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LIMIT AND SHAKEDOWN ANALYSIS OF RC ROD CROSS-SECTIONS

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Abstract. This paper presents the calculation of the cross-section of an RC rod element strength under the quasistatic low-cyclic loadings, considering the non-linear stress-strain relations of materials without cracks. Some mathematical models of the limit and shakedown analysis proposed by the authors involve the technique of calculating the cross-section under one-path loadings, considering the non-uniqueness of problem solutions. The plasticity conditions for the materials of the cross-section are formulated either in stress or in strain space. Simple solutions of two types, direct and inverse, corresponding to the limit states for alternating plasticity or progressive failure, are considered for the non-linear optimization problems obtained.

Keywords: reinforced concrete, cross-section, rod, low-cyclic loading, limit analysis, shakedown, optimization.

1. Introduction

Load-carrying structures of buildings and constructions are exposed to actions (static, thermal, kinematical etc.) which may vary in random manner during the period of their lifetime. As a result, there are repeated alternating cross-sectional forces, which change arbitrarily within the specified area.

At present, only separate design combinations of loads and actions are usually taken into account in the analysis and design procedures. In this case, the cross-section strength is considered to be guaranteed if all possible forces are situated within or on the boundary of the domain of the section carrying capacity. However, combinations of loads causing the residual stress and strain accumulation, that are directly left out of the design account, occur iteratively during the lifetime of the construction.

The strength of the RC rod element cross-sections, maintained within a certain history of variations of repeated loads, was investigated in a number of works (CEB 1996; Павлинов 1999; Korentz 2005). It was ascertained that the strength criteria of the element essentially depend on the repeated force interaction.

Another way to analyze a whole class of loads influencing a construction at once is the method, described in the theory of shakedown. In that case, the strength analysis is guaranteed and does not depend on the possible sequence of forces that are not very perilous. Strength conditions in generalized forces for the cross-sections under different load cycles were stated in investigations of Guralnick and Yala (1998), Alyavdin and Simbirkin (1999), Rizzo *et al.* (2000), Aliawdin (Алявдин 2005); for the similar structures see the approach of Gawęcki and Kruger (1995).

This paper presents the strength conditions for the cross-section of the RC elements obtained on the basis of the shakedown theory. Low-cyclic repeated loads when material strength parameters insignificantly vary during the maintenance period are discussed. It is assumed that concrete works anywhere without cracks.

The mathematical model of the limit analysis under repeated loads includes a technique for calculating the cross-section strength under one-path loadings. At that, the known deformation model was complemented and improved. Calculations are made in accordance with the rigid centroid of the cross-section, the technique of obtaining all solutions of the convex non-smooth problem is proposed, regions with more than one solution are analyzed.

2. The mathematical model of the limit analysis of a cross-section under low-cyclic loadings

2.1. The plasticity conditions formulated in the stress space

In this sub-chapter the plasticity conditions of a reinforced concrete cross-section are formulated in stress space. The mathematical model suggested here is a modified variant of the existing one (Alyavdin, Simbirkin 1999).

Let's consider the cross-section of a RC rod element of an arbitrary form with given physical and geometric characteristics (Fig. 1). Internal forces are imposed towards the principal central axises of the section *XOY* (see Chapter 3).

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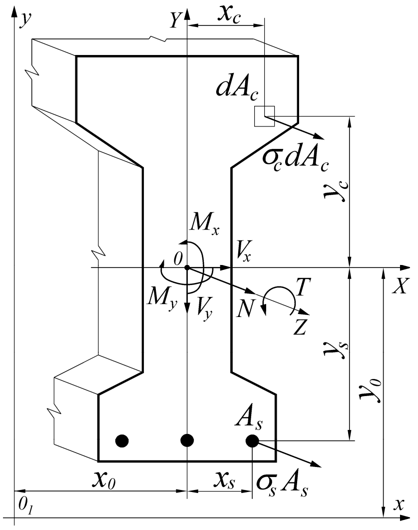

Fig. 1. RC rod cross-section

Reinforced steel is presumed to be a hardening elastic-plastic material. The behaviour of the compressed and tensioned concrete is also non-linear and the elastic-plastic material is harderning or softening. Materials of the element are assumed to be cyclically stable.

Let the cross-section of the reinforced concrete element (Fig. 1) be subjected to the vector of variable repeated forces $\boldsymbol{S} = (N, M_x, M_y, T, V_x, V_y)$ which change arbitrarily within the given domain $\boldsymbol{\Omega_S}$. This domain can be simulated by the polyhedron

$$\Omega_S = \left(\boldsymbol{S} \in \boldsymbol{R}^6 : \boldsymbol{S} = \sum_{l \in L} \alpha_l \boldsymbol{S}_l, \sum_{l \in L} \alpha_l = 1, \ \alpha_l \geq 0, \ l \in L \right), \quad (1)$$

where $\boldsymbol{S}_l$ is the vector of design combinations of the cross-section forces due to the l-th combination of the external loadings (static, thermal and kinematic); a_l is a component of the barycentric coordinate vector, $l \in L$; L is a set of loadings or force combinations. Note that the null-vector or origin of force space $\boldsymbol{S} = \boldsymbol{0} \in \boldsymbol{R}^6$ belongs to the domain $\boldsymbol{\Omega_S}$, $\boldsymbol{0} \subset \boldsymbol{\Omega_S}$.

The stresses $\sigma = (\sigma_z, \tau_{zx}, \tau_{zy})$ appear in surfaces dA of the concrete area A_c with coordinates $\boldsymbol{x} = (x, y)$; the stresses σ_x, σ_y, τ_{xy} are neglected; normal stresses σ_z in the reinforcing steel of the area A_s are only considered. Subscript "z" for stresses σ_z is omitted and subscripts "c" and "s" for the concrete and steel respectively are used below, if necessary.

To check the plasticity of the concrete in compression and the strength of the concrete in tension, a general Balandin-Geniev criterion (Alyavdin, Simbirkin 1999; Алявдин 2005) in terms of principal stresses for the three-dimensional stress state is adopted. It can be written as follows:

$$\begin{gathered}\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - (\sigma_1 \cdot \sigma_2 + \sigma_2 \cdot \sigma_3 + \sigma_2 \cdot \sigma_3 + \sigma_3 \cdot \sigma_1) + \\ + (f_c - f_{ct}) \cdot (\sigma_1 + \sigma_2 + \sigma_3) - f_c \cdot f_{ct} \leq 0, \forall \boldsymbol{x} \in A_c,\end{gathered} \quad (2)$$

where f_c and f_{ct} are the ultimate compressive and tensile stresses in the concrete respectively. The other possible plasticity theory for concrete is presented, for instance, in books of Życzkowski (1981), Yu Mao-Hong (2005).

For a state of plane stress (Fig. 1), the quadratic inequality (2) may be substituted by linear inequalities for concrete in compression and for concrete in tension respectively:

$$\begin{cases} -\sigma + R_l^c \leq 0, & \forall \boldsymbol{x} \in A_c^c, \\ \sigma - R_l^t \leq 0, & \forall \boldsymbol{x} \in A_c^t, \end{cases} \quad (3)$$

where R_l^c and R_l^t are the radicals of functions located on the left side of (3), which depend on shear stresses τ_{zx}, τ_{zy}. They are given by

$$\begin{aligned} R_l^c &= (f_{ct} - f_c - D_l)/2, \\ R_l^t &= (f_{ct} - f_c + D_l)/2, \\ D_l &= \sqrt{(f_{ct} + f_c)^2 - 12 \cdot (\tau_{zx}^2 + \tau_{zy}^2)}. \end{aligned} \quad (4)$$

Their absolute values are the equivalent strengths of concrete; A_c^c and A_c^t are the concrete areas in compression and tension respectively, $A_c = A_c^c \cup A_c^t$.

The total strains in concrete are presented as a sum of elastic ε^e and residual ε^r components (the case of cracks in the RC element is not taken into account here):

$$\varepsilon = \varepsilon^e(\boldsymbol{S}) + \varepsilon^r, \ \forall \boldsymbol{x} \in A_c. \quad (5)$$

The dependence between strains and normal stresses in concrete is known

$$\varepsilon = f_\sigma(\sigma), \quad (6)$$

and reversible (it's not of necessity in sub-chapter 2.2 and so on)

$$\sigma = f_\varepsilon(\varepsilon), \ f_\varepsilon = f_\sigma^{-1}. \quad (7)$$

Furthermore, the residual shear stresses and strains in concrete are neglected, i.e. $\tau_{zx}^r = \tau_{zy}^r = 0$.

The total strains in reinforcing steel are also presented as a sum of elastic ε^e and residual ε^r components, similarly to Eq (5), for $\boldsymbol{x} \in A_s$.

The stress-strain relationship for steel in elastic stage is given by Hooke's law $\sigma_s^e = E_s \cdot \varepsilon_s$, and conditions of ideal plasticity are given by

$$-f_{sy} \leq \sigma_s \leq f_{sy}, \ \boldsymbol{x} \in A_s, \quad (8)$$

where f_{sy} is the steel stress at yield.

General case of non-ideal and elastic-plastic response of materials (with strain hardening or softening) can be considered using approach (Алявдин 2005).

It is obvious that both inequalities (8) may be active at the same point $\boldsymbol{x}$ of the steel area of the cross-section. Then, after transformations, we obtain the inequality

$$\sigma_s^+ - \sigma_s^- - 2 \cdot f_{sy} \leq 0, \quad (9)$$

which confines the cross-section ultimate capacity by the condition of the alternating steel yielding. Subscript "+" and "–" for the ultimate tension and ultimate compression respectively are used below, if necessary.

The plasticity conditions (3) and (9) for all l-th combinations may be written in the following forms:

$$\begin{cases} \min\limits_{l\in L}\left(-\sigma_c(\boldsymbol{S}_l)+R_l^c\right)\le 0, & \boldsymbol{x}\in A_c^c, \\ \min\limits_{l\in L}\left(\sigma_c(\boldsymbol{S}_l)-R_l^t\right)\le 0, & \boldsymbol{x}\in A_c^t, \\ \min\limits_{l\in L}\left(-\sigma_s(\boldsymbol{S}_l)+f_{sy}\right)\le 0, & \boldsymbol{x}\in A_s^c, \\ \min\limits_{l\in L}\left(\sigma_s(\boldsymbol{S}_l)-f_{sy}\right)\le 0, & \boldsymbol{x}\in A_s^t. \end{cases} \quad (10)$$

Besides, the following equilibrium equations must be satisfied:

$$\begin{cases} \int\limits_A \sigma_i^r\left(\varepsilon_i^r\right)dA=0, \\ \int\limits_A \sigma_i^r\left(\varepsilon_i^r\right)\cdot\left(y_{nl}-y_i\right)dA=0, \\ \int\limits_A \sigma_i^r\left(\varepsilon_i^r\right)\cdot\left(x_{nl}-x_i\right)dA=0, \end{cases} \quad (11)$$

where (x_{nl}, y_{nl}) are the parameters of the neutral axis (see sub-chapter 2.2).

It is assumed that the strength of the reinforced concrete element cross-section is ensured if there are fields of residual strains $\varepsilon_c^r(\boldsymbol{x})$, $\boldsymbol{x}\in A_c$, and $\varepsilon_s^r(\boldsymbol{x})$, $\boldsymbol{x}\in A_s$, provided that inequalities (10) and equalities (11) hold. This assertion is equivalent per se to the shakedown theorem for the above-mentioned problem.

The mathematical problem for the ultimate carrying capacity of the element cross-section can be formulated, when the vectors $\boldsymbol{S}_l$ of the section force combinations depend only on one parameter of the load F_0:

$$\boldsymbol{S}_l = F_0\cdot\boldsymbol{S}_{vl},\ l\in L. \quad (12)$$

Thus, the following infinite-dimensional non-linear programming problem of the cross-section limit analysis is derived: the parameter of the load should be maximized,

$$F_0 \to \max, \quad (13)$$

while constraints (10–12) dependent on F_0 are satisfied.

The variables of this problem are the fields of the optimal control variables $\varepsilon_c^r(\boldsymbol{x})$, $\boldsymbol{x}\in A_c$, $\varepsilon_s^r(\boldsymbol{x})$, $\boldsymbol{x}\in A_s$, and parameter F_0.

2.2. The plasticity conditions formulated in the strain space

In this sub-chapter the strength conditions of the reinforced concrete cross-section are formulated in the strain space, as it was proposed by Bykovcev and Ivlev in 1970 (Быковцев, Ивлев 1998), and then analysed by Życzkowski (1981). The vector of variable repeated forces $\boldsymbol{S}$ contains only 3 components here, $\boldsymbol{S}=\left(N,M_x,M_y\right)\in\boldsymbol{R}^3$; the criteria of the optimization problem are still a bit complicated.

The vector $\boldsymbol{S}$ is arbitrarily changed within the given domain $\boldsymbol{\Omega_S}$ (1) as before, when $\boldsymbol{S}\in\boldsymbol{R}^3$.

The limit analysis problem of the RC cross-section is the maximization of the linear function of vectors $\boldsymbol{S}_l$ of all l-th loadings

$$\sum_{l\in L}\boldsymbol{T}_{\boldsymbol{S}_l}^T\boldsymbol{S}_l \to \max, \quad (14)$$

under the conditions:

$$\begin{cases} f_1(\sigma_i)=\boldsymbol{S}, \\ \sigma_i=f_2(\varepsilon_i),\quad \varepsilon_i=\varepsilon_i^e+\varepsilon_i^r, \\ \varepsilon_i=\varepsilon_z+\varphi_x\cdot\left(y_0-y_i\right)+\varphi_y\cdot\left(x_0-x_i\right), \\ \varphi_i\left(\varepsilon_i,\lambda_i,\boldsymbol{K}_{\varepsilon\, i}\right)\le 0, \\ \lambda_i\ge 0,\quad \varphi_i^T\cdot\lambda_i=0,\quad \varepsilon_i^r=\partial\phi_i\cdot\lambda_i, \\ \boldsymbol{S}\in\Omega_{\boldsymbol{S}}\left(\boldsymbol{S}_l, l\in L\right), \end{cases} \quad (15)$$

where $\boldsymbol{T}_{\boldsymbol{S}_l}^T$ are the vectors of the weighting coefficients in accordance with the vectors of l-th loadings $\boldsymbol{S}_l$;

ε_i, ε_i^e, ε_i^r are total, elastic and residual strains for materials; $i\in(c,s)$ are subscripts; c and s refer to the concrete and reinforced steel respectively;

$f_1(\sigma_i)=\boldsymbol{S}$ in $(15)_1$ are the equilibrium conditions, in vector form, of the cross-section under one-path load (for more details see chapter 3);

$\sigma_i=f_2(\varepsilon_i)$ are the material stress-strain relationships, initial or modified under quasi-static low-cyclic loadings;

λ_i, $\boldsymbol{K}_{\varepsilon i}$ are Lagrangian (plastic) multipliers and the vector of yield strains for the RC element materials; this vector contains the values of the ultimate compressive and tensile strains $\boldsymbol{K}_{\varepsilon i}=(\varepsilon_{iu}^-,\varepsilon_{iu}^+)$;

φ_i, ϕ_i are the functions of yielding and plastic potentials for concrete and steel.

Eqs (14)–(15) present an alternative to the limit analysis (10)–(13) infinite-dimensional problem of nonlinear mathematical programming. The variables of this problem are the fields of optimal control variables $\varepsilon_c^r(\boldsymbol{x})$, $\boldsymbol{x}\in A_c$, $\varepsilon_s^r(\boldsymbol{x})$, $\boldsymbol{x}\in A_s$, and the vectors $\boldsymbol{q}_l=\left(\varepsilon_z,\varphi_x,\varphi_y\right)_l$, $\boldsymbol{S}_l$, $l\in L$.

Mathematical models of the limit state problems (14)–(15), so as (10)–(13), could be also formulated using the energy principles. For example, the first model (14)–(15) will be written as follows: find the minimum of RC cross-section energy

$$W(\cdot):=\int\limits_A dA\sum_{i\in(c,s)}\int\limits_0^{\varepsilon_i} f_2(\varepsilon_i)\delta\varepsilon_i-\boldsymbol{q}^T\boldsymbol{S}\to\min, \quad (15a)$$

under the conditions (14), $(15)_{2\text{-}9}$, where $\boldsymbol{q}$ is the vector of strains, $\boldsymbol{q}=\left(\varepsilon_z,\varphi_x,\varphi_y\right)\in\boldsymbol{R}^3$.

The problem (14)–(15) for the repeated alternating loading, which depends on one parameter F_0 (see the sub-chapter 2.1), allows for simple solutions of 2 types, direct and inverse, that correspond to the condition of alternating plasticity or progressive failure of the rod cross-section. These two mentioned cases allow for an analytical solution. In the first case, the plastic failure occurs not simultaneously (isochronously) in the whole section or in one of its parts, changing it into a plastic hinge. In the second case, the cross-section remains inconvertible and collapsing starts in the finite set of points or regions. The alternating plasticity occurs in each of these points or regions, when more than one external action influences it.

An inverse method is applied for the progressive section failure. The full cross-section area A is divided into a tensioned area A^c and compressed area A^t by the neutral axis with parameters (x_{nl}, y_{nl}, α); α is the axis angle of slope. For these areas the only actual inequality $(15)_4$ is true:

$$\begin{cases} \varepsilon_{cu}^{-} \le \varepsilon_c^e + \varepsilon_c^r \le \varepsilon_{cu}^{+}, \\ \varepsilon_{su}^{-} \le \varepsilon_s^e + \varepsilon_s^r \le \varepsilon_{su}^{+}. \end{cases} \quad (16)$$

Let us exclude the residual strains ε^r from each area A^t, A^c in Eqs (16), $\varepsilon_i^r = \varepsilon_{iu}^{\pm} - \varepsilon_i^e$, and then substitute it in the equilibrium condition (11) for residual stresses of the cross-section.

As a result, we can form a system of 3 non-linear algebraic equations to determine the relations between the variables of this problem.

The solving procedure for the inverse problem is an iterative determination process of the neutral axes position with correcting the parameter F_0 and further checking, if the result obtained agrees with conditions (15).

The final domain of interaction of the generalized forces is formed by intersection of the regions that correspond to the considered cases, i.e. a domain $\boldsymbol{\Omega}_{\boldsymbol{S}_c}$ of the guaranteed cross-section carrying capacity under all possible design combinations of repeated low-cyclic loadings is created.

However, for the case of alternating plasticity, both inequalities (16) for concrete and/or steel are not active here and, therefore, the second kind of the collapse for the RC section is not realized.

3. The mathematical model of the limit analysis of the cross-section under one-path loadings

The technique for calculation of the cross-section strength described in this chapter is influenced by the monotonously increasing one-path loading. It is similar to Bich (Біч 1991), Zvezdov *et al.* (Звездов и др. 2002), Bonet *et al.* (2004), Zupan and Saje (2005), but at the same time contains some differences, which are necessary to apply this technique, as the basis to the mathematical model for the repeated loadings.

The mathematical model for the RC section (Fig. 1) under one-path live and long duration forces (N, M_x, M_y) consists of the following equations set:

$$\begin{cases} \begin{cases} \int_A \sigma_i dA - N = 0, \\ \int_A \sigma_i (y_0 - y) dA - M_x = 0, \\ \int_A \sigma_i (x_0 - x) dA - M_y = 0, \end{cases} \\ \varepsilon_i = \varepsilon_z + \varphi_x \cdot (y_0 - y_i) + \varphi_y \cdot (x_0 - x_i), \\ \sigma_i = f_i(\varepsilon_i), \\ \varepsilon_i \le \varepsilon_{iu}, \end{cases} \quad (17)$$

where $(17)_1$ is the stress equilibrium equation; $(17)_2$ is the relative strain compatibility equation in accordance with the plane cross-section hypothesis $\forall (x_i, y_i) \in A_i, i \in (c,s)$; $(17)_3$ are constitutive laws, which define the relation between stress and strain for concrete and steel in the form of stress-strain diagrams; $(17)_4$ are inequalities for the limits of maximum relative axial strains, which define the area of permissible solution of the equation set; σ_c, σ_s are normal stresses along the Z-axis; $A \supseteq A_c \cup A_s$ is the cross-sectional area; ε_z is the unit axial strain at the stiffness centre (centroid) in point O (Fig. 1); φ_x, φ_y are curvatures about the appropriate axes; ε_{iu} are the limit values of the unit longitudinal deformation of concrete and steel.

Any internal forces and strains are calculated with regard to the principal centroidal axes X, Y, Z, which pass through the section stiffness centroid "0". The location of the principal axes does not depend on the section internal forces. In the general case for non-linearly deformed materials, the axes X, Y can be defined as neutral lines (x_0, y_0) of the section in the absence of the axial force N and infinitesimal of the bending moments M_x, M_y:

$$\begin{matrix} \{N = 0,\ M_x \to 0_{\pm},\ \varepsilon(x_0, y_0) = 0\} \Rightarrow X = (x_0, y_0), \\ \{N = 0,\ M_y \to 0_{\pm},\ \varepsilon(x_0, y_0) = 0\} \Rightarrow Y = (x_0, y_0), \end{matrix} \quad (18)$$

where symbol $\to 0_{\pm}$ means vanishing of the moment value M to zero in "+" or "–" region. Then, the stiffness centroid of the cross-section with the coordinates (x_0, y_0) is situated on the intersection of X, Y axes.

In the case of smooth relations $(17)_3$ the approach (18) leads us to the known formulas with the initial tangent modulus of elasticity (for example, in Біч (1991)). The set of stiffness centroid points in the cross-section forms a rod (element) axis.

The problem (17) is equivalent to the system of non-linear equations, which are written in the vector form:

$$f(\boldsymbol{q}, \boldsymbol{S}) = \boldsymbol{0}. \quad (19)$$

They include a three-dimensional vector of unknown section strains $\boldsymbol{q} = (\varepsilon_z, \varphi_x, \varphi_y) \in \boldsymbol{R}^3$ and the force vector $\boldsymbol{S} = (N, M_x, M_y) \in \boldsymbol{R}^3$ corresponding to it energy-wise, so the degree of kinematical indetermination is three.

The equation set (19) describes all possible states of the RC element section until its collapse. These equations contain non-convex and non-smooth relations, forming the vector function $\boldsymbol{f}(\cdot) \in \boldsymbol{R}^3$ (Alyavdin, Simbirkin 1999). The RC cross-section strength of a rod element is safe, if there is at least one solution to the equation set (19), when the conditions $(17)_4$ are satisfied.

4. Methods of solution

The linear iterative technique (Біч 1991; Bonet *et al.* 2004; Тур, Рак 2003; Zupan, Saje 2005;) is commonly used to solve the non-linear equation system (17) or (19). During its calculation the secant or tangent elasticity modulus of materials are corrected taking into consideration the changing properties of the section till the collapse of concrete or steel.

A disadvantage of the iteration method is its inefficiency for a non-monotonous function. This method does not allow to find the whole set of solutions for Eq. (19). It is suitable only for the analysis of the section design strength. An additional simplification and algorithm modification are required to analyze the cross-section state at all levels of loadings of the construction by the iteration methods.

The most suitable method for solving these problems is the one which allows to find the set of all possible solutions. If the system has a potential, then the global extremums of all the local ones of an objective function will be searched for

$$f_0(\boldsymbol{q},\boldsymbol{S}) \rightarrow \min, \qquad (20)$$

where $f_0(\cdot)$ is the potential of the equation set (19), $f_0 \in \boldsymbol{R}^1$ (see also function $W(\cdot)$, (15a), in sub-chapter 2.2). The function f_0 can also be taken up as:

$$f_0(\boldsymbol{q},\boldsymbol{S}) = \sum_{i=1}^{3} f_i^2(\boldsymbol{q},\boldsymbol{S}). \qquad (21)$$

The objective function f_0 (21) is non-smooth and non-differentiable and can have several local minimums within an accessible region.

The necessary (first-order) condition for the strict local minimum of the non-smooth function $f_0(\boldsymbol{q}, \boldsymbol{S})$ in point $\boldsymbol{q}$ is:

$$f_0^{\downarrow}(\boldsymbol{q}) > 0, \qquad (22)$$

where the left part is "the quickest descent speed" (Алявдин 2005) of function f_0 in point $\boldsymbol{q}$.

A set of solutions for the problem (19) and for the global minimum in (20) should be found using random search methods and the straight enumerative technique. In reference to Aliawdin (Алявдин 2005), a modified Newton-Raphson method with specific choice of initial points and genetic algorithm (GA) are proposed for finding the non-unique solutions of non-smooth problems for the reinforced concrete structures.

This modified Newton-Raphson method with a proper initial approximation has more rapid convergence than GA. But such method in its usual form does not permit to find possible solutions and is not applied to the non-monotonic functions $\boldsymbol{f}(\cdot)$ in the equations (19).

The genetic algorithm convergence has non-uniform, but stable behaviour. The algorithm allows to find all local and global extremums of the objective function, which may be non-convex and non-smooth. But the accuracy of the solution obtained by the simple genetic algorithm is lower than in case, when iteration numeric methods are applied.

To solve the non-linear equations (19) in this paper, the following hybrid algorithms combining the standard GA and gradient methods are offered below:

– Hybrid algorithm. An initial approximation localized in the extremum area is performed by the standard GA, and then the result is defined more precisely using numeric methods.

– A genetic algorithm with additional training of the leader (Тененев, Паклин 2003). The feature of the algorithm is the best individual leader created by the GA, which is trained using the gradient method and selected from the population (or from a random set of possible solutions). After that, the operation of gradient and genetic methods is realized in the parallel mode. The algorithm sequence is next:

1) $k = 0$. Population, which consists of m individuals $\{C^s, s = 1, m\}^k$, is generated by the GA. Number one took up an individual C^1 with the best index (minimum value of function (20)), which corresponds to $\boldsymbol{q}_b^k$ and $\boldsymbol{q}^k = \boldsymbol{q}_b^k$.

2) $k = k + 1$. Using the gradient method, next, the approximation of vector $\boldsymbol{q}^k$ is calculated. With the GA, next, the population $\{C^s, s = 1, m\}^k$ is created and the best individual is searched out, corresponding to the next vector $\boldsymbol{q}_b^k$.

3) If $f_0(\boldsymbol{q}_b^k) < f_0(\boldsymbol{q}^k)$, then $\boldsymbol{q}^k = \boldsymbol{q}_b^k$.

4) If $f_0(\boldsymbol{q}_b^k) \geq f_0(\boldsymbol{q}^k)$, then $C^1 = \boldsymbol{q}^k$.

5) If the stop condition is met, then *end,* else go to item 2.

During the equations solving, a more suitable way is to divide the force-strain relation into main parts. Then, it is better to use gradient methods for monotonic parts and hybrid methods described above as non-monotonic parts.

5. Features of calculating the cross-section under one-path loadings

The rectangle sections of the RC rod element, subjected to the forces (N, M_x), are considered for the strength analysis of the section under one-path loadings.

The calculations were made, for example, for the cross-sections with the following properties: the section with the dimension 400×400 mm, made from concrete with f_c = 28 MPa, f_{ct} = 2.2 MPa, E_b = 37 GPa, ε_{cu1} = –3.5‰, ε_{c1} = –2.0‰, ε_{ctu1} = 0.4‰, and from steel 4Ø28S500 (f_y = 500 MPa, E_s = 200 GPa, ε_{su} = 10‰).

The stress-strain diagram with limited descending branch of the concrete (Тур, Рак 2003) and bilinear stress-strain relation for the bar reinforcement were used (Fig. 2).

For the construction of the domain of the admissible internal forces in the cross-section (Fig. 4a) it is necessary to define the acceptable strains set (Fig. 4b), and to determine the corresponding strength condition of this problem.

The required strains domain is bounded by linear functions $(17)_4$ for the characteristic point of the cross-section. For this example such functions are: $\varepsilon_{cu1} = \varepsilon_z + \varphi_x \cdot (y_0 - h)$ for the most compressed concrete fibre (Fig. 4. 2c), and $\varepsilon_{su} = \varepsilon_z + \varphi_x \cdot (y_0 - a)$ for tensile reinforcement (Fig. 4. 1s), where h is a cross-sectional height and a – a value of concrete cover. Other borders are similarly evaluated for the main parts of the concrete and reinforcement work in the next characteristic point of the cross-section: cracking in tensile concrete for up and bottom fibre (Figs. 4. 3, 4), steel strain at yield stress (Fig. 4. 6), concrete strain corresponds to peak stress (Fig. 4. 5). Strains borders for maximal carrying strength (where two solutions exist) were computed by solving the optimization problem (20) for Eq. (19).

The domains of the admissible strength of the cross-section under one-path loadings for the internal forces (Fig. 4a) were computed by the substitution values of the boundary strain (Fig. 4b) in the stress equilibrium equation $(17)_1$.

The following features stipulated by non-monotonic stress-strain relations for materials are determined.

1) Non-uniqueness of solution. A typical diagram of the relation „*M*-φ" for one of the cross-sections is presented in Fig. 3. There are 2 regions, where the function is non-monotonic and has more than one solution for (20): one is in the part of concrete cracking (points 3 and 3*, Fig. 3) and the other is in the part of maximum (points 2 and 2*, Fig. 3), i.e. in the main parts of the function which characterize the section behaviour.

2) Two solutions exist in the part of the maximum value of the function through the whole length of the bearing capacity boundary. At that, the deviation between the collapse point and the maximum strength point is increasing at $M \to \min$, and $|N| \to \max$. The region with 2 solutions for (20) is shaded in Fig. 4.

In approaches Bich (Біч 1991), Zvezdov *et al.* (Звездов и др. 2002), the strength calculation by the deformation model is made using the iteration method until the section achieves the limit strains (collapse) of compressed concrete or tensioned steel. Otherwise, the conditions $(17)_4$ serve as a criterion for the RC section strength. In this case, the maximum strength of the section for certain initial secants can be up to 1…25% (maximum with $M = 0$) larger than the strength of the section under collapse (see the shaded area in Fig. 4a). Though the deviation between the maximum and collapse strength is less than 1% for the considerable part of length of the carrying capacity boundary, the presence of non-monotonic part of the function significantly worsens the convergence of the iteration method.

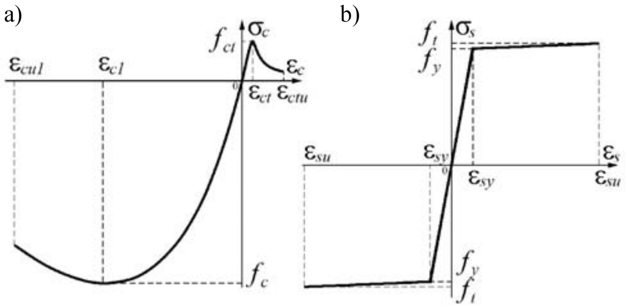

Fig. 2. Stress-strain relationships for concrete (a) and reinforced steel (b)

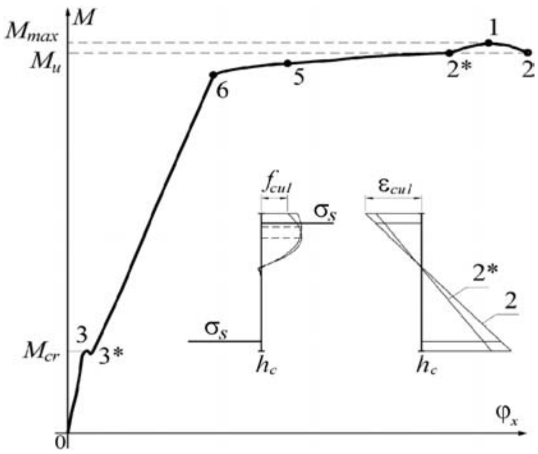

Fig. 3. Unified moment–curvature relation $y = M(\varphi_x)$ with $N \neq 0$ and section stress and strain distribution in the points with 2 solutions

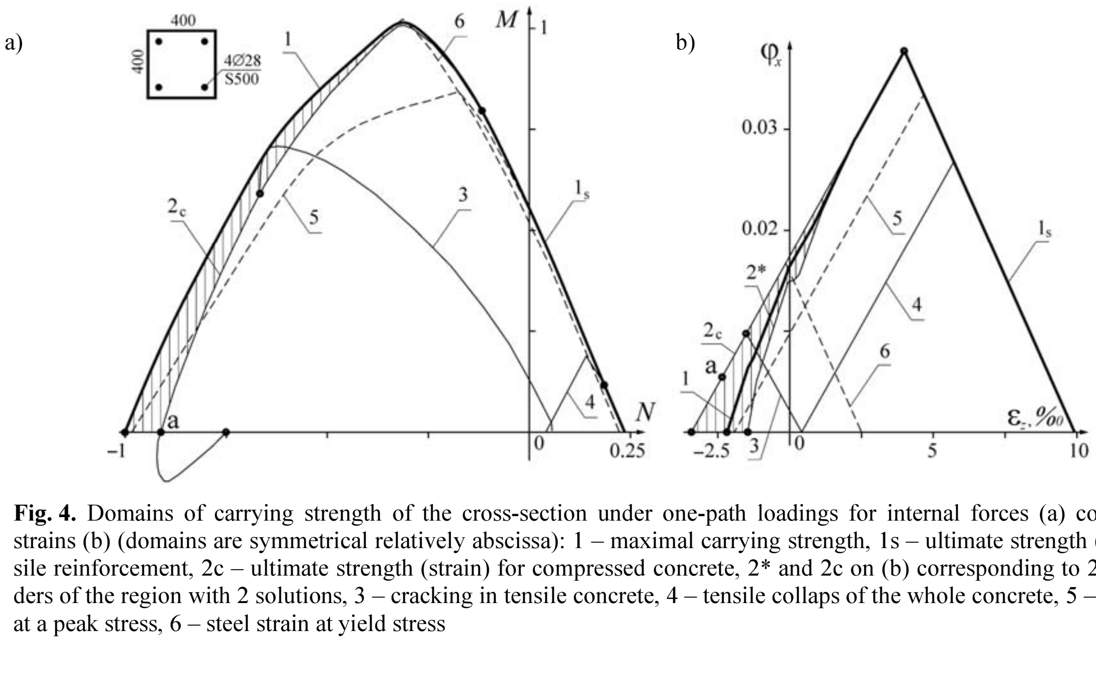

Fig. 4. Domains of carrying strength of the cross-section under one-path loadings for internal forces (a) corresponding to strains (b) (domains are symmetrical relatively abscissa): 1 – maximal carrying strength, 1s – ultimate strength (strain) for tensile reinforcement, 2c – ultimate strength (strain) for compressed concrete, 2* and 2c on (b) corresponding to 2c on (b) – borders of the region with 2 solutions, 3 – cracking in tensile concrete, 4 – tensile collaps of the whole concrete, 5 – concrete strain at a peak stress, 6 – steel strain at yield stress

Each solution of the set of Eq (19) is necessary for calculating the single cross-section strength in the general case for a statically indeterminate structure. However, for such a structure and for the whole element displacements we may take into account only the values of extreme (minimum) strains in the cross-section.

3) It was determined, that there are additional "false" solutions of Eq (19) due to a large size of the elemental area of the b) section which appears in the process of numerical evaluating of the integrals from the Eq (19).

4) There are more than 2 solutions for the set of equations (19), when $|N|\rightarrow$max and $M\rightarrow 0$; from the point *a* (Fig. 4) solutions like (ε_z, $\varphi \neq 0$) and (ε_z, $\varphi \neq 0$) exist. One solution (ε_z, φ) may have 2 couples of forces. In the part of cracking, up to 3 solutions of the equations set (19) are possible.

The results of the analytical model (17) proposed for calculating the strength of the element cross-section under one-path loads coincide with the ones realized in the commercial software created on the bases of normative documents. However, this model (17) allows to use the standardized relations „σ-ε" without iteration for the section stiffness whose coefficients depend on general, not standardized relation „*E*-ε".

6. Examples of shakedown analysis for the RC section

As an example for limit and shakedown analysis (on the basis of sub-chapter 2.2), we examine a square section of the statically determinated element described in the section 5.

Steel is presumed to be a cyclically stable material. For this example we assumed, firstly, that the stress-strain relationships for concrete under repeated loads are similar to the ones under monotonic loads, taking into account certain conditions (CEB 1996; Павлинов 1999; Korentz 2005) and that the cracks do not appear in the concrete under cyclic loading.

The RC element cross-section is influenced by the compression (tension) force *N* and the bending moment *M* in one of the principal planes. The values of the loadings are unknown at any specific time moment, but they vary within the limits $N^- \leq N \leq N^+$, $M^- \leq M \leq M^+$; for the strains it looks like: $\varepsilon^{N-} \leq \varepsilon \leq \varepsilon^{N+}$, $\varphi^{M-} \leq \varphi \leq \varphi^{M+}$, where ε and φ belong to the domain of the allowed strains under one-path loading (Fig. 4b). The case of the symmetric cyclic loadings is analyzed, when the maximal strain in compression and tension parts of the section is realized for the same combination of the axial force and the bending moment.

The final domain $\mathbf{\Omega}_{S_c}$ of the cross-section carrying capacity under repeated low-cyclic loadings is obtained as a region corresponding exactly to the progressive failure conditions (Fig. 5).

The example of a fictitious limit strain distribution, residual strain distribution and residual normal stresses for the case of the cross-section progressive failure in one of the points for vector $\boldsymbol{S}$ is given in Fig. 6.

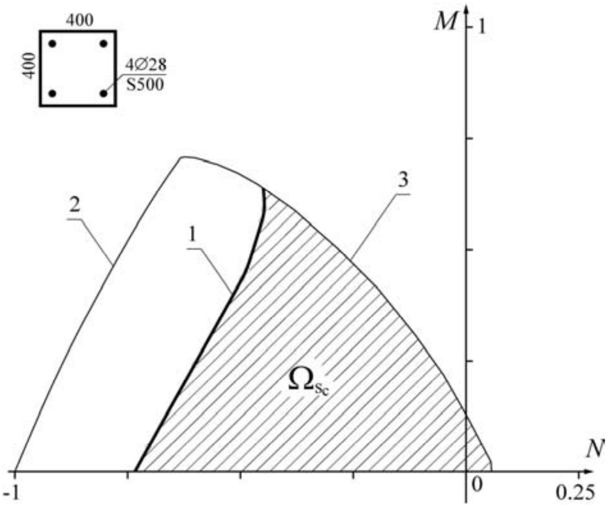

Fig. 5. Domain $\mathbf{\Omega}_{Sc}$ of the RC cross-section carrying strength under different loading conditions (domain is symmetrical relatively abscissa): 1 – low-cyclic loadings, progressive failure; 2 – one-path characteristic strength; 3 – the boundary of cracking in tensile concrete

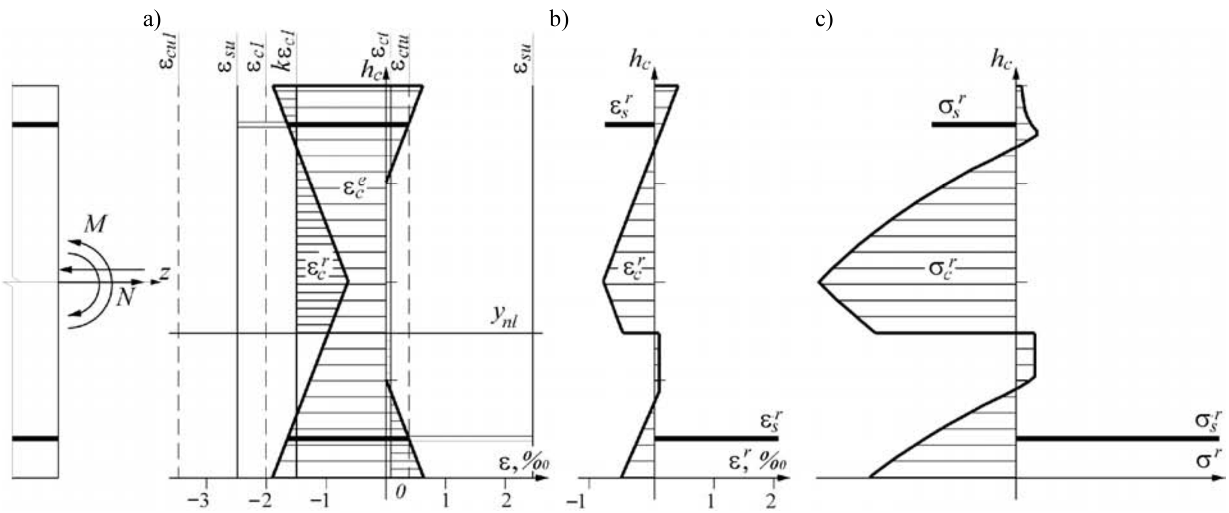

Fig. 6. Example of fictitious limit strain distribution (a), residual strain distribution (b) and residual normal stresses (c) for the cross-section progressive failure

The low-cyclic loads effect, for certain zero-to-compression or zero-to-tension cycle, reduces the cross-section strength degradation up to 25…30% of the section strength for one-path loading (Fig. 5, boundary 2), defined by the deformation model with the full stress-strain diagram (Fig. 2). The percentage area ratio for the section low-cyclic strength $\mathbf{\Omega}_{S_c}$ and characteristic one-path section strength domain (Fig. 5, bound 2) amounts to 58%.

7. Conclusions

An analytical model used to predict the ultimate strength of the RC element cross-sections under repeatedly alternating low-cyclic loadings are proposed in this study. The essential effect of the variable forces interaction on the values of the elements carrying capacity is determined.

The offered mathematical model also allows for the strength evaluating of the element cross-section under one-path loading. To obtain more than one solution of this non-smooth optimization problem it is advisable to apply the hybrid method combining the standard genetic algorithms and the gradient algorithms.

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GELŽBETONINIO STRYPO SKERSPJŪVIŲ RIBINĖ PUSIAUSVYRA IR PRISITAIKYMAS

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S a n t r a u k a

Nagrinėjamas nepleišėjančio gelžbetoninio elemento skerspjūvio stiprio skaičiavimas veikiant mažaciklei apkrovai. Naudojama netiesinė medžiagos įtempių-deformacijų priklausomybė. Pateikiami ribinės pusiausvyros ir prisitaikomumo analizavimo uždavinių matematiniai modeliai. Naudojama metodika skerspjūvio parinkimui vienos trajektorijos apkrovimo atvejais, nagrinėjamas uždavinio sprendinių nevieninteliškumas. Medžiagos plastiškumo sąlygos formuluojamos įtempiais arba deformacijomis. Sprendžiant netiesinio optimizavimo uždavinius gaunami paprasti dviejų tipų sprendiniai: tiesioginis ir atvirkštinis, atitinkantys progresuojančio arba kintamo plastiškumo ribinius plastinio suirimo atvejus.

Reikšminiai žodžiai: gelžbetonis, skerspjūvis, strypas, mažaciklis apkrovimas, ribinio būvio analizė, prisitaikymas, optimizacija.

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