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OUT-OF-PLANE VIBRATIONS OF CURVED NONPRISMATIC BEAM UNDER A MOVING LOAD

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Abstract. The forced vibrations of a curved-in-plane nonprismatic beam with a variable cross section and any curvature, generated by a load moving at a variable velocity are analyzed. Approximation with Chebyshev series and a generalized eigentransformation were used to solve the system of the partial differential equations describing the considered problem. The derived equations in their final form enable one to determine displacement and rotation functions for any beam. In order to verify the derived formulas the eigenproblem solution (used in the eigentransformation method) was compared with the one obtained by the finite element method.

Keywords: curved beam, nonprismatic beam, out-of-plane vibrations, moving load.

1. Introduction

The vibration of curved-in-plane beams is an important problem considering that such beams are often used in civil and mechanical engineering applications. The solution of the curved-in-plane beam problem is especially complicated when the system parameters, e.g. curvature and material and geometrical characteristics, are variable. In such a case, the coupled partial differential equations with two unknown functions, describing the problem cannot be separated in a simple way.

Even though the curved-in-plane beam problem has been investigated by many researchers, the literature on the dynamics of such system is not so voluminous as that on the vibration of arches. The free vibration of curved-in-plane prismatic beams was studied by Tufekci and Dogruer (2006), Lee *et al.* (2008), Kawakami *et al.* (1995) and Lee and Chao (2000). In paper of Tufekci and Dogruer (2006) the circular prismatic beam vibration problem was examined taking into consideration the effects of cross-sectional shear and rotational inertia. The first five eigenfrequencies of the system were determined using analytical methods. A similar model was adopted in Lee *et al.* (2008) where the eigenproblem was solved for differently supported beams with variable curvature (parabolic, elliptical and sinusoidal). The Runge-Kutta method was used to solve the derived differential equations. In Kawakami *et al.* (1995) the eigenproblem was solved using the discrete Green function. In the paper by Lee and Chao (2000) dealing with the vibration of a circular girder with a variable cross section, the power series method was employed. The differential quadrature element method (DQEM) was used to solve the eigenproblem in Chen (2008) and Li *et al.* (2008). The vibration of a thin-walled circular girder with a stepwise variable cross section was studied in Piovan *et al.* (2000). The analysis was limited to eigenfrequencies and eigenforms. Paper of Challamel *et al.* (2010) is an example in which the finite element method (FEM) was employed. The authors defined a finite element with 10 and 12 degrees of freedom and then used it to determine the eigenfrequencies of a curved composite beam.

The problem of the aperiodic vibration of a beam with any curvature and a variable cross section was solved using the Frobenius method combined with the dynamic stiffness method and the Laplace transformation in Huang *et al.* (1998). The inverse Laplace transform was determined using the Durbin method.

The vibration of curved-in-plane beams under a moving load is the subject of papers by Yang *et al.* (2001), Huang *et al.* (2000) and Wu and Chiang (2003). In paper of Yang *et al.* (2001) the vibration of a prismatic girder, generated by a concentrated harmonically variable load moving at a constant speed, were analyzed. The problem was solved analytically using the approximation method. Trigonometric Fourier series were used in the approximation. The solution was limited to the first term of the series. The results were compared with the ones yielded by FEM. Huang *et al.* (2000) studied the eigenproblem and aperiodic vibrations of a parabolic girder with a variable cross section, produced by a load moving at a constant rate. The dynamic stiffness matrix method was used to solve the problem. The solutions (within an element) were approximated by Taylor series. In research of Wu and Chiang (2003) a finite element with six degrees of freedom was defined and used to solve the considered problem. The results were compared with the results obtained by Yang *et al.* (2001).

In the present paper the problem of vibrations generated by a load moving at a variable speed is solved.

Since the eigentransformation method was used in the solution also the eigenproblem was solved. The obtained solutions apply to girders with arbitrarily variable parameters, such as curvature and material and geometrical characteristics.

No highly complex loads, such as inertia loads (moving sprung and unsprung masses or their complex combinations) are considered in this paper since its principal aim was to present a method for solving nonprismatic girders. The vibration problem has been extensively investigated for prismatic girders, for example by Fryba (1972), Szcześniak (1991) and Śniady (1976). The algorithm presented here can, of course, be used (after some modifications) to analyze the more complex problems mentioned above.

The considered problem was solved using the method employed by the author in his earlier papers to solve the problem of the eigenvibrations of the Euler beam 1999 and the Timoshenko beam 2006. The method uses Chebyshev series to approximate the differential equations and it is based on the method of the approximate solution of ordinary differential equations, described in Paszkowski (1975). It should be noted that the final solution for the analyzed form of the differential equation has a universal character and can be used to solve a system with any geometrical and material parameters.

The algorithm was employed to solve a numerical example, i.e. the problem of a curved-in-plane beam with a linearly variable cross section, whose axis is described by a chain curve. Four cases of load, differing in their loading rate were considered. Since the eigentransformation method was used in the solution of the problem, also the eigenfrequencies of the system were determined. In order to verify the derived formulas, the frequencies were compared with the ones determined by FEM.

The proposed method can be used to analyze real nonprismatic bar structures. In the case of reinforced concrete structures, it may be difficult to determine the variable substitute geometric and strength characteristics of their cross sections. The theoretical and experimental determination of the characteristics is the subject of papers by, among others, Bywalski and Kamiński (2011) and Kamiński and Pawlak (2011). In the authors' opinion the proposed method can also be used to solve more complex problems related to wave phenomena in heterogeneous nonprismatic bar systems, e.g. presented by M. Major and I. Major (2010) the problem of wave propagation in a laminar nonprismatic bar. The method will be applied to solve such problem in the next papers by the authors.

2. Formulation of problem

The vibrations of a curved-in-plane nonprismatic beam described in terms of the Bernoulli-Euler theory are considered. Loads: $Q_z(S,t)$, $M_y(S,t)$, $M_x(S,t)$ move on the beam at variable velocity $V(t)$ (Fig. 1). The beam's axis is a flat curve lying in plane XY, having length $2a$. It is also assumed that the distribution of the beam's geometrical and material parameters is symmetric relative to plane XY.
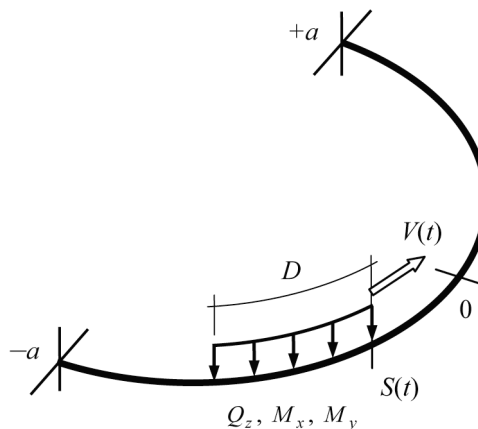

Fig. 1. Scheme of the system

Under the above assumptions the equations describing the out-of-plane vibrations of the beam have the form:

$$
\begin{cases}
-EI_y \frac{\partial^4 w}{\partial s^4} - 2 \frac{\partial EI_y}{\partial s} \frac{\partial^3 w}{\partial s^3} + \left(\kappa^2 GJ_s - \frac{\partial^2 EI_y}{\partial s^2} \right) \frac{\partial^2 w}{\partial s^2} + \frac{\partial}{\partial s}\left(\kappa^2 GJ_s \right) \frac{\partial w}{\partial s} \\
\quad + \left(\kappa GJ_s + \kappa EI_y \right) \frac{\partial^2 \vartheta}{\partial s^2} + \left(\frac{\partial}{\partial s}\left(\kappa GJ_s \right) + 2 \frac{\partial}{\partial s}\left(\kappa EI_y \right) \right) \frac{\partial \vartheta}{\partial s} = -f\left(q_z + \frac{\partial m_y}{\partial s} \right) + g\,\rho \frac{\partial^2 w}{\partial t^2}, \\[2ex]
\left(\kappa GJ_s + \kappa EI_y \right) \frac{\partial^2 w}{\partial s^2} + \frac{\partial}{\partial s}\left(\kappa GJ_s \right) \frac{\partial w}{\partial s} \\
\quad + GJ_s \frac{\partial^2 \vartheta}{\partial s^2} + \frac{\partial GJ_s}{\partial s} \frac{\partial \vartheta}{\partial s} - \kappa^2 EI_y\, \vartheta = -f\, m_x + g\, J_s^m \frac{\partial^2 \vartheta}{\partial t^2},
\end{cases}
\tag{1}
$$

and the internal forces are defined by the formulas:
– the bending moments:

$$\hat{m}_y = \frac{\hat{M}_y}{P_0\,a} = \frac{1}{f}\left(EI_y\left(-\frac{\partial^2 w}{\partial s^2} + \kappa \vartheta\right)\right), \qquad (2)$$

– the torsional moment:

$$\hat{m}_x = \frac{\hat{M}_x}{P_0\,a} = \frac{1}{f}\left(GJ_s\left(\kappa\frac{\partial w}{\partial s} + \frac{\partial \vartheta}{\partial s}\right)\right). \qquad (3)$$

The quantities in Eqs (1)–(3) are: $\vartheta(s,t) = \vartheta(S,t)$ – the angle of torsion of the cross section relative to axis X; $w(s,t) = W(S,t)/a$ – dimensionless displacements (determined in right-handed local coordinate system $x = X/a$, $y = Y/a$) perpendicular to the girder's plane; $s = S/a$ – a dimensionless parameter describing girder axis $s \in \langle -1, 1\rangle$; $\kappa(s) = a\,K(S)$ – the dimensionless curvature of the girder; dimensionless material and geometrical characteristics: $\rho = \overline{\rho}/\rho_0$ – density per unit length, $J_s^m = \rho_0\,a^2\,\overline{J}_s^m$ – a solid polar moment of inertia, $GI_y = \overline{GI}_y/EI_0$ – torsional rigidity, $EI_y = \overline{EI}_y/EI_0$ – flexural rigidity ($\overline{GJ}_s$ is a dimensional characteristic corresponding to GJ_s, etc.); $Q_z = P_0\,q_z/a$, $M_x = P_0\,a\,m_x/a$ – dimensionless load; and constants:

$$f = P_0\,a^2\big/EI_0, \qquad g = a^4\rho_0\big/EI_0, \qquad (4)$$

where parameters ρ_0, EI_0, P_0 are reference quantities. In the case of a curved bar, the cross section's generalized (dimensional) moment of inertia $\overline{I}_y$ occurring in the above is expressed by:

$$\overline{I}_y = \int_A \frac{a^4 z^2}{1 - \kappa(s)\,y}\,dx\,dy. \qquad (5)$$

The symbols representing local axes, displacements, external forces and internal forces are shown in Fig. 2.
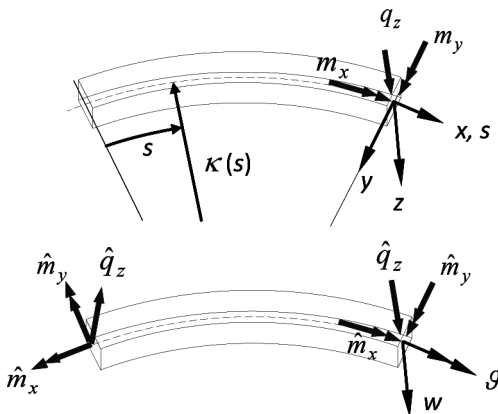

Fig. 2. Local coordinate system, external forces, displacements and internal forces

3. Solution of problem

The solutions of system of differential Eqs (1) are sought in the form of the following Chebyshev series:

$$w(x,t) = \sum_{l=0}^{\infty}{}' a_l[w]\,\mathrm{T}_l(x) = \sum_{l=0}^{\infty}{}' w_l(t)\,\mathrm{T}_l(x),$$

$$\vartheta(x,t) = \sum_{l=0}^{\infty}{}' a_l[\vartheta]\,\mathrm{T}_l(x) = \sum_{l=0}^{\infty}{}' \vartheta_l(t)\,\mathrm{T}_l(x), \qquad (6)$$

where: $\sum_{l=0}^{\infty}{}' a_l[f] = \frac{1}{2}a_0[f] + a_1[f] + a_2[f] + \ldots;$ $\mathrm{T}_l(x)$ is the l-th Chebyshev polynomial of the first kind.

The method presented in Appendix A and in the Ruta's papers (Ruta 1999, 2006) will be used to solve the problem. When reduced to a matrix form, system of Eqs (1) is expressed by the formula:

$$\hat{\mathbf{P}}_0(s)\,\mathbf{f}^{(4)}(s,t) + \hat{\mathbf{P}}_1(s)\,\mathbf{f}^{(3)}(s,t) + \hat{\mathbf{P}}_2(s)\,\mathbf{f}^{(2)}(s,t) +$$
$$\hat{\mathbf{P}}_3(s)\,\mathbf{f}^{(1)}(s,t) + \hat{\mathbf{P}}_4(s)\,\mathbf{f}(s,t) = \hat{\mathbf{P}}(s,t) + \hat{\mathbf{R}}_4(s)\,\ddot{\mathbf{f}}(s,t), \qquad (7)$$

where matrix functions $\hat{\mathbf{P}}_m(s)$, $m = 0, 1, 2, 3, 4$; and $\hat{\mathbf{R}}_4(s)$ vector $\hat{\mathbf{P}}(s,t)$ are expressed by the formulas (where simplified notation $f^{(p)}(s) = \partial^p f(s)/\partial s^p$ is used):

$$\hat{\mathbf{P}}_0 = \begin{bmatrix} -EI_y & 0 \\ 0 & 0 \end{bmatrix}, \qquad \hat{\mathbf{P}}_1 = \begin{bmatrix} -2\left(EI_y\right)^{(1)} & 0 \\ 0 & 0 \end{bmatrix},$$

$$\hat{\mathbf{P}}_2 = \begin{bmatrix} \kappa^2 GJ_s - \left(EI_y\right)^{(2)} & \kappa\,GJ_s + \kappa\,EI_y \\ \kappa\,GJ_s + \kappa\,EI_y & GJ_s \end{bmatrix},$$

$$(8)$$

$$\hat{\mathbf{P}}_3 = \begin{bmatrix} \left(\kappa^2 GJ_s\right)^{(1)} & \left(\kappa\,GJ_s\right)^{(1)} + 2\left(\kappa\,EI_y\right)^{(1)} \\ \left(\kappa\,GJ_s\right)^{(1)} & \left(GJ_s\right)^{(1)} \end{bmatrix},$$

$$\hat{\mathbf{P}}_4 = \begin{bmatrix} 0 & \left(\kappa EI_y\right)^{(2)} \\ 0 & -\kappa^2 EI_y \end{bmatrix}, \quad \hat{\mathbf{R}}_4 = \begin{bmatrix} g\,\rho & 0 \\ 0 & g\,J_s^m \end{bmatrix},$$

$$\hat{\mathbf{P}} = -f \begin{bmatrix} q_z + (m_y)^{(1)} \\ m_x \end{bmatrix}. \qquad (9)$$

Then matrix functions $\mathbf{Q}_m$ and $\mathbf{S}_m$ (see formulas (A.3)) are determined ($\mathbf{S}_m$ is calculated similarly as $\mathbf{Q}_m$ by replacing functions $\hat{\mathbf{P}}_m$ in formula (A.3) with functions $\hat{\mathbf{R}}_m$). Having substituted the coefficients of expansion of functions $\mathbf{Q}_m$, $\mathbf{S}_m$ and (9) into Chebyshev series into Eq. (A.2) one gets the infinite system of ordinary differential equations:

$$\sum_{l=0}^{\infty}{}' \begin{bmatrix} k_{11}(k,l) & k_{12}(k,l) \\ k_{21}(k,l) & k_{22}(k,l) \end{bmatrix} \begin{bmatrix} w_l \\ \vartheta_l \end{bmatrix} = \begin{bmatrix} P_1(k) \\ P_2(k) \end{bmatrix} +$$

$$\sum_{l=0}^{\infty}{}' \begin{bmatrix} b_{11}(k,l) & 0 \\ 0 & b_{22}(k,l) \end{bmatrix} \begin{bmatrix} \ddot{w}_l \\ \ddot{\vartheta}_l \end{bmatrix}, \qquad (10)$$

$$k = 0,1,2,3,\ldots$$

At this solution stage, elements $k_{ij}(k,l)$; $i,j=1,2$ of Eq. (10) in the above system contain the coefficients of expansion of functions: EI_y, $\kappa\, EI_y$, $\kappa^2\, EI_y$, GJ_s, $\kappa\, GJ_s$, $\kappa^2\, GJ_s$ as well as the coefficients of expansion of their first and second derivatives. After complex transformations are performed using the relation (Paszkowski 1975):

$$a_l = \frac{1}{2l}\left(a_{l-1}^{(1)} - a_{l+1}^{(1)}\right), \qquad l \neq 0\ , \qquad (11)$$

where $a_l = a_l[f]$ and $a_l^{(p)} = a_l[\partial^p f/\partial x^p]$, the coefficients of Eq. (10) assume the final form:

$$
\begin{aligned}
k_{11}(k,l) = &-8(k^2-9)(k^2-4)l\big((l+1)\big((k-1)e_{k-l}+(k+1)e_{k+l}\big)-2\sum_{j=0}^{l}(k+2j-l)e_{k+2j-l}\big)+ \\
&2l(k^2-9)\big((k+1)(k+2)\big(\kappa^2 g_{k-l-2}-\kappa^2 g_{k+l-2}\big)-2(k^2-4)\big(\kappa^2 g_{k-l}-\kappa^2 g_{k+l}\big)+ \\
&(k-1)(k-2)\big(\kappa^2 g_{k-l-2}-\kappa^2 g_{k+l-2}\big)\big),
\end{aligned}
\tag{12.1}
$$

$$
\begin{aligned}
k_{12}(k,l) = &\,2l(k^2-9)\big((k+1)(k+2)\big(\kappa^1 g_{k-l-2}-\kappa^1 g_{k+l-2}\big)-2(k^2-4)\big(\kappa^1 g_{k-l}-\kappa^1 g_{k+l}\big) \\
&+(k-1)(k-2)\big(\kappa^1 g_{k-l+2}-\kappa^1 g_{k+l+2}\big)\big) \\
&+2(k^2-9)(k^2-4)\big((k+1)\big(\kappa^1 e_{k-l-2}+\kappa^1 e_{k+l-2}\big)-2k\big(\kappa^1 e_{k-l}+\kappa^1 e_{k+l}\big) \\
&+(k-1)\big(\kappa^1 e_{k-l+2}+\kappa^1 e_{k+l+2}\big)\big),
\end{aligned}
\tag{12.2}
$$

$$
\begin{aligned}
k_{21}(k,l) = &\,2l\Big((k+1)(k+2)(k+3)\big((l-1)\kappa^1 e_{k-l-2}+(l+1)\kappa^1 e_{k+l-2}\big) \\
&-\big(2(k-2)(k-3)\big((k+5)l+6\big)\kappa^1 e_{k-l}+2(k+2)(k+3)\big((k-5)l-6\big)\kappa^1 e_{k+l}\big) \\
&+(k-1)(k-2)(k-3)\big((l+1)\kappa^1 e_{k-l+2}+(l-1)\kappa^1 e_{k+l+2}\big)-120\sum_{j=0}^{l}(k+2j-l)\kappa^1 e_{k+2j-l}\Big) \\
&-2l(k^2-9)\big((k+1)(k+2)\big(\kappa^1 g_{k-l-2}-\kappa^1 g_{k+l-2}\big)-2(k^2-4)\big(\kappa^1 g_{k-l}-\kappa^1 g_{k+l}\big) \\
&+(k-1)(k-2)\big(\kappa^1 g_{k-l+2}-\kappa^1 g_{k+l+2}\big)\big),
\end{aligned}
\tag{12.3}
$$

$$
\begin{aligned}
k_{22}(k,l) = &\,2l\,(k^2-9)\big((k+1)(k+2)\big(g_{k-l-2}-g_{k+l+2}\big) \\
&-2(k^2-4)\big(g_{k-l}-g_{k+l}\big)+(k-1)(k-2)\big(g_{k-l+2}-g_{k+l+2}\big)\big) \\
&-\tfrac{1}{2}\big((k+1)(k+2)(k+3)\big(\kappa^2 e_{k-l-4}+\kappa^2 e_{k+l-4}\big)-4(k+3)(k^2-4)\big(\kappa^2 e_{k-l-2}+\kappa^2 e_{k+l-2}\big) \\
&+6k(k^2-9)\big(\kappa^2 e_{k-l}+\kappa^2 e_{k+l}\big)-4(k-3)(k^2-4)\big(\kappa^2 e_{k-l+2}+\kappa^2 e_{k+l+2}\big) \\
&+(k-1)(k-2)(k-3)\big(\kappa^2 e_{k-l+4}+\kappa^2 e_{k+l+4}\big)\big),
\end{aligned}
\tag{12.4}
$$

$$
\begin{aligned}
b_{11}(k,l) = &\,\tfrac{1}{2}g\big((k+1)(k+2)(k+3)\big(\rho_{k-l-4}+\rho_{k+l-4}\big)-4(k^2-4)(k+3)(k^2-4)\big(\rho_{k-l-2}+\rho_{k+l-2}\big) \\
&+6k(k^2-9)\big(\rho_{k-l}+\rho_{k+l}\big)-4(k^2-4)(k-3)(k^2-4)\big(\rho_{k-l+2}+\rho_{k+l+2}\big) \\
&+(k-1)(k-2)(k-3)\big(\rho_{k-l+4}+\rho_{k+l+4}\big)\big),
\end{aligned}
\tag{13.1}
$$

$$
\begin{aligned}
b_{22}(k,l) = &\,\tfrac{1}{2}g\big((k+1)(k+2)(k+3)\big(i_{k-l-4}+i_{k+l-4}\big)-4(k^2-4)(k+3)(k^2-4)\big(i_{k-l-2}+i_{k+l-2}\big) \\
&+6k(k^2-9)\big(i_{k-l}+i_{k+l}\big)-4(k^2-4)(k-3)(k^2-4)\big(i_{k-l+2}+i_{k+l+2}\big) \\
&+(k-1)(k-2)(k-3)\big(i_{k-l+4}+i_{k+l+4}\big)\big),
\end{aligned}
\tag{13.2}
$$

$$
\begin{aligned}
P_1(k,l) = &-f\big((k+1)(k+2)(k+3)\,q_{k-4}-4(k+3)(k^2-4)\,q_{k-2} \\
&+6k(k^2-9)\,q_k-4(k-3)(k^2-4)\,q_{k+2}+(k-1)(k-2)(k-3)q_{k+4}\big) \\
&-(k^2-9)\big((k+1)(k+2)\,m_{k-3}-3(k-1)(k+2)(k^2-4)\,m_{k-1} \\
&+3(k+1)(k-2)\,m_{k+1}-(k-1)(k-2)\,m_{k+3}\big),
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
P_2(k,l) = &-f\big((k+1)(k+2)(k+3)t_{k-4}-4(k+3)(k^2-4)\,t_{k-2} \\
&+6k(k^2-9)\,t_k-4(k-3)(k^2-4)\,t_{k+2}+(k-1)(k-2)(k-3)\,t_{k+4}\big).
\end{aligned}
$$

The coefficients in formulas (12)–(14) are the coefficients of expansion of the following functions into Chebyshev series:

$$e_l = a_l[EI_z], \quad \kappa^P e_l = a_l[\kappa^P EI_z], \quad g_l = a_l[GJ_s],$$

$$\kappa^P g_l = a_l[\kappa^P GJ_s], \quad \rho_l = a_l[\rho], \quad i_l = a_l[J_s^m],$$

$$-q_l = a_l[q_z], \quad t_l = a_l[m_x], \quad m_l = a_l[m_y].$$

The first four blocks of system of Eqs (10) – $k = 0, 1, 2, 3$ are satisfied identitywise. The blocks' equations are replaced with equations describing the boundary conditions. When formulating equations stemming from the boundary conditions on the beam's end ($s = \mp 1$) one uses the expansions of function (6), formulas (2)–(3) for internal forces and the following formulas for calculating the Chebyshev polynomials in points $s = \mp 1$ (Paszkowski 1975):

$$T_n^{(m)}(1) = \begin{cases} 1 & \text{for } m = 0, \\ n^2 & \text{for } m = 1, \\ n^2(n^2 - 1) & \text{for } m = 2, \end{cases}$$

$$T_n^{(m)}(-1) = (-1)^{n-m} T_n^{(m)}(1). \tag{15}$$

The equations in the case of girder clamping are expressed by the formulas (for the girder's left and right end, respectively):

$$\vartheta(-1,t) = \sum_{l=0}^{\infty}{}' (-1)^l \vartheta_l = 0, \quad \vartheta(1,t) = \sum_{l=0}^{\infty}{}' \vartheta_l = 0,$$

$$w(-1,t) = \sum_{l=0}^{\infty}{}' (-1)^l w_l = 0, \quad w(1,t) = \sum_{l=0}^{\infty}{}' w_l = 0, \tag{16}$$

$$\frac{\partial w(-1,t)}{\partial s} = -\sum_{l=0}^{\infty}{}' (-1)^l l^2 w_l = 0,$$

$$\frac{\partial w(1,t)}{\partial s} = \sum_{l=0}^{\infty}{}' l^2 w_l = 0.$$

In the case of girder hinged fixing, the equations are expressed by the formulas (for the girder's left and right end, respectively):

– hinged fixing with rotation possible only around axis y, the first two equations are identical with Eqs (16), the third equation follows from condition $m_x(-1,t) = 0$, $m_x(1,t) = 0$ and it has the form:

$$\frac{\partial^2 w(-1,t)}{\partial s^2} = \frac{1}{3}\sum_{l=0}^{\infty}{}' (-1)^l l^2 (l^2 - 1) w_l = 0,$$

$$\frac{\partial^2 w(1,t)}{\partial s^2} = \frac{1}{3}\sum_{l=0}^{\infty}{}' l^2 (l^2 - 1) w_l = 0. \tag{17}$$

– hinged fixing with possible rotation around axes x and y:

$$\frac{\partial \vartheta(-1,t)}{\partial s} = -\sum_{l=0}^{\infty}{}' (-1)^l l^2 \vartheta_l = 0,$$

$$\frac{\partial \vartheta(1,t)}{\partial s} = \sum_{l=0}^{\infty}{}' l^2 \vartheta_l = 0, \tag{18}$$

the other two equations are Eqs (16) and (17).

The modified system of Eqs (10) limited to finite system $N = 2(m+1)$ of equations and with its terms rearranged is expressed by the formula:

$$\mathbf{K}\mathbf{q}(t) + g\,\mathbf{B}\ddot{\mathbf{q}}(t) = \mathbf{F}(t), \tag{19}$$

where: $\mathbf{q}(t) = [\mathbf{w}(t) \quad \vartheta(t)]^{\mathrm{T}}$; $\mathbf{w} = [w_0, w_1, ..., w_m]^{\mathrm{T}}$; $\vartheta = [\vartheta_0, \vartheta_1, \vartheta_2, ..., \vartheta_m]^{\mathrm{T}}$. The system is then transformed by multiplying its left side by $\mathbf{K}^{-1}$. Subsequently, matrix $\mathbf{K}^{-1}\mathbf{B}$ is reduced to Jordan's form $\mathbf{J} = \mathbf{S}(\mathbf{K}^{-1}\mathbf{B})\mathbf{S}^{-1}$. Since the solution of the eigenproblem for the considered system:

$$(g\mathbf{K}^{-1}\mathbf{B} - \lambda\mathbf{I})\mathbf{q} = \mathbf{0}, \tag{20}$$

leads to single eigenvalues λ, the transformation matrix assumes the form $\mathbf{S} = \mathbf{W}^{-1}$, where $\mathbf{W}$ is an eigenmatrix obtained by solving eigenproblem (20), and matrix $\mathbf{J} = \{\lambda\} = \mathrm{diag}[\lambda_1, \lambda_2, ..., \lambda_n]$ becomes a diagonal matrix. By substituting $\mathbf{q}(t) = \mathbf{S}^{-1}\mathbf{r}(t) = \mathbf{W}\mathbf{r}(t)$ into Eq. (19), multiplying the latter's left side by $\mathbf{S} = \mathbf{W}^{-1}$, introducing a damping term into Eq. (19) and performing simple transformations, one reduces system (19) to the following system of separated equations:

$$\{\lambda\}\ddot{\mathbf{r}}(\tau) + \{2\alpha\sqrt{\lambda}\}\dot{\mathbf{r}}(\tau) + \mathbf{r}(\tau) = \mathbf{W}^{-1}\mathbf{K}^{-1}\mathbf{F} = \mathbf{f}(\tau). \tag{21}$$

Parameter $\tau = t/\sqrt{g} = t\sqrt{EI_0/a^4\rho_0}$ in Eq. (21) represents dimensionless time.

In order to reduce the computing time and reject the vibration forms loaded with errors (the eigenforms corresponding to higher vibration frequencies), the incorrect (or insignificant) eigenforms are rejected. In this case, $k < N$ eigenforms are taken into account, and matrices $\mathbf{W}$ and $\mathbf{W}^{-1}$ then become rectangular matrices with dimensions respectively $N \times k$ and $k \times N$ (Kleiber 1995), and vector $\mathbf{r} = [r_1, r_2, ..., r_k]^{\mathrm{T}}$.

When the load function has the form $f_j(\tau) = 1\delta(\tau)$, solution $h_j(\tau)$ of a single equation of separated system of Eqs (21) (the Green function) is expressed by the formula:

$$h_j(\tau) = \frac{\omega_j}{\sqrt{1-\alpha_j^2}} \exp(-\alpha_j \omega_j \tau) \sin \omega_j' \tau, \tag{22}$$

where: $\omega_j = 1/\sqrt{\lambda_j}$, $\omega_j' = \omega_j\sqrt{1-\alpha_j^2}$. When the load is described by an arbitrary function, the solution is determined by calculating the convolution load with the Green function:

$$r_j(\tau) = \frac{n\omega_j}{\sqrt{1-\alpha_j^2}} \times \int_0^\tau f_j(s) \exp(-\alpha_j\omega_j(\tau-s)) \sin\omega_j'(\tau-s)\,ds. \tag{23}$$

Having calculated all the components of vector $\mathbf{r}(t)$, one calculates the sought vector $\mathbf{q}(t) = \left[\mathbf{w}(t), \vartheta(t)\right]^{\mathrm{T}} = \mathbf{W}\,\mathbf{r}(t)$.

4. Numerical example

The above algorithm was used to solve the following problem. A curved-in-plane nonprismatic beam in the form of a Catenary arch is loaded with a uniformly distributed load moving at a variable velocity. The beam's ends are connected with a foundation: hinged in point $S = -a$ and clamped in point $S = a$. The parametric equations of the Catenary arch as a function of arc length are:

$$X = A\operatorname{arcsinh} S/A, \quad Y = F + A - \sqrt{A^2 + S^2}, \quad (24)$$

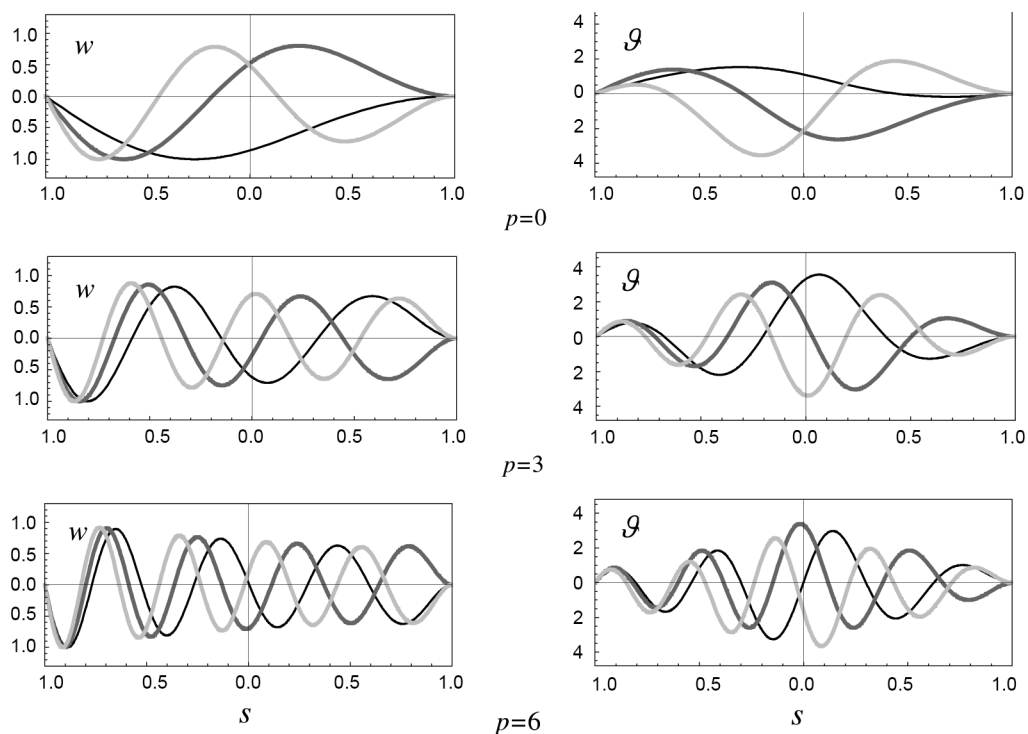where: $F = A\left(\cosh L/A - 1\right)$ is the height and $2L$ is the arch span (Fig. 3).
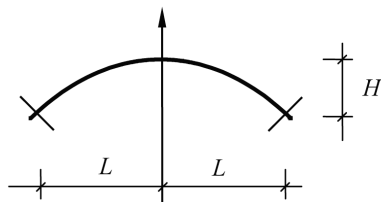
Fig. 3. Hinged-clamped Catenary arch

The beam's curvature is expressed by the equation $K(S) = A\Big/\sqrt{A^2 + S^2}$. When solving the problem in its dimensionless form it was assumed that the beam's cross section is a rectangle with constant width b and linearly variable height $h(S) = b\big(\mathrm{T}_0\left(S/a\right) + \eta\,\mathrm{T}_1\left(S/a\right)\big)$, where $\eta = 0.4$ and $S \in \langle -a, a \rangle$, and furthermore $H/L = 1.5$. The other parameters assume the following values: $\rho_0 = \rho(0)$, $EI_0 = EI(0)$. Because of the adopted method, first the eigenproblem was solved and the obtained eigenfrequencies were compared with the ones yielded by FEM (in which the system was divided into forty 3D beam elements with a linearly variable cross section) in order to verify the algorithm. The values of the first 10 eigenfrequencies obtained by the two methods are shown in Fig. 4. In order to check the convergence of the proposed method, the eigenproblem was also solved for different approximation base sizes: $m = 20, 30, 40, 50$. The obtained results are presented in Tables 1 and 2.

Table 1. Non-dimensional vibration frequencies $\omega = \Omega\,a^2\sqrt{\rho_0/EI_0}$ of H-C Catenary arch

	FEM	This paper
ω_1	3.35137	3.35160
ω_2	10.7146	10.7165
ω_3	23.3796	23.3832
ω_4	41.0290	41.0375
ω_5	63.4671	63.4836
ω_6	90.6378	90.6664
ω_7	122.532	122.579
ω_8	159.143	159.211
ω_9	200.467	200.569
ω_{10}	221.779	222.909

$p=0$

$p=3$

$p=6$

Fig. 4. Diagrams of eigenforms for H-C Catenary arch Form denotations: $p+1$ ——— ; $p+2$ ——— ; $p+3$ ——— ; where $p = 0, 3$ or 6

Table 2. Non-dimensional vibration frequencies $\omega = \Omega\, a^2 \sqrt{\rho_0/EI_0}$ of H-C Catenary arch for different approximation base sizes

	ω_1	ω_2	ω_3	ω_4	ω_5
$m = 20$	3.35351	10.7207	,23.3723	41.0299	63.5431
$m = 30$	3.35160	10.7165	23.3832	41.0375	63.4836
$m = 40$	3.35153	10.7164	23.3842	41.0379	63.4830
$m = 50$	3.35160	10.7165	23.3834	41.0376	63.4839
	ω_6	ω_7	ω_8	ω_9	ω_{10}
$m = 20$	90.6465	122.424	160.423	199.576	222.914
$m = 30$	90.6664	122.579	159.211	200.569	222.909
$m = 40$	90.6660	122.578	159.215	200.579	222.919
$m = 50$	90.6661	122.577	159.215	200.581	222.911

Then the obtained eigenforms were used to solve the forced vibration problem. The dimensionless function defining the considered moving load (Fig. 5) is expressed by the formula:

$$
q_z(x,\tau) = \begin{cases} q\big(\mathrm{H}(\min(1,s(\tau))-x) \\ \quad -\mathrm{H}(\max(-1,s(\tau)-d)-x)\big) \\ \qquad \text{when } \; -1 \le s(\tau) < 1+d \\ 0 \qquad \text{when} \qquad s(\tau) \ge 1+d, \end{cases} \tag{25}
$$

where d is the load length and $s(\tau)$ defines the position of the load's front.
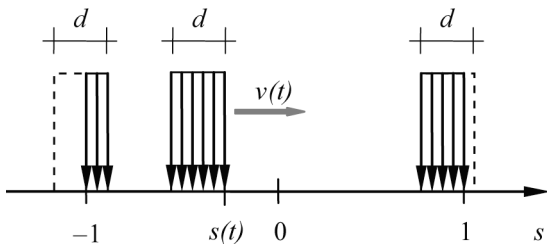
Fig. 5. Form of moving load

This (dimensionless) function is expressed by the formula:

$$
s(\tau) = -1 + \int_0^\tau \frac{v(\tau)}{v_c}\, d\tau, \tag{26}
$$

where: $v_c = \sqrt{EI_0/a^2\rho_0}$ is a reference velocity and $v(\tau) = V(t) = V(\tau\sqrt{g})$ ($V(t)$ is a dimensional velocity). Dimensionless load $q = 1000$. In the example, four cases differing in the distribution of moving load velocity were analyzed. Diagrams of relative velocity $v(\tau)/v_c$ for the particular cases are shown in Fig. 6a while the corresponding diagrams of function $s(\tau)$ are shown in Fig. 6b.

For the time-variable spatial distribution of the load the coefficients of expansion of function $q_z(x,\tau)$ are functions dependent on τ. Using the following relations:
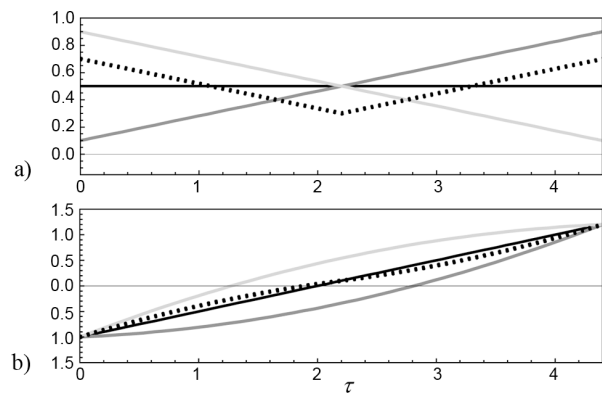
Fig. 6. Diagrams of: a) relative velocity $v(\tau)$ and b) function $s(\tau)$ for considered load cases

$$
a_0[\mathrm{H}(s_0 - s)] = 2 - \frac{2}{\pi}\arccos s_0,
$$

$$
\begin{aligned}
a_k[\mathrm{H}(s_0 - s)] &= -\frac{2}{\pi k}\sin(k\arccos s_0) = \\
&-\frac{2}{\pi k}\mathrm{U}_{k-1}(s_0)\sqrt{1-s_0^2}, \quad k = 1,2,3,\dots,
\end{aligned} \tag{27}
$$

the coefficients of expansion of function $q_z(x,\tau)$ were determined:

$$
\begin{aligned}
q_0(\tau) &= -q\frac{2}{\pi}\big(\arccos(\min(1,s(\tau))) \\
&- \arccos(\max(-1,s(\tau)-d))\big),
\end{aligned}
$$

$$
\begin{aligned}
q_k(\tau) &= -q\frac{2}{\pi k}\Big(\mathrm{U}_{k-1}(\min(1,s(\tau)))\sqrt{1-\min(1,s(\tau))^2} \\
&- \mathrm{U}_{k-1}(\max(-1,s(\tau)-d))\sqrt{1-(\max(-1,s(\tau)-d))^2}\Big), \\
&\qquad k = 1,2,3,\dots.
\end{aligned} \tag{28}
$$

where $\mathrm{U}_k(s)$ is a Chebyshev polynomial of the first kind. Fig. 7 shows the results in the form of displacement function $w(s,\tau)$ for the case when $-1 \le s \le 1$ and τ assumes such values for which the front of the moving load
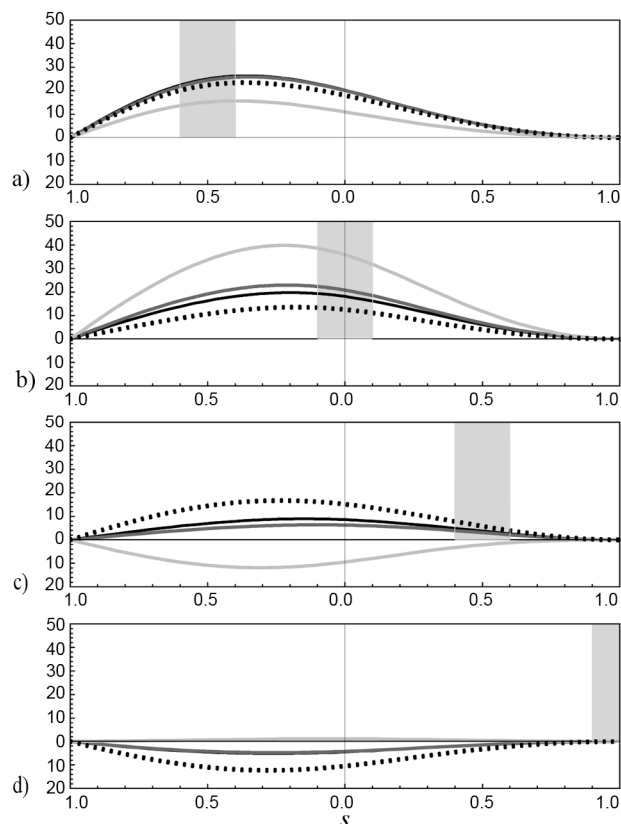
Fig. 7. Diagrams of beam displacements $w(s, \tau_i)$ at instants when load front is in respective points s_i: a) $s_1 - 0.5 + d/2$; b) $s_2 = 0.0 + d/2$; c) $s_3 = 0.5 + d/2$; d) $s_4 = 1.0 + d/2$ for considered load cases differing in velocity distribution. Denotations as in Fig. 6
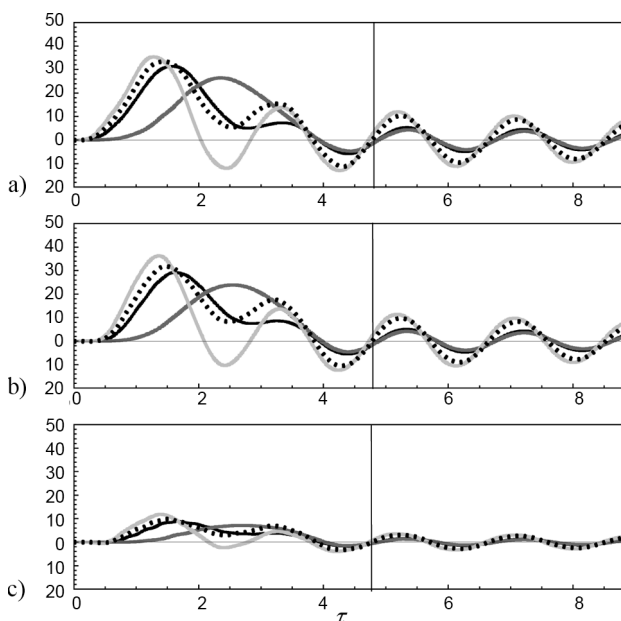
Fig. 8. Diagrams of displacements $w(s_j, \tau)$ for different beam points: a) $s_1 = -0.5$; b) $s_1 = 0.0$; c) $s_1 = 0.5$ for considered load cases differing in velocity distribution. Denotations as in Fig. 6

is respectively in points $s(\tau) = -0.5 + d/2$, $0.0 + d/2$, $0.5 + d/2$, $1.0 + d/2$. Fig. 8 shows diagrams of displacements $w(s_j, \tau)$ of the beam's particular points $s_j = -0.5$; 0.0; 0.5 for $0 \le \tau \le 2\tau_a$, where τ_a is the time of travel of the load on the girder, i.e. the time in which the front of the load covers distance $l = 1.0 + d - (-1.0) = 2.0 + d$.

5. Conclusions

The system's eigenfrequencies obtained in the example (Table 1) are in good agreement with the ones yielded by the finite element method. Table 2 shows that the solutions quickly converge towards the final result. It appears that for approximation base $m = 30, 40, 50$ the obtained solutions are slightly more accurate than the ones obtained for $m = 20$.

Problems like the one considered in this paper can be effectively solved and the computing time significantly reduced by transforming the matrix of system-of-equations coefficients to Jordan's form. In many cases this transformation reduces the coefficients matrix to a diagonal form.

Using the system of equations in its final form derived here one can directly solve the vibration problem for a girder with any parameters. It is enough to substitute appropriate coefficients of the expansions of the beam parameter functions into Chebyshev series into the formulas for the coefficients of system of equations (12)–(14). One should also remember to modify the equations for the boundary conditions. It should be noted that the proposed solution method can be directly used to solve the problem of aperiodic vibrations (not necessarily generated by a moving load. The numerical example shows that a rather small approximation base is needed to obtain accurate solutions.

Appendix A. Approximation method of solving differential equations with variable coefficients

A generalization of the theorem about ordinary differential equations (Paszkowski 1975) is used to solve the system of differential equations in this paper. The generalization consists in the application of the method of the approximate solution of a linear differential equation with variable coefficients to systems of linear differential equations (Paszkowski 1975):

$$\sum_{m=0}^{n} \hat{\mathbf{P}}_{\mathbf{m}}(x)\ \mathbf{f}^{(n-m)}(x) = \hat{\mathbf{P}}(x), \qquad (A.1)$$

where: coefficients $\hat{\mathbf{P}}_{\mathbf{m}}(x)$ are square matrices of degree N, and $\mathbf{f}(x)$ and $\hat{\mathbf{P}}(x)$ are N-element vectors.

In a special case, when system of Eqs (A.1) is a 4th-order ($n = 4$) system, the sought coefficients $a_l[\mathbf{f}]$ of the Chebyshev expansion of vector function $\mathbf{f}$ satisfy the following infinite system of algebraic equations:

$$\sum_{l=0}^{\infty}{}'\Big\{ 8(k^2-9)(k^2-4)(k^2-1)\,k\,\big(a_{k-l}[\mathbf{Q}_0]+a_{k+l}[\mathbf{Q}_0]\big)$$

$$+4(k^2-9)(k^2-4)(k^2-1)\,\big(a_{k-l-1}[\mathbf{Q}_1]+a_{k+l-1}[\mathbf{Q}_1]-a_{k-l+1}[\mathbf{Q}_1]-a_{k+l+1}[\mathbf{Q}_1]\big)$$

$$+2(k^2-9)(k^2-4)\big((k+1)\,\big(a_{k-l-2}[\mathbf{Q}_2]+a_{k+l-2}[\mathbf{Q}_2]\big)-2k\,\big(a_{k-l}[\mathbf{Q}_2]+a_{k+l}[\mathbf{Q}_2]\big)$$

$$+(k-1)\,\big(a_{k-l+2}[\mathbf{Q}_2]+a_{k+l+2}[\mathbf{Q}_2]\big)\big)$$

$$+(k^2-9)\big((k+1)(k+2)\,\big(a_{k-l-3}[\mathbf{Q}_3]+a_{k+l-3}[\mathbf{Q}_3]\big)-3(k-1)(k+2)\big(a_{k-l-1}[\mathbf{Q}_3]+a_{k+l-1}[\mathbf{Q}_3]\big)$$

$$+3(k+1)(k-2)\,\big(a_{k-l+1}[\mathbf{Q}_3]+a_{k+l+1}[\mathbf{Q}_3]\big)-(k-1)(k-2)\,\big(a_{k-l+3}[\mathbf{Q}_3]+a_{k+l+3}[\mathbf{Q}_3]\big)\big)$$

$$+\frac{1}{2}\big((k+1)(k+2)(k+3)\,\big(a_{k-l-4}[\mathbf{Q}_4]+a_{k+l-4}[\mathbf{Q}_4]\big)-4(k+3)(k^2-4)\,\big(a_{k-l-2}[\mathbf{Q}_4]+a_{k+l-2}[\mathbf{Q}_4]\big)$$

$$+6k(k^2-9)\,\big(a_{k-l}[\mathbf{Q}_4]+a_{k+l}[\mathbf{Q}_4]\big)-4(k-3)(k^2-4)\,\big(a_{k-l+2}[\mathbf{Q}_4]+a_{k+l+2}[\mathbf{Q}_4]\big)$$

$$+(k-1)(k-2)(k-3)\,\big(a_{k-l+4}[\mathbf{Q}_4]+a_{k+l+4}[\mathbf{Q}_4]\big)\big)\Big\}\,a_l[f]= \tag{A.2}$$

$$= (k+1)(k+2)(k+3)\,a_{k-4}[\hat{\mathbf{P}}]-4(k+3)(k^2-4)\,a_{k-2}[\hat{\mathbf{P}}]+6k(k^2-9)\,a_k[\hat{\mathbf{P}}]$$

$$-4(k-3)(k^2-4)\,a_{k+2}[\hat{\mathbf{P}}]+(k-1)(k-2)(k-3)\,a_{k+4}[\hat{\mathbf{P}}]\quad,\qquad\qquad k=0,\ 1,\ 2,\ 3,\ \dots$$

where:

$$\mathbf{Q}_0=\hat{\mathbf{P}}_0\,,\qquad \mathbf{Q}_1=-4\hat{\mathbf{P}}_0^{(1)}+\hat{\mathbf{P}}_1\,,\qquad\qquad \mathbf{Q}_2=6\hat{\mathbf{P}}_0^{(2)}-3\hat{\mathbf{P}}_1^{(1)}+\hat{\mathbf{P}}_2\,, \tag{A.3}$$

$$\mathbf{Q}_3=-4\hat{\mathbf{P}}_0^{(3)}+3\hat{\mathbf{P}}_1^{(2)}-2\hat{\mathbf{P}}_2^{(1)}+\hat{\mathbf{P}}_3\,,\qquad\qquad \mathbf{Q}_4=\hat{\mathbf{P}}_0^{(4)}-\hat{\mathbf{P}}_1^{(3)}+\hat{\mathbf{P}}_2^{(2)}-\hat{\mathbf{P}}_3^{(1)}+\hat{\mathbf{P}}_3\,.$$

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