hat{\beta}$) | $\bar{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) | $\bar{\beta}$ | Bias($\hat{\beta}$) | MSE($\hat{\beta}$) |
| $n = 50$ | 0.54017 | 0.04017 | 0.01723 | 30.23429 | 0.23429 | 8.47979 | 0.52818 | 0.02818 | 0.01590 | 49.62395 | -0.37604 | 22.73702 |
| $n = 75$ | 0.49757 | -0.00242 | 0.01090 | 29.90316 | -0.09683 | 5.02064 | 0.51290 | 0.01290 | 0.01067 | 49.65691 | -0.34308 | 14.04159 |
| $n = 100$ | 0.50270 | 0.00270 | 0.00817 | 30.07504 | 0.07504 | 4.31725 | 0.50107 | 0.00107 | 0.00661 | 49.61829 | -0.38170 | 10.81860 |
| $n = 200$ | 0.51140 | 0.01140 | 0.00399 | 30.07122 | 0.07122 | 2.50826 | 0.50500 | 0.00500 | 0.00366 | 49.63920 | -0.36079 | 5.79705 |
| | $\alpha = 0.75,\ \beta = 5.0$ | | | | | | $\alpha = 0.75,\ \beta = 15.0$ | | | | | |
| | $\bar{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) | $\bar{\beta}$ | Bias($\hat{\beta}$) | MSE($\hat{\beta}$) | $\bar{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) | $\bar{\beta}$ | Bias($\hat{\beta}$) | MSE($\hat{\beta}$) |
| $n = 50$ | 0.77261 | 0.02261 | 0.01108 | 5.14181 | 0.14181 | 0.10502 | 0.76265 | 0.01265 | 0.01080 | 14.98277 | -0.01722 | 1.30089 |
| $n = 75$ | 0.79058 | 0.04058 | 0.00758 | 5.08393 | 0.08393 | 0.05860 | 0.75276 | 0.00276 | 0.00883 | 14.82471 | -0.17528 | 0.75398 |
| $n = 100$ | 0.77772 | 0.02772 | 0.00569 | 5.12895 | 0.12895 | 0.05536 | 0.76035 | 0.01035 | 0.00677 | 14.85640 | -0.14359 | 0.63830 |
| $n = 200$ | 0.77060 | 0.02060 | 0.00299 | 5.05898 | 0.05898 | 0.03181 | 0.75300 | 0.00300 | 0.00331 | 14.97875 | -0.02124 | 0.37163 |
| | $\alpha = 0.75,\ \beta = 30.0$ | | | | | | $\alpha = 0.75,\ \beta = 50.0$ | | | | | |
| | $\bar{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) | $\bar{\beta}$ | Bias($\hat{\beta}$) | MSE($\hat{\beta}$) | $\bar{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) | $\bar{\beta}$ | Bias($\hat{\beta}$) | MSE($\hat{\beta}$) |
| $n = 50$ | 0.75275 | 0.00275 | 0.01435 | 29.72675 | -0.27324 | 4.63122 | 0.75440 | 0.00440 | 0.01147 | 49.56909 | -0.43090 | 12.88186 |
| $n = 75$ | 0.75409 | 0.00409 | 0.01229 | 29.90485 | -0.09514 | 4.10721 | 0.75472 | 0.00472 | 0.00983 | 50.20775 | 0.20775 | 12.23846 |
| $n = 100$ | 0.76961 | 0.01961 | 0.00621 | 30.15211 | 0.15211 | 2.87633 | 0.76778 | 0.01778 | 0.00620 | 50.40140 | 0.40140 | 8.06633 |
| $n = 200$ | 0.75002 | 0.00002 | 0.00423 | 29.92855 | -0.07144 | 1.40879 | 0.74707 | -0.00292 | 0.00378 | 49.79454 | -0.20545 | 4.16414 |

**Proposition 5.** *Let $X$ be a random variable having a Muth distribution with parameter $\alpha \in (0,1]$. Let $Y = \beta X$ with $\beta > 0$. Then, we have the following.*

(i) *The moment generating function of* $Y$ *is* $M_Y(t;\alpha,\beta) = M(\beta t;\alpha)$, $t \in \mathbb{R}$.

(ii) *The moments of* $Y$ *are given by* $E[Y^k] = \beta^k E[X^k]$, $k = 1, 2, \dots$.

(iii) *The quantile function of* $Y$ *is* $Q_Y(u;\alpha,\beta) = \beta\, Q(u;\alpha)$, $0 < u < 1$.

(iv) *The mode of* $Y$ *is:* $\text{mode}(Y) = \beta\text{mode}(X)$ *for* $\alpha \in (1/\varphi^2, 1]$*; and* $\text{mode}(Y) = 0$ *for* $\alpha \in (0, 1/\varphi^2]$.

(v) *Let* $Y_1, \dots, Y_n$ *be* $n$ *independent random variables with cumulative distribution function given by Eq.* (8.1)*. Then, the moments of the minimum order statistic* $Y_{1:n}$ *are given by* $E[Y_{1:n}^k] = \beta^k E[X_{1:n}^k]$, $k = 1, 2, \dots$.

As in Section 7, the parameters $\alpha$ and $\beta$ of the scaled Muth distribution can be estimated by ML. Denoting by $\log L(\alpha,\beta)$ the log-likelihood function based on a random sample from $Y$, the ML estimates of $\alpha$ and $\beta$ are the values that maximize $\log L(\alpha,\beta)$. We solved the system of equations $\partial \log L(\alpha,\beta)/\partial\alpha = 0$ and $\partial \log L(\alpha,\beta)/\partial\beta = 0$, obtaining $(\hat{\alpha}, \hat{\beta})$, and we checked that this pair corresponds to a global maximum. The details are omitted here. Table 4 displays the results of a Monte Carlo simulation study, where we generated $N = 100$ random samples of different sizes $n$ for selected values of $\alpha$ and $\beta$. As it can be seen, ML provides acceptable estimates of the parameters.

## 9 A Real Data Application

In this section, we use a real data set to illustrate that the scaled Muth distribution can be a more appropriate model than other traditional distributions, such as the exponential, the gamma, the lognormal and the Weibull distributions.

The data set was taken from the website of the Bureau of Meteorology of the Australian Government (www.bom.gov.au). It contains the monthly total rainfall (in mm) collected from January of 2000 to February of 2007 in the rain gauge station of Carrol, located in the State of New South Wales on the east coast of Australia. Table 5 displays the data.

**Table 5.** Carrol data set ($n = 83$).

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 12.0 | 22.7 | 75.5 | 28.6 | 65.8 | 39.4 | 33.1 | 84.0 | 41.6 | 62.3 | 52.5 | 13.9 | 15.4 | 31.9 |
| 32.5 | 37.7 | 9.5 | 49.9 | 31.8 | 32.2 | 50.2 | 55.8 | 20.4 | 5.9 | 10.1 | 44.5 | 19.7 | 6.4 |
| 29.2 | 42.5 | 19.4 | 23.8 | 55.2 | 7.7 | 0.8 | 6.7 | 4.8 | 73.8 | 5.1 | 7.6 | 25.7 | 50.7 |
| 59.7 | 57.2 | 29.7 | 32.0 | 24.5 | 71.6 | 15.0 | 17.7 | 8.2 | 23.8 | 46.3 | 36.5 | 55.2 | 37.2 |
| 33.9 | 53.9 | 51.6 | 17.3 | 85.7 | 6.6 | 4.7 | 1.8 | 98.7 | 62.8 | 59.0 | 76.1 | 67.9 | 73.7 |
| 27.2 | 39.5 | 6.9 | 14.0 | 3.0 | 41.6 | 49.5 | 11.2 | 17.9 | 12.7 | 0.8 | 21.1 | 24.5 | |

The scaled Muth distribution was fitted to the data. The ML estimates were $\hat{\alpha} = 0.4608$ and $\hat{\beta} = 33.9049$. Figure 2 represents the cumulative relative frequency versus the theoretical cumulative probabilities. Graphically, it can be seen that the theoretical probabilities fit the empirical ones quite well. In fact, the associated correlation coefficient between them is 0.9986.

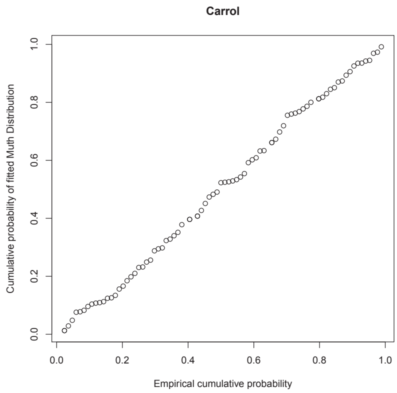


**Figure 2.** Fitted scaled Muth distribution to Carrol data set.

Although from Figure 2 we see that the scaled Muth distribution provides a good fit, we also applied several goodness of fit tests, specifically, the Cramér von Mises statistic $W^2$, the Watson statistic $U^2$, the Anderson–Darling statistic $A^2$, the Kolmogorov–Smirnov statistics $D^+$, $D^-$ and $D$, and the Kuiper statistic $V$. A detailed definition together with simple formulae for calculating these statistics can be found in D'Agostino and Stephens [11, Chapter 4]. To get the $p$-values we applied a parametric bootstrap by generating 1000 bootstrap samples (cf. Stute et al. [23] and Babu and Rao [3] for full details). The results obtained are shown in Table 6 and clearly suggest that the scaled Muth distribution provides a satisfactory fit.

**Table 6.** Carrol data set: Goodness-of-fit tests.

| | $W^2$ | $U^2$ | $A^2$ | $D^+$ | $D^-$ | $D$ | $V$ |
|---|---|---|---|---|---|---|---|
| $p$-value | 0.8690 | 0.8690 | 0.3920 | 0.8450 | 0.5170 | 0.7050 | 0.7290 |

Finally, we compared the scaled Muth distribution with other models commonly used to fit non-negative data such as the exponential, gamma, lognormal and Weibull distributions. For this aim, we calculated the Akaike information criterion AIC (cf. Akaike [2]) and the Bayesian information criterion BIC (cf. Schwarz [22]), which are defined as follows

$$\mathrm{AIC} = 2r - 2\log L, \qquad \mathrm{BIC} = -2\log L + r\big(\log n - \log(2\pi)\big),$$

where $r$ is the number of parameters and $L$ denotes the maximized value of the likelihood function. The model with lower values of AIC and/or BIC is preferred. Table 7 shows the ML estimated parameters and the AIC and BIC values for each distribution. Accordingly, it is clear that the scaled Muth distribution provides a better fit.

**Table 7.** Carrol data set: Model, ML estimates, AIC and BIC values.

| Model | ML estimates | AIC | BIC |
|---|---|---|---|
| Muth($\alpha,\beta$) | $\hat{\alpha}=0.4608$, $\hat{\beta}=33.9049$ | 740.3600 | 741.5220 |
| Exponencial($\lambda$) | | | |
| $f(x;\lambda)=\lambda e^{-\lambda x}$ | $\hat{\lambda}=0.0295$ | 753.0520 | 753.6330 |
| Gamma(a,b) | | | |
| $f(x;a,b)=\dfrac{1}{b^a\Gamma(a)}\,x^{a-1}e^{-x/b}$ | $\hat{a}=1.5160$, $\hat{b}=22.3838$ | 747.3087 | 748.4706 |
| Lognormal($\mu$,$\sigma$) | | | |
| $f(x;\mu,\sigma)=\dfrac{1}{x\sigma\sqrt{2\pi}}\,e^{-(\log x-\mu)^2/2\sigma^2}$ | $\hat{\mu}=3.1597$, $\hat{\sigma}=1.0253$ | 767.1983 | 768.3602 |
| Weibull(a,b) | | | |
| $f(x;a,b)=\dfrac{a}{b}\left(\dfrac{x}{b}\right)^{a-1}\,e^{-(x/b)^a}$ | $\hat{a}=1.3665$, $\hat{b}=36.9120$ | 744.4891 | 745.6511 |

## 10 Conclusions

The Muth distribution is a model for non-negative continuous random variables introduced in the Seventies in the context of reliability theory. For decades, this probability model has been overlooked in the literature with the exception of a paper by Leemis and McQueston [15], where its relation with the exponential distribution was pointed out. In the current paper, various mathematical properties of the Muth distribution are derived. More precisely, the variate generation property is shown by using the Lambert $W$ function, the mode is given in closed form as a function of the golden ratio and tractable expressions for computing the moments are obtained in terms of the generalized integro-exponential functions, which are also useful to calculate the moments of the order statistics. Parameter estimation is performed and a Monte Carlo simulation study reveals that the maximum likelihood method provides acceptable estimates. In addition, a scaled version of the Muth distribution is considered and a real data set application illustrates its usefulness.

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## References

[1] K.M. Abadir. The mean-median-mode inequality: counterexamples. *Economet. Theor.*, **21**(2):477–482, 2005. http://dx.doi.org/10.1017/S0266466605050267.

[2] H. Akaike. A new look at statistical model identification. *IEEE Trans. Autom. Control*, **19**(6):716–723, 1974. http://dx.doi.org/10.1109/TAC.1974.1100705.

[3] G.J. Babu and C.R. Rao. Goodness-of-fit tests when parameters are estimated. *Sankhyā*, **66**(1):63–74, 2004.

[4] L.J. Bain. Analysis for the linear failure-rate life-testing distribution. *Technometrics*, **16**(4):551–559, 1974. http://dx.doi.org/10.1080/00401706.1974.10489237.

[5] N. Balakrishnan and C.R. Rao (Eds.). *Order Statistics: Theory and Methods*, volume 16 of *Handbook of Statistics.* Elsevier, Amsterdam, 1998.

[6] V. Barnett. Probability plotting methods and order statistics. *J. Roy. Statist. Soc. Ser. C Appl. Statist.*, **24**(1):95–108, 1975.

[7] R. Bartoszyński and M. Niewiadomska-Bugaj. *Probability and Statistical Inference, 2nd edition.* Wiley–Interscience, Hoboken, NJ, 2008.

[8] P.J. Bickel and K.A. Doksum. *Mathematical Statistics, Basic Ideas and Selected Topics, vol. I, 2nd edition.* Prentice Hall, New Jersey, 2001.

[9] L.K. Chan. On a characterization of distributions by expected values of extreme order statistics. *Am. Math. Mon.*, **74**(8):950–951, 1967. http://dx.doi.org/10.2307/2315271.

[10] R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey and D.E. Knuth. On the Lambert $W$ function. *Adv. Comput. Math.*, **5**(4):329–359, 1996. http://dx.doi.org/10.1007/BF02124750.

[11] R.B. D'Agostino and M.A. Stephens (Eds.). *Goodness-of-Fit-Techniques.* Marcel Dekker, New York, 1986.

[12] H.A. David and H.N. Nagaraja. *Order Statistics.* John Wiley, New York, 2003.

[13] G.S. Fishman. *Monte Carlo: Concepts, Algorithms and Applications.* Springer Ser. Oper. Res. Springer-Verlag, New York, 1996. http://dx.doi.org/10.1007/978-1-4757-2553-7.

[14] P. Jodrá. Computer generation of random variables with Lindley or Poisson–Lindley distribution via the Lambert $W$ function. *Math. Comput. Simul.*, **81**(4):851–859, 2010. http://dx.doi.org/10.1016/j.matcom.2010.09.006.

[15] L.M. Leemis and J.T. McQueston. Univariate distribution relationships. *Am. Stat.*, **62**(1):45–53, 2008.

[16] J.W. Meijer and N.H.G. Baken. The exponential integral distribution. *Statist. Probab. Lett.*, **5**(3):209–211, 1987.

[17] M.S. Milgram. The generalized integro-exponential function. *Math. Comp.*, **44**(170):443–458, 1985.

[18] J.E. Muth. Reliability models with positive memory derived from the mean residual life function. In C.P. Tsokos and I. Shimi (Eds.), *The Theory and Applications of Reliability*, volume 2, pp. 401–435. Academic Press, Inc., New York, 1977.

[19] F.W.F. Olver, D.W. Lozier, R.F. Boisvert and C.W. Clark (Eds.). *NIST Handbook of Mathematical Functions.* Cambridge University Press, Washington, DC, 2010.

[20] N. Ozalp and E. Bairamov. Uniform convergence and computation of the generalized exponential integrals. *J. Math. Chem.*, **49**(2):520–530, 2011.

[21] E. Parzen. Quantile probability and statistical data modeling. *Stat. Sci.*, **19**(4):652–662, 2004.

[22] G. Schwarz. Estimating the dimension of a model. *Ann. Statist.*, **6**(2):461–464, 1978.

[23] W. Stute, W. González Manteiga and M. Presedo Quindimil. Bootstrap based goodness-of-fit tests. *Metrika*, **40**(3):243–256, 1993.