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QUALITY ASSESSED EIGENFREQUENCY ANALYSIS

R. Baušys

1. Introduction

Over recent decades, the finite element method as one of the discretized numerical methods, has become a rapidly developing technique for the solution of a wide range of physical problems such as solid mechanics, fluid mechanics, heat transfer, vibrations etc. The primary goal of comprehensive numerical simulation procedures is fully utilise the computer resources in engineering decision-making. If engineering decisions are made using the results of numerical simulation, we must be able to determine the reliability of the modelling process. Therefore, the quality of results, obtained from finite element computations is a cornerstone of the quality of engineering decisions based on them. In order to establish the reliability of the numerical simulation, two requirements must be satisfied:

1. The mathematical model used to represent an engineering problem must account for all essential attributes of the system.

2. Numerical approximation of the solution of the mathematical model must be sufficiently accurate.

This paper is concerned with techniques by which the errors arised due to the numerical solution of the mathematical models is controlled.

The starting point in the finite element formulation is differential equations and arbitrary boundary conditions associated with a mathematical model representing an engineering system or process. In finite element approach these differential equations are discretized through Galerkin functional with assumed shape functions over each element. Finally, the separate element equations are assembled to form the system matrix equation which is solved to determine the unknown parameters of interest. Unless the shape functions include the exact solution, the simulated results will be only approximate. The automatic evaluation of finite element discretization errors is the most important ingredient of a comprehensive analysis. Based on the observed discretization errors, the analyst can construct with the available mesh generation procedures new refined meshes and thus continue the analysis process until the required accuracy is achieved.

Error estimation techniques evaluate the amount of solution errors due specifically to mesh discretization. By now a considerable success has been achieved mainly on problems of linear elliptic type, such as linear elastostatics and stationary heat conduction problems [1] - [4].

The most important ingredient of the error estimation is the construction of the new solution of a higher quality since the exact solution of complex engineering problems is generally unknown. Typically, this new improved solution is obtained by a *posteriori* procedure which utilise the original finite element solution itself.

We focus our attention on the construction of the eigensolutions of higher quality by model improvement for the case of the unstructured meshes. The enhancements of the solution are provided by the local/global error control approaches. Within a framework of the local updating, the Superconvergent Patch Recovery technique for displacements (SPRD) is applied to free vibration problems for improving eigenmodes and eigenfrequencies [5], [6]. This method shows excellent results for the lower eigenfrequencies, but for the higher eigenfrequencies the improvement of the eigenpairs is still not enough to provide a reliable error estimation. In order to improve the higher frequencies, we employ a preconditioned conjugate gradient scheme to optimise successive deflated Rayleigh quotients. This technique has been studied in, for example, [7] -[10]. The idea of global updating is to improve the FE solution of order p by the SPRD method and then use it as the starting trial eigenvector in the preconditioned conjugate gradient scheme to obtain a solution similar to the FE solution of order $p+1$. When we have obtained the global updated solution of order $p+1$, we apply again the SPRD technique to get an improved solution of order $p+2$, thus we have a global-local approach.

Numerical examples show the nice properties of local/global updated solution as a basis for an error estimator and an error indicator in an adaptive finite element strategies.

2. Basic equations

We study eigenvalue problems based on differential equations of the form

$$\lambda\rho\, u(x) + \tilde{\nabla}^T D\tilde{\nabla} u(x) = 0 \ , \ in\,\Omega \tag{1}$$

with boundary conditions

$$u(x) = u_b \qquad on\,\Gamma_u \tag{2}$$

$$\tilde{\nabla}_n{}^T D\tilde{\nabla} u(x) = \sigma_b \qquad on\,\Gamma_\sigma \tag{3}$$

where $\tilde{\nabla}$ is the differential operator, $\tilde{\nabla}_n$ is the boundary operator, D is the constitutive matrix and ρ is the mass density. The Ω is a spatial domain with the boundary $\Gamma = \Gamma_u \bigcup \Gamma_\sigma$ $(\Gamma_u \bigcap \Gamma_\sigma = \varnothing)$, where Γ_u is the boundary with essential boundary conditions and Γ_σ is the boundary with natural boundary conditions.

The finite element approximation

$$u(x) = N u^h \tag{4}$$

where N contains the basis functions and u^h the corresponding nodal displacement values obtained from the Galerkin procedure as applied to equation (1). In FE context, we obtain the following equation

$$(K - \lambda^h M)u^h = 0 \ , \ \lambda^h = (\omega^h)^2 \tag{5}$$

where K is the stiffness matrix and M is the consistent mass matrix of the structure. Equation (5) is of the form of a linear generalised eigenvalue problem for the finite dimensional discrete problem, with eigenvalues λ_i^h being equal to the squares of the eigenfrequencies ω_i^h .

In order to assess the discretization errors, we may assume the improved eigenfrequencies (of higher accuracy order) can be constructed using Rayleigh quotient

$$(\omega_i^*)^2 = \frac{\sum\limits_K \int\limits_{\Omega_e} (\tilde{\nabla} u_i^*)^T D\tilde{\nabla} u_i^* \, dx}{\sum\limits_K \int\limits_{\Omega_e} (u_i^*)^T \rho u_i^* \, dx} \tag{6}$$

where K is summed over the total number of elements and u_i^* is a displacement field over the elements which has a higher order of accuracy. The recovered displacement field of the eigenmode u_i^* will be determined by a postproccessed updating technique.

The quality of any error estimator is measured by effectivity index giving the ratio of the estimated errors to the actual ones as follows

$$\theta_i = \frac{\Delta\bar{\omega}_i^h}{\Delta\omega_i^h} \tag{7}$$

where the error in eigenfrequencies of the original finite element solution

$$\Delta\omega_i^h = \omega_i^h - \omega_i \tag{8}$$

where ω_i are the exact eigenfrequencies and the estimated error of the finite element solution

$$\Delta\bar{\omega}_i^h = \omega_i^h - \omega_i^* \tag{9}$$

The error estimator is said to be asymptotically exact if the estimated errors tend to exact ones as the finite element mesh is refined.

3. Error indicators for eigenfrequency analysis

3.1. Error indicators based on local projection

This approach is based on the fact that the nodal points of the finite element approximation are found to be exceptional points at which the prime variables (displacements) have higher order accuracy in respect of the global accuracy [11]. These points are called the superconvergent points of the finite element solution. The SPRD technique is based on a higher order displacement field u_i^* fitted to super-convergent values from the FE-solution in a least square sense over local element patches. From the higher order accuracy displacement field an improved kinetic energy and a strain energy can be calculated and thus an improved eigenfrequency can be obtained. A separate patch recovery must be made for each eigenmode of interest. This approach is a local updating method, so no global system of equations has to be constructed and solved. Details are available in [5], [6]. This method does not show sufficient improvement for higher eigenfrequencies to provide a reliable error estimation.

3.2. Error indicators based on global updating

In order to improve the higher eigenfrequencies, we employ a preconditioned conjugate gradient scheme to optimise successive deflated Rayleigh quotients. The convergence profiles of this technique are characterised by two phases. The initial phase which may require a large number of iterations and then asymptotic phase where the rate of convergence is proportional to the relative separation between the eigenfrequency being computed and the next higher one. We emphasise that by choosing the SPRD improved FE solution as starting eigenvector we can get directly to the asymptotic phase or at least not too far from it. To accelerate the speed of convergence of the global updating, we use a preconditioning matrix that is a diagonal of the sum of the stiffness and mass matrices. This allows us to keep the computations on an element level which gives a high computational efficiency. We do not seek to determine a fully converged eigenpairs, so we restrict the number of the iterations to in advance prescribed quantity. When we have obtained the global updated solution of order $p+1$, we apply again the SPRD technique to get an improved solution of order $p+2$, thus we have a global-local updating. In detail this approach is described below.

3.2.1. The global-local updating algorithm

Here follows a description of the algorithm we have used for the global-local updating of each eigenpair.

1. Compute the preconditioning matrix C as the diagonal elements of $K - \lambda_j^* M$. Note that we will have to compute a new preconditioning matrix for each eigenpair.

2. The initial vector $u_j^{(0)}$ is chosen by projecting the SPRD improved FE solution of order p onto the M-orthogonal complement of V_j, i.e. $V_{j-1}^T M u_j^{(0)} = 0$. We have used a limited number of iterations NITMAX. Set the iteration counter $k=0$ and take the vector $p^{(-1)}$ arbitrary.

3. Loop start as long as $k<$NITMAX execute steps 4-9, otherwise go to step 10.

4. If $k = 0$ then set $\beta^{(0)} = 0$, else

$$\beta^{(k)} = -\frac{p^{(k-1)T} K C^{-1} g_j^{(k)}}{p^{(k-1)T} K p^{(k-1)}} \tag{10}$$

5. Compute

$$\tilde{p}^{(k)} = C^{-1} g_j^{(k)} + \beta^{(k)} p^{(k-1)} \tag{11}$$

where

$$g_j^{(k)} = \frac{2}{u_j^{(k)T} M u_j^{(k)}} (K u_j^{(k)} - R(u_j^{(k)}) M u_j^{(k)}) \tag{12}$$

6. Compute $p^{(k)}$ by M-orthogonalizing $\tilde{p}^{(k)}$ in respect of V_j.

7. Compute the coefficient $\alpha^{(k)}$ by minimising the Rayleigh quotient

$$R(u_j^{(k)} + \alpha^{(k)} p^{(k)}) \tag{13}$$

Minimum requires $dR/d\alpha = 0$. From which we obtain

$$\alpha^{(k)} = \frac{nd - mb + \sqrt{\Delta}}{2(bc - ad)} \tag{14}$$

where

$$\begin{aligned} a &= p^{(k)T} K u_j^{(k)} \quad & m &= u_j^{(k)T} M u_j^{(k)} \\ d &= p^{(k)T} M p^{(k)} \quad & c &= p^{(k)T} M u_j^{(k)} \\ d &= p^{(k)T} K p^{(k)} \quad & m &= u_j^{(k)T} K u_j^{(k)} \\ \Delta &= (nd - mb)^2 - 4(bc - ad)(ma - nc) \end{aligned} \tag{15}$$

8. Evaluate

$$\tilde{u}_j^{(k+1)} = u_j^{(k)} + \alpha^{(k)} p^{(k)} \tag{16}$$

9. The new approximation vector $u_j^{(k+1)}$ is determined by M-normalising $\tilde{u}_j^{(k+1)}$. Increase the iteration counter and return to step 3.

10. Perform SPRD improvement of order $p+2$ on the global updated solution.

11. Proceed with the next eigenpair at step 1.

Originally convergence as a stopping criteria is considered when the relative increment of the Rayleigh quotient and the relative residual are less than a prescribed value. Since we only seek to improve eigenpairs, we will not continue the iteration until convergence occurred, but break after a small number of iterations.

4. Numerical example

In order to test performance of the proposed method, we consider vibrational problems of the elastic two-dimensional structures which, in equilibrium position, lie in plane. We will pay attention to the transverse vibrations of thin membranes of uniform thickness. A square thin membrane, shown in Fig 1, is considered.

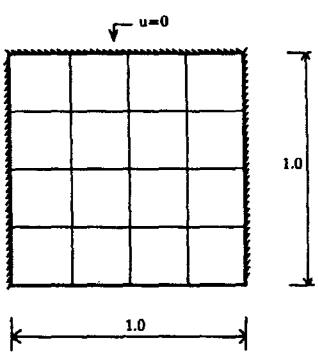
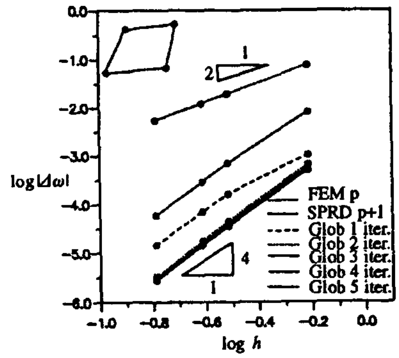
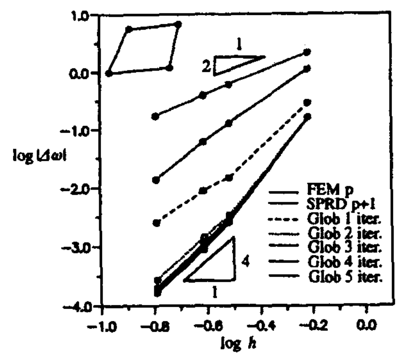


Fig 1. A finite element model of a square membrane

For simplicity, the wave propagation velocity $c = \sqrt{T/\rho}$ is assumed to be $1.0\frac{m}{s}$, where T is the uniform tension in the membrane. The analysis has been conducted for most commonly used linear quadrilateral and triangular finite elements. The study of the convergence rate of the proposed global updating procedure has been performed for the different eigenfrequencies as shown in Figs 2-3.

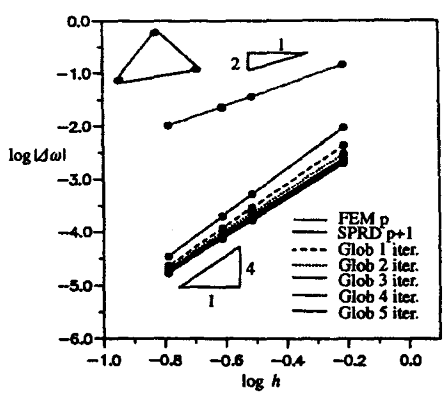


a) 1st eigenfrequency

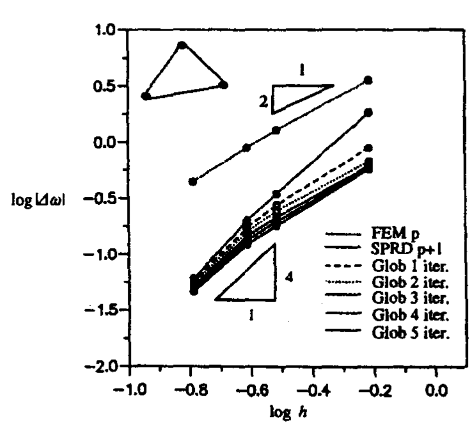


b) 10th eigenfrequency

Fig 2. Convergence rates in iterations of global updating procedure using quadrilateral elements

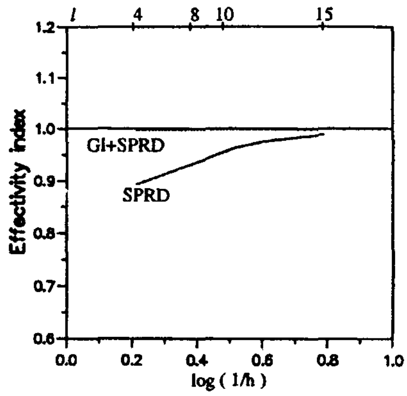


a) 1st eigenfrequency

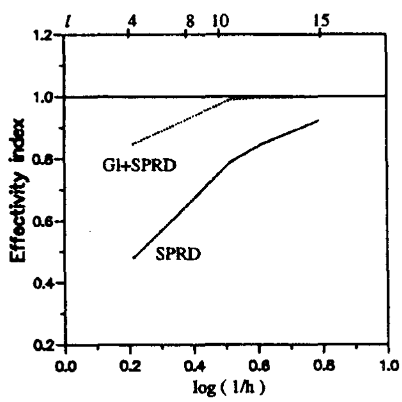


b) 10th eigenfrequency

Fig 3. Convergence rates in iterations of global updating procedure using triangular elements

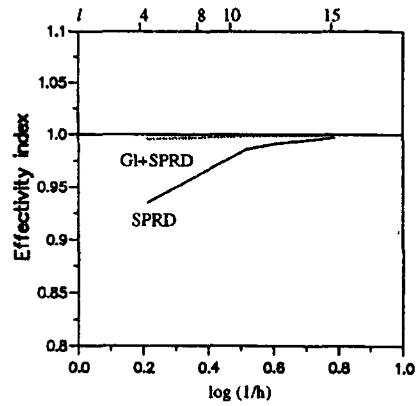


a) 1st eigenfrequency

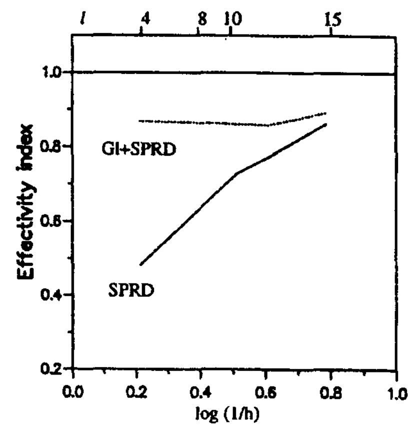


b) 10th eigenfrequency

Fig 4. Effectivity indices by quadrilateral elements



a) 1st eigenfrequency



b) 10th eigenfrequency

Fig 5. Effectivity indices by triangular elements

One can observe that the sufficient accuracy is achieved after a few global updating iterations. In order to keep computational efficiency, the number of the iterations is therefore set to be *3* for all numerical experiments.

The convergence of the effectivity indices are plotted in Figs 4-5.

We observe that the effectivity indices converge to one rapidly for all quadrilateral and triangular elements tested when the finite element mesh is refined. The superiority of the global updating coupled with SPRD technique (GI+SPRD) over the local updating (SPRD) is clearly demonstrated. The global updating is a more time-consuming procedure than SPRD.

5. Conclusions

The local and global updating techniques have been presented to improve solution of the generalised eigenvalue problem encountered in the finite element analysis of structural dynamics problems. The SPRD improved solution gives a good initial trial eigenvector for the modified conjugate gradient scheme, which immediately put us on the asymptotic phase of the convergence profile. The most attractive feature of the approach is that all operations of global updating can be efficiently performed on the element level: no global stiffness and mass matrices have to be assembled. The numerical study of the performance of the error estimator demonstrates the asymptotic exactness that shows the reliability of the proposed procedure.

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KOKYBIŠKAI ĮVERTINTA LAISVŲJŲ SVYRAVIMŲ ANALIZĖ

R. Baušys

S a n t r a u k a

Inžinerinių uždavinių sprendimo rezultatai, gauti naudojant baigtinių elementų metodą, kaip ir kitus diskrečius metodus, yra apytiksliai. Todėl projektuotojui labai aktualu įvertinti atliktų skaičiavimų kokybę. Straipsnyje pateikiami metodai, suteikiantys galimybę įvertinti baigtinių elementų diskretizacijos kokybę laisvųjų svyravimų uždaviniams. Kadangi daugelio inžinerinių uždavinių atveju analitinis sprendinys nėra galimas, tai pagrindinis šios procedūros sudėtingumas yra rasti aukštesnės tikslumo klasės sprendinį. Šis poprocesorinis pagerintas sprendinys yra nustatomas naudojant pradinio baigtinių elementų sprendinio informaciją. Darbe pateikiami lokalus ir globalus apibendrinto nuosavųjų reikšmių uždavinio sprendinio pagerinimo būdai. Lokalusis būdas yra paremtas superkonvergencinėmis pradinio sprendinio savybėmis. Globalaus sprendinio pagerinimo būdo esmė - susietų gradientų metodas, naudotas p+1 baigtiniams elementams. Globalus sprendinio pagerinimo būdas yra efektyvesnis aukštesnėms nuosavoms reikšmėms, ir sprendinio kokybės įvertinimo procedūra yra patikimesnė platesnei nuosavų reikšmių spektro daliai.

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