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STRUCTURE OF CONSTITUTIVE RELATIONS FOR ISOTROPIC ELASTIC MEMBRANES WITH VOIDS

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1. Introduction

A non-linear theory of flexible membranes with vacuous pores or voids [1] may be considered as a two-dimensional analogue of the theory of elastic materials with voids first developed by Nunziato & Cowin [2] and subsequently extensively studied in the literature (see [2, 3] and references cited therein). While the approach adopted in [1] was commonly called the direct approach in the shell literature (cf. [4–6]), ie being essentially independent on any three-dimensional theory, the mechanical balance laws postulated in [1] was largely motivated by the physical arguments underlying the three-dimensional theory [2, 3]. However, the theory of membranes differs in many ways from the three-dimensional theory. One source of differences lies in the definition and structure of material symmetry groups. These differences have been exploited in detail by Cohen & Wang [4] and Murdoch & Cohen [5] within the classical theory of elastic membranes, ie membranes without voids to which the more general theory of [1] reduces under certain assumptions.

This paper is concerned with the constitutive equations for elastic membranes with voids. The general form of such constitutive equations and restrictions due to the frame-indifference principle, the energy imbalance principle and possible material symmetries are considered. Special attention is devoted to the particular case of isotropic membranes. The general structure of constitutive equations in this special case is examined in details.

The coordinate-free (absolute) tensor notation for surface vector and tensor fields introduced in [5, 6] is used consistently throughout the paper. This makes the considerations not only simpler and more transparent but also entirely suppresses the need to examine the invariance of the constitutive equations under the coordinate transformations.

2. Equations of motion and constitutive relations

A theory of membranes with vacuous pores or voids may be considered the simplest generalisation of the classical theory (see eg [4–6]). Moreover, as in the classical theory, a membrane is mathematically modelled as a two-dimensional continuum consisting of point-like particles, which at each time instant t are smoothly distributed over a smooth geometric surface $M(t)$ in the physical space. A theory of membranes with voids is richer than the classical theory in this that every spatial configuration $M(t)$ of the membrane is additionally characterised by a single scalar function $\varphi=\varphi(\boldsymbol{y},t)$. Here $\boldsymbol{y}\in M(t)$ denotes the spatial place currently occupied by a typical membrane particle. Physical interpretation of φ may slightly vary dependently on the intended application of the theory. In general, it is assumed that the surface mass density $m(\boldsymbol{y},t)$ measured per unit area of $M(t)$ may be written in the form

$$m(\boldsymbol{y},t)=\gamma(\boldsymbol{y},t)\varphi(\boldsymbol{y},t), \qquad (1)$$

where $\gamma(\boldsymbol{y},t)$ is interpreted as the matrix mass density (cf. the three-dimensional theory [7]). Thus $\varphi(\boldsymbol{y},t)$ may be called the area fraction function. Moreover, this physical interpretation requires that

$$m(\boldsymbol{y},t)>0,\quad 0<\gamma(\boldsymbol{y},t)\le m(\boldsymbol{y},t),\quad 0<\varphi(\boldsymbol{y},t)\le 1.$$

In this theory, $\varphi(\boldsymbol{y},t)$ is regarded as the independent kinematics variable so that the classical laws of mechanics must be suitably reformulated in order to obtain the complete set of governing field equations [1].

In the analysis of special problems, it is often convenient to formulate the field equations and constitutive relations expressed entirely in terms of referential field variables. Accordingly, in the subsequent

considerations it will be assumed that the membrane has been assigned a particular, time-independent configuration M. This configuration serves to identify membrane particles and the motion of the membrane is described by a time-dependent mapping χ which carries each membrane particle whose reference place was $x \in M$ into its spatial place $y = \chi(x,t)$ at present time t. Then $\varphi(y,t)$ may be regarded as a function of x and t so we write $\varphi(y,t)$ for $\varphi(\chi(x,t),t)$. Assuming that the deformation mapping $y = \chi(x,t)$ is differentiable in respect of both arguments, the velocity of the membrane and the surface deformation gradient are defined by

$$v(x,t) = \dot{\chi}(x,t), \quad F(x,t) = \nabla\chi(x,t). \tag{2}$$

Throughout this paper $\nabla \equiv \nabla_s$ and $Div \equiv Div_s$ denote the surface gradient and divergence operators on M (see [5, 6]).

In the referential description, the initial surface mass density $m_0(x)$ is time independent and the law of mass conservation is satisfied identically. The remaining three laws of balance, the balance of linear and angular momentum and the balance of equilibrated forces yield the following field equations [1]

$$Div\boldsymbol{T} + \boldsymbol{f} = m_0\dot{\boldsymbol{v}}, \quad \boldsymbol{T}\boldsymbol{F}^T - \boldsymbol{F}\boldsymbol{T}^T = \boldsymbol{0} \tag{3}$$

and

$$Div\boldsymbol{h} + g + f = m_0(\kappa\ddot{\varphi} + \dot{\kappa}\dot{\varphi}) \,. \tag{4}$$

The equations (3) are classical (cf. [5, 6]), where $\boldsymbol{T}(x,t)$ denotes the surface stress tensor (of the first Piola-Kirchhoff type) and $\boldsymbol{f}(x,t)$ is the surface body force.

The mechanical variables appearing in the equations (4), which represents the balance law of equilibrated forces, are named as follows (the terminology adopted from the three-dimensional theory [2, 3]): $\boldsymbol{h}(x,t)$ – the equilibrated surface force vector, $g(x,t)$ – the intrinsic equilibrated force, $f(x,t)$ – the extrinsic equilibrated force, $\kappa(x,t)$ – the equilibrated inertia. Moreover, in consistency with (1), the initial mass density is written as $m_0(x) = \gamma_0(x)\varphi_0(x)$. Thus $\gamma_0(x)$ and $\varphi_0(x)$ are the initial matrix mass density and the initial area fraction function.

The general constitutive equations for elastic membrane with voids relate surface stress tensor $\boldsymbol{T}$, the equilibrated surface stress vector $\boldsymbol{h}$ and the intrinsic equilibrated surface force g to the surface deformation gradient $\boldsymbol{F}$, the area fraction function φ, the surface gradient $\nabla\varphi$ of φ and the rate of change $\dot{\varphi}$ of φ. Such constitutive equations must also be consistent with the laws of thermodynamics and the principle of frame-indifference described in detail in [1, 8]. In the purely mechanical theory of elastic membranes with voids considered in this paper, the balance law of energy and the principle of irreversibility may be combined into the energy imbalance principle, which implies the following local dissipation inequality

$$\dot{\Phi} - \boldsymbol{T} \bullet \dot{\boldsymbol{F}} - \boldsymbol{h} \bullet \nabla\dot{\varphi} + (g - \frac{1}{2}\dot{\kappa}\dot{\varphi})\dot{\varphi} \le 0 \,. \tag{5}$$

Here $\Phi(x,t)$ denotes an elastic potential, which may be identified with the internal energy for adiabatic processes or with the free energy under isothermal conditions.

The constitutive equations that are consistent with the principle of frame-indifference can be most simply expressed through the change of both independent and dependent variables. As it is know from the classical theory of membranes, the surface deformation gradient $\boldsymbol{F}(x,t)$ may be written in the form $\boldsymbol{F} = \boldsymbol{I}\mathbf{F}$, where $\mathbf{F}(x,t)$ is the tangential deformation gradient and $\boldsymbol{I}(y,t)$ denotes the inclusion operator in the current configuration of the membrane [4, 5]. Then, introducing the tangential surface stress tensor (of the second Piola-Kirchhoff type) $\mathbf{S}(x,t)$, which is defined by $\boldsymbol{T} = \boldsymbol{F}\mathbf{S}$, the equations (3) may be expressed in the equivalent form

$$Div(\boldsymbol{F}\mathbf{S}) + \boldsymbol{f} = m_0\dot{\boldsymbol{v}}, \quad \mathbf{S} = \mathbf{S}^T \,. \tag{6}$$

By virtue of the principle of frame-indifference the constitutive equations depend on the surface deformation gradient only through the surface deformation tensors $\mathbf{C}(x,t)$ being defined by

$$\mathbf{C} = \boldsymbol{F}^T\boldsymbol{F} = \mathbf{F}^T\mathbf{F} \,. \tag{7}$$

Then, the frame-indifferent constitutive equations for an elastic membrane with voids are

$$\Phi = \tilde{\Phi}(e,\dot{\varphi}), \; \mathbf{S} = \tilde{\mathbf{S}}(e,\dot{\varphi}), \; \boldsymbol{h} = \tilde{\boldsymbol{h}}(e,\dot{\varphi}), \; g = \tilde{g}(e,\dot{\varphi}) \,. \tag{8}$$

Here for simplicity of writing e is short notation for

$$e \equiv (\varphi_0, \mathbf{C}, \varphi, \nabla\varphi).$$

The constitutive equations must (8) satisfy the dissipation inequality (5) which may by rewritten in the form

$$\dot{\Phi} - 2\mathbf{S}\bullet\dot{\mathbf{C}} - \boldsymbol{h}\bullet\nabla\dot{\varphi} + (g - \frac{1}{2}\dot{\kappa}\dot{\varphi})\dot{\varphi} \le 0. \qquad (9)$$

Assuming that the response function $\tilde{\Phi}(e,\dot{\varphi})$ is differentiable in respect of all its arguments and making use of (8) the inequality (9) reads

$$\begin{aligned}&(\partial_F\tilde{\Phi} - \frac{1}{2}\tilde{\mathbf{S}})\bullet\dot{\boldsymbol{F}} + (\partial_{\nabla\varphi}\tilde{\Phi} - \tilde{\boldsymbol{h}})\bullet\nabla\dot{\varphi} + \\ &+ (\partial_{\varphi}\tilde{\Phi} - \tilde{g} + \frac{1}{2}\dot{\kappa}\dot{\varphi})\dot{\varphi} + (\partial_{\dot{\varphi}}\tilde{\Phi})\ddot{\varphi} \le 0.\end{aligned} \qquad (10)$$

It then follows that the response function $\tilde{\Phi}$ and, by implication, the response functions $\tilde{\mathbf{S}}$ and $\tilde{\boldsymbol{h}}$ may not depend on the rate of change $\dot{\varphi}$ of the area fraction function. Thus the consistency with the laws of thermodynamics requires that the constitutive relations (8) must assume the following forms

$$\begin{aligned}&\Phi = \tilde{\Phi}(e;\boldsymbol{x}), \quad \mathbf{S} = \tilde{\mathbf{S}}(e;\boldsymbol{x}),\\ &\boldsymbol{h} = \tilde{\boldsymbol{h}}(e;\boldsymbol{x}), \quad g = \tilde{g}(e,\dot{\varphi};\boldsymbol{x}).\end{aligned} \qquad (11)$$

Moreover, the response functions $\tilde{\mathbf{S}}$ and $\tilde{\boldsymbol{h}}$ are determined by an elastic potential by the relations

$$\tilde{\mathbf{S}}(e;\boldsymbol{x}) = 2\partial_{\mathbf{C}}\tilde{\Phi}(e;\boldsymbol{x}), \quad \tilde{\boldsymbol{h}}(e;\boldsymbol{x}) = \partial_{\nabla\varphi}\tilde{\Phi}(e;\boldsymbol{x}). \qquad (12)$$

The intrinsic equilibrated force $g(\boldsymbol{x},t)$ is not derivable from the elastic potential and therefore requires individual treatment. In particular, the dependence of the response functions $\tilde{g}$ on $\dot{\varphi}$ is excluded by the dissipation inequality (9).

The constitutive equations (11) supplemented by the constitutive equation for the equilibrated inertia

$$\kappa = \tilde{\kappa}(\varphi_0, \varphi; \boldsymbol{x}) \qquad (13)$$

provide the complete set of relations, which determines the elastic response of membranes with voids. In view of (12) and (13) the inequality (10) reduces to the form

$$\tilde{\Gamma}(e,\dot{\varphi};\boldsymbol{x})\dot{\varphi} \le 0, \qquad (14)$$

where

$$\tilde{\Gamma}(e) \equiv \partial_{\varphi}\tilde{\Phi}(e) - \tilde{g}(e) + \frac{1}{2}\partial_{\varphi}\tilde{\kappa}(e)\dot{\varphi}^2 \qquad (15)$$

may be called the dissipation function. It is seen from (14) that $\tilde{\Gamma}(e,\dot{\varphi})$ must be opposite in sign to $\dot{\varphi}$ and must vanish whenever $\dot{\varphi} = 0$, that is

$$\tilde{\Gamma}(e,0;\boldsymbol{x}) \equiv \partial_{\varphi}\tilde{\Phi}(e;\boldsymbol{x}) - \tilde{g}(e,0;\boldsymbol{x}) = 0. \qquad (16)$$

Moreover,

$$\partial_{\dot{\varphi}}\tilde{\Gamma}(e,0;\boldsymbol{x}) \equiv -\partial_{\dot{\varphi}}\tilde{g}(e,0;\boldsymbol{x}) \le 0. \qquad (17)$$

Within purely mechanical theory the constitutive equations (11) and (13) together with the equations of motion (6) and (4) provide the complete set of field equations governing the motion and deformation of an elastic membrane with voids.

3. Material symmetry groups and representation of response functions

The constitutive equations (11) determine a mechanical response of a membrane with voids relative to a particular reference configuration M. The same constitutive relations may be formulated equally well relative to any other reference configuration M'. If $\boldsymbol{y} = \chi(\boldsymbol{x},t)$ and $\varphi = \varphi(\boldsymbol{x},t)$ describe the motion of the membrane with voids relative to the reference configuration M, then the same motion relative to another reference configuration M' may be described by $\boldsymbol{y} = \chi'(\boldsymbol{x}',t)$ and $\varphi = \varphi'(\boldsymbol{x}',t)$. Let further $\lambda : M \to M'$ be a deformation of a membrane from M to M'. Then, the two mappings χ' and χ are related by $\chi' = \lambda \circ \chi$ so that $\boldsymbol{x}' = \lambda(\boldsymbol{x})$ and $\varphi(\boldsymbol{x},t) = \varphi'(\lambda(\boldsymbol{x}),t)$. Moreover, by the chain rule we have the following transformation rules

$$\mathbf{F}' = \mathbf{F}\mathbf{H}, \quad \nabla\varphi' = \mathbf{H}^T\nabla\varphi. \qquad (18)$$

Here $\mathbf{H}(\boldsymbol{x}) = \nabla\lambda(\boldsymbol{x})^{-1}$ and $\nabla\lambda(\boldsymbol{x}) : T_{\boldsymbol{x}}M \to T_{\boldsymbol{x}'}M'$ is the tangential surface gradient of the deformation from M to M'. Moreover, with the use of (18) we have

$$\mathbf{C}' = \mathbf{F}'^T\mathbf{F}' = \mathbf{H}^T\mathbf{F}^T\mathbf{F}\mathbf{H} = \mathbf{H}^T\mathbf{C}\mathbf{H}. \qquad (19)$$

In general, when the reference configuration is changed from M to M', the response functions will also change. Denoting by a prime all variables and response functions defined relative to M', the form of corresponding constitutive equations equivalent to (11) may formally be obtained by replacing unprimed quantity by primed quantity. Thus, if the response of a

membrane particle in respect of one reference configuration is known, then with the use of (18) and (19) it can be determined in respect of any other reference configuration.

It may happen that the mechanical response of a membrane is indistinguishable for two different reference configurations. If this a case, the symmetry group of a membrane may be described as that set of linear transformations of local reference configuration which leave unaltered the response of the membrane to all deformations. Let M' and M' denote two configurations of the membrane such that given a membrane particle share the same place $\boldsymbol{x}$ and the same tangent spaces $T_xM = T_xM'$. Then, the relative deformation gradient $\mathbf{H}(\boldsymbol{x})$ is a linear map of T_xM into itself. Let $\mathbf{H}(\boldsymbol{x}) \in GL(T_xM)$ be such that

$$\begin{aligned}
&\tilde{\Phi}(\varphi_0, \mathbf{C}, \varphi, \nabla\varphi) = \tilde{\Phi}(\varphi_0, \mathbf{H}^T\mathbf{C}\mathbf{H}, \varphi, \mathbf{H}^T\nabla\varphi),\\
&\tilde{\mathbf{S}}(\varphi_0, \mathbf{C}, \varphi, \nabla\varphi) = \mathbf{H}\tilde{\mathbf{S}}(\varphi_0, \mathbf{H}^T\mathbf{C}\mathbf{H}, \varphi, \mathbf{H}^T\nabla\varphi)\mathbf{H}^T,\\
&\tilde{h}(\varphi_0, \mathbf{C}, \varphi, \nabla\varphi) = \mathbf{H}^T\tilde{h}(\varphi_0, \mathbf{H}^T\mathbf{C}\mathbf{H}, \varphi, \mathbf{H}^T\nabla\varphi), \qquad (20)\\
&\tilde{g}(\varphi_0, \mathbf{C}, \varphi, \nabla\varphi, \dot{\varphi}) = \tilde{g}(\varphi_0, \mathbf{H}^T\mathbf{C}\mathbf{H}, \varphi, \mathbf{H}^T\nabla\varphi, \dot{\varphi}),
\end{aligned}$$

for all tangential deformation tensors $\mathbf{C}(\boldsymbol{x},t): T_xM \to T_xM$ and all $\nabla\varphi(\boldsymbol{x},t) \in T_xM$. The set of all authomorphisms $\mathbf{H}(\boldsymbol{x}) \in GL(T_xM)$ satisfying (20) forms a subgroup of the general linear group $GL(T_xM)$. This subgroup, denoted by $\mathscr{G}$, is called the symmetry group of an elastic membrane with voids.

It is important to notice that the symmetry group $\mathscr{G}$ depends on the reference configuration. Since the symmetry group will change with the reference state, it is important to fix the reference configuration relative to which the material symmetry holds. It seems natural, therefore, that the undistorted state of vanishing deformation for which $|\det\mathbf{H}| = 1$ should be chosen as the reference state that identifies the inherent symmetries of material.

The conditions (20) appear to be the definition of the symmetry group $\mathscr{G}$. In application, however, the response functions are rarely given explicitly, while the group $\mathscr{G}$ can often be assumed. As a result, (20) becomes a condition upon the unspecified response functions. In other words, an assignment of the symmetry group narrows the class of response functions.

4. Isotropic elastic membranes

An isotropic membrane is one whose mechanical properties are identical in all directions tangent to the surface M. This may be the case if the matrix material is isotropic and the initial void distribution is uniform in the reference configuration. Formally, an isotropy conditions require that the material symmetry group $\mathscr{G}$ contain the orthogonal group $O(T_xM)$. However, the condition $O(T_xM) \subset \mathscr{G}$ is not invariant under change of local reference configurations (cf. [4, 5]). Local reference configurations for which this condition is satisfied are called undistorted reference configurations. Relative to such a reference configuration the reduced forms of the constitutive equations for isotropic elastic membranes with voids may be obtained from the representation theorems of isotropic functions in two-dimension [7].

The requirement that the response functions in the constitutive equations (11) are isotropic functions of their arguments implies that they depend on $\mathbf{C}$ and $\nabla\varphi$ only through their joint invariants. The complete set of independent principal invariants consists of

$$\boldsymbol{i} \equiv (i_1, i_2, k_1, k_2). \qquad (21)$$

The first two invariants, i_1 and i_2, are the same as in the classical theory of membranes without voids. These are the principal invariants of the surface deformation tensor $\mathbf{C}$ defined by [5]

$$i_1 \equiv \mathrm{tr}\,\mathbf{C}, \quad i_2 \equiv \det\mathbf{C} = j^2, \qquad (22)$$

where $j = |\det\mathbf{F}|$. The two additional invariants k_1 and k_2 are

$$k_1 = \nabla\varphi \bullet \nabla\varphi, \quad k_2 = \nabla\varphi \bullet \mathbf{C}\nabla\varphi. \qquad (23)$$

That there are only four functionally independent invariants of $\mathbf{C}$ and $\nabla\varphi$ is obvious if we note that at any point $\boldsymbol{x} \in M$, the tensor $\mathbf{C}$ may be expressed in the spectral form so that its off-diagonal component is zero. Thus the joint invariants of $\mathbf{C}$ and $\nabla\varphi$ can be expressed in terms of two components of $\mathbf{C}$ and two components of $\nabla\varphi$. It may be noted further that the assumption $\det\mathbf{F} > 0$ implies that $\mathbf{C}$ is a positive-definite tensor. Thus, the invariants (22) and (23) satisfy the conditions

$$i_1^2 - 4i_2 \geq 0, \quad k_1 \geq 0, \quad k_2 \geq 0, \tag{24}$$

which define the natural domain of the response functions for isotropic membranes with voids.

Now the reduced form of the response functions may be obtained from the representation theorems of tensor functions [7, 8]. By the general theorem on scalar-valued isotropic tensor functions, the response function $\tilde{\Phi}(\varphi_0, \varphi, \mathbf{C}, \nabla\varphi; x)$ is an isotropic function if and only if it depends on $\mathbf{C}$ and $\nabla\varphi$ only through the joint invariants (21). Hence the constitutive equation for the elastic potential takes the form

$$\Phi = \tilde{\Phi}(\varphi_0, \varphi, i_1, i_2, k_1, k_2). \tag{25}$$

The relevant theorems on tensor-valued and vector-value tensor functions in two-dimensions imply that the response function for the surface stress tensor must be of the form

$$\begin{aligned}\tilde{\mathbf{S}}(\varphi_0, \mathbf{C}, \varphi, \nabla\varphi) = \tau_0 \mathbf{1}_0 + \tau_1 \mathbf{C} + \tau_2 \nabla\varphi \otimes \nabla\varphi + \\ + \tau_3 (\mathbf{C}\nabla\varphi \otimes \nabla\varphi + \nabla\varphi \otimes \mathbf{C}\nabla\varphi),\end{aligned} \tag{26}$$

while the response function for the equilibrated stress vector takes the form

$$\tilde{\boldsymbol{h}}(\varphi_0, \mathbf{C}, \varphi, \nabla\varphi) = (\alpha_0 \mathbf{1}_0 + \alpha_1 \mathbf{C})\nabla\varphi. \tag{27}$$

Here the response coefficients τ_Γ, $\Gamma = 0,1,2,3$, and α_Λ, $\Lambda = 0,1$, are isotropic functions of the invariants (21):

$$\begin{aligned}\tau_\Gamma &= \tau_\Gamma(\varphi_0, \varphi, i_1, i_2, k_1, k_2), \\ \alpha_\Lambda &= \alpha_\Lambda(\varphi_0, \varphi, i_1, i_2, k_1, k_2).\end{aligned} \tag{28}$$

The constitutive equation for the intrinsic equilibrated force, like constitutive equation for the elastic potential, is of the form

$$g = \tilde{g}(\varphi_0, \varphi, i_1, i_2, k_1, k_2, \dot{\varphi}). \tag{29}$$

Further restrictions on the form of the response functions (26) and (27) may be derived from the consistency conditions (12). In particular, it is not difficult to see that the response coefficients τ_Γ and α_Λ are determined by the derivatives of the elastic potential in respect of the invarinats (21). To this end we note that the derivatives of the invariants (22) and (23) in respect of $\mathbf{C}$ to $\nabla\varphi$ tand are given by the following formulae

$$\begin{aligned}&\partial_{\mathbf{C}} i_1 = \mathbf{1}_0, \ \partial_{\mathbf{C}} i_2 = i_1 \mathbf{1}_0 - \mathbf{C}, \ \partial_{\mathbf{C}} k_1 = \mathbf{0}, \\ &\partial_{\mathbf{C}} k_2 = \nabla\varphi \otimes \nabla\varphi, \ \partial_{\nabla\varphi} i_1 = \partial_{\nabla\varphi} i_2 = \mathbf{0}, \\ &\partial_{\nabla\varphi} k_1 = 2\nabla\varphi, \ \partial_{\nabla\varphi} k_2 = 2\mathbf{C}\nabla\varphi.\end{aligned} \tag{30}$$

This can be shown by direct calculation. The differentiation of (25) in respect of $\mathbf{C}$ to $\nabla\varphi$ together with the use of (30) yields

$$\begin{aligned}&\partial_{\mathbf{C}} \tilde{\Phi} = (\Phi_1 + i_1 \Phi_2)\mathbf{1}_0 - \Phi_2 \mathbf{C} + \Phi_2' (\nabla\varphi \otimes \nabla\varphi), \\ &\partial_{\nabla\varphi} \tilde{\Phi} = 2(\Phi_1' \mathbf{1}_0 + \Phi_2' \mathbf{C})\nabla\varphi,\end{aligned} \tag{31}$$

where

$$\Phi_\alpha \equiv \partial\tilde{\Phi}/\partial i_\alpha, \ \Phi_\alpha' \equiv \partial\tilde{\Phi}/\partial k_\alpha.$$

The comparison of (31) with (26) and (27) together with the use of general relations (12) shows that

$$\Phi_\alpha \equiv \partial\tilde{\Phi}/\partial i_\alpha, \ \Phi_\alpha' \equiv \partial\tilde{\Phi}/\partial k_\alpha. \tag{32}$$

As a result of the above considerations we have

$$\begin{aligned}&\mathbf{S} = \tau_0 \mathbf{1}_0 + \tau_1 \mathbf{C} + \tau_2 \nabla\varphi \otimes \nabla\varphi, \\ &\boldsymbol{h} = (\alpha_0 \mathbf{1}_0 + \alpha_1 \mathbf{C})\nabla\varphi,\end{aligned} \tag{33}$$

which together with the constitutive equation (29) for the intrinsic equilibrated force and the constitutive equation (13) for the equilibrated inertia characterise completely an isotropic elastic membrane with voids.

5. Closing remarks

The derived representations of the response functions for elastic membranes with voids provide the most general form of the constitutive equations, which are consistent with the basic laws of mechanics and thermodynamics, the principle of material frame-indifference and the assumed material symmetry group. The derived representations provide valuable theoretical guidance for the formulation of specific constitutive equations for the particular classes of membranes.

It is not difficult to see that under the assumption $\varphi = 1$, the considered theory reduces to the classical theory of membranes without voids, in which case the constitutive equation for the tangential stress tensor coincide with the result obtained in [4]. This may be regarded as a partial verification of the correctness of the results derived in this paper.

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IZOTROPINIŲ TAMPRIŲ MEMBRANŲ SU TUŠTUMOMIS FIZINIŲ PRIKLAUSOMYBIŲ STRUKTŪRA

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Santrauka

Straipsnis skirtas tamprių membranų su tuštumomis fizinėms priklausomybėms sudaryti. Nagrinėjama tokių priklausomybių bendroji forma, apribojimai, kuriuos lemia atskaitos sistemos indiferentiškumas, energijos disbalanso principai ir galima medžiagos simetrija. Gautos tamprios membranos su tuštumomis būvį aprašančios funkcijos pateikia fizinių priklausomybių bendriausiąją formą, į kurią įeina pagrindiniai mechanikos, termomechanikos dėsniai, atskaitos sistemos indiferentiškumo principas ir pasirinkta medžiagos simetrijos grupė. Šios funkcijos išraiškos duoda vertingas teorines nuorodas formuluojant specifines fizines priklausomybes įvairioms membranų klasėms. Straipsnyje daugiausia dėmesio skiriama izotropinėms membranoms, išsamiai ištirta jų fizinių priklausomybių bendroji struktūra. Laikant, kad funkcija $\varphi = 1$, nesunku pastebėti, kad pateiktą teoriją galima traktuoti kaip klasikinę membranų (be tuštumų) teoriją.

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