AN ANALYSIS OF TRENDS AND CYCLES OF LAND PRICES USING WAVELET TRANSFORMS

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ABSTRACT. This paper analyses the low-frequency temporal variation of unit land prices in two single localities, the municipalities of Espoo and Nurmijärvi, of the Finnish real estate markets by the use of the wavelet transforms. These transforms are nonparametric orthogonal series estimators, which are capable of providing the necessary time and frequency information of land prices simultaneously in a highly flexible fashion. In the empirical section of this paper both the raw and the quality-adjusted unit price series are analysed. The estimated cycles and trends are all nonlinear and, in particular, the behavior of cyclical component is highly curvelinear and transient in time. The findings strongly suggest that the sub-markets in question are not in a steady state, but are continuously evolving in time. It seems that much of the temporal variation present in the untransformed series is, in fact, explained by the quality differences in the attribute variables. The use of quality corrections produced significant improvements in the internal reliability of results.

KEYWORDS: Unit land price; Trend; Cycle; Wavelet transform; Quality-adjustment

1. INTRODUCTION

Valid and reliable measurement of changes underlying property prices over time is a highly important practical and theoretical issue that has traditionally encountered major obstacles in the hedonic pricing methodology. One common solution is to use the time dummy variable technique, or so-called covariance analysis method, in order to provide the necessary quality-adjustments that stem from temporal variation. Albeit widely applied, this approach has a multitude of serious drawbacks. The core problem of using the indicator variable procedure seems to be its inflexibility as the effect of time is merely represented by a series of fixed discrete jumps, which are highly inaccurate in practice. The second problem is the lack of sufficient degrees of freedom, since estimation involves an extensive set of time-indexed dummy variables along with other regressors, at least one for each time period. The third problem is the necessity to choose the correct time interval, e.g. a period of one-year, that somehow reflects a typical decision-making horizon, although economic agents operate and actions take place simultaneously at various different time scales. The final major limitation of dummy variable technique is that the estimated model structure potentially suffers from acute multicollineary problem, which distorts the estimated parameters, when used to represent time evolution of property prices.

There are several other and more sophisticated approaches to account for the effects of time variability than indicator variables in the hedonic context. The structural time series approach is often a viable tool, which can sepa-

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rate long-term price movements (trends and cycles) from seasonal and irregular price variability. They are suitable for the analysis of non-stationary features of price series, in which time interval need not be equispaced. However, they seem to posses a serious disadvantage, if the observable series contain many outlying or influential orservations as is typical of many land value studies (see Hannonen, 2005a); the unobserved component estimates tend to overshoot (undershoot) the effect of levels. In those cases, the estimated trends and cycles can be non-informative per se, i.e. they fluctuate at too high or low a level to be meaningful as such, although they are importart part of the overall model structure in conjuction with the estimated hedonic prices. Fourier-based methods are also one interesting flexible alternative to standard indicator variable method. However, they have a hard time in reproducing non-stationary elements of the price series that are common in land markets. In essence, wavelet estimators extend the main ideas of Fourier analysis to situations, in which different forms of non-stationary behavior are expected.

The application of *wavelets* has rapidly increased over the last 20 years with over 1000 reviewed papers now appearing each year, and with the total number of over 16 000 articles being published to date in diverse fields of study (Addison, 2004; Crowley, 2005). The theoretical underpinnings of wavelets were completed in the late 1980's, whereas the 1990's witnessed a rapid increase in the number of different practical applications. These applied fields include (Graps, 1995), among others, signal and image processing, data compression, astronomy, acoustics, fractals, partial differential equations, medicine, seismology, speech discrimination, optics and nuclear physics. At the moment they are at the verge of entering mainstream econometrics (Schleicher, 2002; Gencay *et al*., 2002) with some applications in different fields of finance and economics (Ramsey, 2002). In real estate markets wavelets represent a totally new perspective of time-series modeling, whose empirical performance has not been investigated so far. This paper tries, in part, to fill that void.

Wavelet-based methods offer a viable alternative to the ubiquitous Fourier analysis (e.g. McMillen & Dombrow, 2001) and the associated modified transforms, such as the windowed (or short-time) Fourier transform, that have serious shortcomings in the modeling of complex, non-stationary phenomena, i.e. when the data-generating process under consideration is itself transient and evolves in time. Wavelet transforms are particularly suitable for analyzing different kinds of economic and financial data, including real estate, as most phenomena observed in these markets are thought to be time-varying and continuously changing. This (quasi) open system's nature with the aperiodic and intermittent temporal variation is a result of dynamic market conditions, which are driven by, inter alia, changes in consumers' preferences, investors' expectations and technological advantages.

There are at least four different uses of wavelet analysis in real estate economics and finance, which are (Ramsey 2000; Crowley, 2005):

1. Explanatory analysis to gain new insight of the data or phenomenon;
2. Density estimation and regression for evaluating spatially inhomogeneous surfaces;
3. Time-scale decomposition of disaggregate series;
4. Disaggregate forecasting.

In explanatory analysis the relevant question that can be tackled with wavelet estimators is the time scale versus frequency: in real estate economics and finance an examination of data to evaluate the presence and ebb and flow of frequency components is potentially valuable. In density estimation and regression wavelet estimators are superior to conventional kernel estimators, whenever local inhomogeneities are present (which are highly typical in the land market). In time-scale decomposition the crucial issue is the recognition that meaningful relationships between real estate economic variables can possible be found at the disaggregate (scale) level rather than an aggregate level. In disaggregate forecasting wavelet estimators provide a basis for establishing global versus local aspects of the series, therefore addressing the question of whether the forecasting is really possible. All of these can potentially lead to different kinds of gains, including improvements in the bias-variance trade-off, new insightful perspectives to the nature of data-generating process and enchainment of robustness to modeling errors (Ramsey, 2002). This paper concentrates on the first and third areas of uses emphasizing dynamic aspects of low-frequency land price formulation processes.

Research Problem and Methodology

Hedonic modelling approaches can be classified by their *flexibility* in discovering data structure to three categories (see, e.g. Pace, 1995):

(1) parametric approaches;
(2) semiparametric approaches;
(3) nonparametric approaches.

Parametric models that represent *data modelling culture* (Breiman, 2001) have formed the conventional dogma of hedonic pricing methods in real estate studies, where prespecified global models are estimated by means of ordinarily least squares or some modification thereof. Benefits of parametric approaches undeniably include: simplicity, interpretability, parsimony and comprehensive statistical theory. The fundamental obstacle, however, underlying the general use of parametric models is their inflexibility, i.e. inability to learn genuine structure about relationships between variables from the evidence in such decision-making settings, where theoretically unknown nonlinearity or nonstationarity is expected. This is the typical case when the effects of variables representing location and time are considered (McMillen and Thorsnes, 2003). The conventional result is that even the best parametric model tends to impose restrictions that substantially reduce the explanatory and predictive power of the hedonic equation (Pace, 1993 & 1995; Anglin and Gencay, 1996; inter alia). Unless the theory-laden parametric model coincides with the data-generating process (this is a strong assumption), profound misspecification errors may result imposing serious threats to their empirical validity.

Semiparametric and nonparametric approaches are representative of *algorithmic modelling culture* (Breiman, 2001) that emphasise aspects of learning the complex structure from the available facts and adaptability to the features underlying the data. They are particularly suitable for many hedonic modelling situations, where incomplete knowledge prevents the exact a priori specification of nonlinear or nonstationary components of functional form. Semiparametric estimators are, more precisely, an intermediate strategy between theory-laden and data-driven estimators that have restricted learning ability, i.e. semiparametric estimators can approximate functions only within some prespecified classes. Their practical relevance is mainly in balancing the dual goals of low specification error and high efficiency (Pace, 1995; Anglin and Gencay, 1996) and in enchaining the interpretability of results. Nonparametric estimators are by their nature highly flexible and, thus, capable of approximating very general classes of functions (e.g. smooth functions, square integrable functions) that neither require any restrictive, unwarranted prespecification of the model's functional form nor any specific error distribution assumption. This renders nonparametric estimators to be powerful data-driven tools, albeit highly sensitive to the problem of undersmoothing or overfitting, if local estimation is implemented unduly.

Wavelet analysis appears best suited to exploratory analysis of complex, non-stationary functions (or signals) (Bruce *et al.*, 1996). Statistically speaking, wavelets can be viewed as *nonparametric* orthogonal series estimators (Fan & Gijbels, 1996, pp. 26-39) that can effectively handle the discontinuities caused by different regime shifts that typically plague the economic and financial data. The main objective of this study is to model the time-series variability of land prices in two single localities (municipalities of Espoo and Nurmijärvi) of the Finnish land markets by using sophisticated techniques of wavelet transforms, whose potential is not realized in current practice. The focus is on *a flexible exploratory analysis of cyclical patterns and trends of unit land prices* in the given sub-markets, i.e. on studying long-term disequilibrium dynamics in these markets without restrictive a priori specifications about the nature of relationships. Therefore by assuming less, we are hopefully able to discover more about the genuine temporal structure of land prices. Wavelets are expected to give new insights into the nature of cycle and trend components of observed land price series as the restrictive assumptions of stationary and linearity can be avoided. They are especially suitable to the comprehensive multiresolution analysis of disaggregate series; the process of data aggregation and the concept of equispaced series do not play any fundamental role in the context of wavelet analysis or synthesis. Both unadjusted or raw price series and quality-adjusted or transformed price series are analysed; the necessary quality-adjustments, to achieve better comparability across the observations in the series, are performed using local regression tools with adaptive bandwidth choice procedure[1].

Previous Related Research

Wavelet transforms have not been studied before in the context of the real estate finance and economics. However, the work on different flexible estimators is immense in the real estate market literature. In real estate research, local polynomial modeling approaches have been popular: the Nadaraya-Watson estimator (Pace, 1993 & 1995; Anglin and Gencay, 1996; Gencay and Yang, 1996; inter alia) and the locally weighted least squares (Wallace, 1996; McMillen, 1996; Case *et al.*, 2004; Clapp, 2003 & 2004; inter alia) represent typical choices that are relatively simple to use, yet effective in their inferences. To summarize, in most cases the use of local modeling tools has yielded to significant improvements in the bias-variance trade-off. These local regression methods are, in theory, very similar to wavelet-based methods in their asymptotic minimax properties so that similar conclusions ought to be expected. The most similar, previous research are, however, (i) the paper of (McMillen & Dombrow, 2001), which uses Fourier approach to the estimation of house price changes, and (ii) the paper of (Wang, 2003) that applies Fourier-based time-series methods to the determination of common cycles in property and related markets. Wavelets analysis, in a way, extends the underlying idea of Fourier analysis into the non-stationary world, where e.g. trends, abrupt changes or chirps and volatility clustering of prices exist.

2. WAVELET TRANSFORMS

Fourier Transform versus Wavelet Transform

The classical Fourier transform's utility lies in its ability to analyse a function in the time domain for its frequency content (Graps, 1995), which has been successfully applied in a healthy amount of different practical applications and theoretical considerations[2]. The main problem of the Fourier approach is, however, that it gives information only about how much

[1] Wavelet and local regression tools have very similar statistical optimality features, both being (nearly) minimax in asymptotic sense (see Fan and Gijbels, 1996).

[2] The origins of Fourier analysis goes as far as the early 19th century, when Joseph Fourier presented that any 2π-periodic function could usefully be represented as an infinitive sum of sine and cosine functions.

of each frequency exists in the function, but it does not tell modeler when in time domain those frequency components appear; in other words, it has only frequency resolution and no time resolution. In practice, the Fourier transform has trouble reproducing transient signals and functions with abrupt changes, such as land prices. This means that the Fourier transform is only applicable to the analysis of *stationary processes*, whose frequency content does not change in time. In the case of stationary series, one does not need to know at what times frequency components appear, since all frequency elements exist at all times. When the time localization of the spectral components is desired, a transform giving the simultaneous time-frequency representation of the function is needed (Polikar, 2001).

Wavelet transform, on the other hand, is capable of providing the time and frequency information simultaneously; wavelets are localized both in the time and the frequency domain. This enables them to escape Heisenberg's curse[3], i.e. the physical principle that tells that one cannot be simultaneously arbitrarily precise with respect to the exact frequency and the exact time occurrence of that frequency of a function (Vidakovic, 1999, p. 35; Schleicher, 2002). Wavelets are designed to give good time resolution and poor frequency resolution at high frequencies and good frequency resolution and poor time resolution at low frequencies (Polikar, 2001). This approach is particularly useful if the function has high frequency bursts for short durations and low frequency components that last a longer period of time. Furthermore, wavelets tend to be much less sensitive to any errors in the data, since they can effectively separate the long-term movements from high-frequency details, whereas in the Fourier transform these errors - that are common in land price studies - can transform a smooth function into a jumpy one and vice versa, which is highly undesirable (Mackenzie, 2001). In essence, the wavelet transform is an ideal method for finding out the information content of signals that nonperiodic, noisy, intermittent and transient (Addison, 2004), which is highly typical of any economic time-series data, including the real estate.

Both transforms have, however, common similarities as well. Fourier analysis consists of breaking up a function into sine and cosine waves of various frequencies. Similarly, wavelet analysis is the breaking up of a function into shifted, dilated or compressed versions of the original wavelet. Fourier and wavelet transforms decompose a function into a weighted sum of its various frequency components (Boggess & Narcowich, 2001, p. 254). Fourier and wavelet transforms are both localized in frequency. The fast Fourier transform and discrete wavelet transform are both linear operations that generate similar data structures and both transforms can be viewed as a rotation in a function space to a different domain (Graps, 1995).

Mathematical Formulation of Wavelets[4]

Continuous Wavelet Transform

In general, wavelets are localized waves that serve as local basis functions in continuous time. *The continuous wavelet transform* of a signal $f(t)$ can be defined as (Kaiser, 1994, pp. 60-77; Goswami & Chan, 1999, pp. 67-72; Boggess & Narcowich, 2001, pp. 254-255; Polikar, 2001; inter alia):

$$\Psi_f^{\psi}(s,\tau) = \frac{1}{\sqrt{|s|}} \int_{-\infty}^{+\infty} f(t)\,\overline{\psi}\left(\frac{t-\tau}{s}\right) dt\,. \qquad (1)$$

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[3] More precisely, this law is known as Heisenberg Indeterminacy Principle, which says the exact position and the exact velocity of an object cannot be simultaneously determined.

[4] In this section a very short overview about the foundations of mathematical theory of wavelets is presented. For a more comprehensive accounts, see e.g. (Burrus *et al.*, 1998; Vidakovic, 1999; Percival & Walden, 2000).

In mathematical terms, the transform represents a convolution of the wavelet function with the signal $f(t)$ (Addison, 2004) and, thus, is a measure of similarity (or correlation) of the observed signal and the local orthonormal[5] basis functions (wavelets). The transformed signal is a function of scale parameter (s) that either dilates or compresses the basis functions and translation parameter (τ) that shifts the basis functions in time domain. $\overline{\psi}$ denotes a complex conjugation of ψ, the transforming function, which is called *the mother wavelet* or analyzing wavelet. It serves as a prototype for all other functions applied in the wavelet analysis. Wavelets are compactly supported, i.e. they have a finite length, which enables temporal localisation of signals features. They are usually quite irregular in shape, which makes them an ideal candidate for modeling different regime shifts (discontinuities) typically encountered with economic data. The degree of regularity depends on the chosen wavelets family and the number of vanishing moments (or approximation order).

The *scale* parameter is the crucial element in wavelet analysis, which is similar to scale used in maps (Polikar, 2001). High scales represent low-frequency variation, which give global information about the function, whereas low scales correspond to high-frequency component of variability, which brings out detailed micro-information about the signal. This varying scale enables the researcher, as Graps (1995) states, to see both the forest and the trees. Change of the scale from high to low means, in this context, zooming in and seeing the trees in the structure.

Each continuous wavelet function is expressible as (Vidakovic, 1999, p. 44; Ramsey, 2002):

$$\psi_{s,\tau} = \frac{1}{\sqrt{|s|}}\psi\left(\frac{t-\tau}{s}\right), \qquad (2)$$

where the term $1/\sqrt{|s|}$ ensures that the norm of $\psi_{s,t}$ is equal to one. The function $\psi_{s,t}$ is centered at τ with scale s; the energy of function is concentrated in a neighborhood of τ with size proportional to s (Crowley, 2005). The transform given in (1) is reversible and the original signal can be reconstructed via wavelet synthesis as (Kaiser, 1994, pp. 60-77; Vidakovic, 1999, p. 45; Boggess & Narcowich, 2001, pp. 254-255; Polikar, 2001; inter alia):

$$f(t) = \frac{1}{c_\psi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{1}{s^2}\Psi_f^\psi(s,\tau)\,\psi_{s,\tau}\,d\tau\,ds\,, \qquad (3)$$

given that the admissibility condition holds (e.g. Goswami & Chan, 1999, p. 69):

$$c_\psi = 2\pi\int_{-\infty}^{+\infty}\frac{|\psi'(\xi)|^2}{|\xi|}\,d\xi < \infty\,, \qquad (4)$$

where $\psi'(\xi)$ is the Fourier transform of ξ. This wavelet synthesis states that the signal $f(t)$ can be decomposed as a weighted sum of its spectral components. The admissibility condition, on the other hand, implies:

$$\int_{\Re}\psi(t)dt = \psi'(0) = 0\,. \qquad (5)$$

This means that wavelet is a waveform of effectively limited duration that has an average value of zero. It is not a restrictive condition in practice; it merely requires that the wavelet function is oscillatory (where its' name also stems from). In addition, wavelets are constructed so that they possess a higher order of vanishing moments (Goswami & Chan, 1999, p. 69).

In this study it is assumed that the set of wavelet functions forms an orthonormal set, i.e.:

$$\int_{\Re}\psi_m(t)\,\overline{\psi}_n(t)\,dt = \delta_{mn}\,, \qquad (6)$$

where δ_{mn} is the Kronecker delta function. This means that the basis functions are

[5] Orthogonality condition is not always fullfilled when using wavelets.

pairwise orthogonal to each other and they have a length of 1 (in other words, the information content of one wavelet is independent of the information content of another wavelet, which makes the calculation of wavelet transform much easier).

Wavelet Series Transform

One of the major problems for many practical applications underlying the continuous wavelet transform is the high redundancy of information. Therefore continuous wavelet function is usually discretized, which produces:

$$\psi_{j,k}(t) = s^{-j/2}\psi(s^{-j}t - k\tau)\,. \tag{7}$$

As orthonormality of wavelets is assumed, we obtain a wavelet series transform (discretized continuous wavelet estimator) (e.g. Polikar, 2001):

$$\Psi_f^{\psi}(j,k) = \int_{\Re} f(t)\overline{\psi}_{j,k}(t)\,dt \tag{8}$$

and

$$f(t) = c_{\psi}\sum_j\sum_k \Psi_f^{\psi}(j,k)\,\psi_{j,k}(t)\,. \tag{9}$$

Now *the exact definition of unit price of land parcel* in terms of wavelets can be given:

$$p_u(t) = \frac{\int_{\Re} f(t)m(t)\,dt}{\int_{\Re} m(t)\,dt}\,, \tag{10}$$

where $m(t)$ denotes the parcel size of land at time t, which is discretized for computational purposes. The discretized continuous wavelet estimator is not really a time-discrete, only the translation and the scale steps are discrete (Valens, 1999).

The wavelet function has, in many cases, a companion with a different gender, the scaling function or *father wavelet*, which represents a smooth baseline trend or the coarsest information of the function. For many functions, the low-frequency content is the most important part of the signal; in a way it identifies the function. Father wavelet also forms a set orthonormal basis as:

$$\phi_{j,k}(t) = s^{-j/2}\phi(s^{-j}t - k\tau) \tag{11}$$

with:

$$\int \phi(t)\,dt = 1\,. \tag{12}$$

This solves the problem of infinitive number of wavelets needed in the analysis.

Now an alternative way of expressing wavelets is in terms of solutions to sets of equations defined by low and high pass filters (Strang & Nguyen, 1997, pp. 22-27; Ramsey, 2002). When using dyadic blocks, we have:

$$\phi(t) = \sqrt{2}\sum_{k=0}^{n} l(k)\phi(2t-k)\,, \tag{13}$$

$$\psi(t) = \sqrt{2}\sum_{k=0}^{n} h(k)\phi(2t-k)\,, \tag{14}$$

where $l(k)$ is a low pass filter and $h(k)$ is a high pass filter. These can be expressed in terms of father and mother wavelets as (Ramsey, 2002):

$$l(k) = \frac{1}{\sqrt{2}}\int \phi(t)\phi(2t-k)\,dt\,, \tag{15}$$

$$h(k) = \frac{1}{\sqrt{2}}\int \psi(t)\psi(2t-k)\,dt\,, \tag{16}$$

$$h(k) = (-1)^k\, l(k)\,\cdot \tag{17}$$

The low pass filter averages and the high pass filter differences, i.e. the low pass filter works as a smoother in manner similar to moving averages that brings the coarsest information and high pass filter delivers the detailed information. These two filters are clearly not independent of each other, but related via equation (17), and are commonly known as a pair of quadrature mirror filters in the signal processing parlance.

Discrete Wavelet Transform

The two-scale relation or multiresolution formulation associates the scaling function and the wavelets; in particular, we have (Goswami & Chan, 1999, p. 96; Valens, 1999):

$$\phi(2^j t) = \sum_{k=0}^{n} l_{j+1}(k)\phi(2^{j+1}t - k), \quad (18)$$

$$\psi(2^j t) = \sum_{k=0}^{n} h_{j+1}(k)\phi(2^{j+1}t - k), \quad (19)$$

and now we can reconstruct the original $f(t)$ signal as:

$$f(t) = \sum_k \lambda_{j-1}(k)\phi(2^{j-1}t - k) + \sum_k \gamma_{j-1}(k)\psi(2^{j-1}t - k) \quad (20)$$

with

$$\lambda_{j-1}(k) = \sum_m l(m - 2k)\lambda_j(m), \quad (21)$$

$$\gamma_{j-1}(k) = \sum_m h(m - 2k)\gamma_j(m). \quad (22)$$

This is the essence of *the discrete wavelet transform* due to (Mallat, 1989), which has solved the problem of the non-existence of analytical solutions, and has offered a digitally implementable version of continuous wavelet transform without specifying any wavelet (Valens, 1999). The one-dimensional discrete (inverse) wavelet transform to is then used for actually computing the cyclical patterns and trends that are hidden in the original time-amplitude representation of the unit prices.

3. EMPIRICAL ANALYSIS OF LAND PRICES

Sample Data

Empirical modeling of land prices is like a complex crystal with many faces. If held up to the light a particular pattern of reflected light can be seen. If the orientation of the crystal is changed, then a completely different pattern of reflected light is formed. In land markets, the sensitivity to changes in data, i.e. different submarket, time period or scale and land use can lead to widely differing results, even in the context of unified methodology. Majority of that variation is explained by spatio-temporal movements: functional forms and parameters tend to vary with location and are not homogeneous throughout the data set, whereas temporally changing market conditions cause data-generating processes to evolve over time. To reduce the sample dependency, i.e. to improve the invariance of empirical study, the paper examines two data sets that are located in different submarkets and associated with partially non-overlapping time frames. The land type (land use) is, in contrast, fixed in order to reduce unnecessary heterogeneity of land prices. It represents undeveloped land not yet reached its highest and best use: vacant sites without a local detailed plan that are reserved for residential housing purposes.

The first sample data involve observations on land prices and the associated characteristics in the municipality of Espoo, a highly polycentric city, which lies inside the Helsinki metropolitan area with circa 225 000 habitants; its population is the second largest of the cities in Finland, which has experienced a rapid growth in its late history. The study period is

Table 1. Some common sample statistics for the municipality of Espoo

Variable (unit)	**Arithmetic mean**	**Minimum**	**Maximum**	**Std. Deviation**
Total price (€)	59126.40	3027.00	756846.00	61976.88
Square price (€/m^2) (unit price)	22.99	0.24	127.55	19.09
Parcel size (m^2)	4207.49	1000.00	28400.00	4613.75
Distance to CBD of Helsinki, L_2-metric (km)	17.22	7.61	27.29	4.34
Quarterly price index of single-family houses	154.06	116.80	187.30	22.35
Parcel type (=0 if whole site; 1 otherwise)	-	0	1	-

from January, 1990 to December, 2001 with total number of observations of 400. In terms of quality, this data set is preferable over the second sample, i.e. it has been pre-checked for any errors and analysed before (Hannonen, 2005b). In Table 1 are documented some standard sample statistics for the study variables in the case of Espoo.

The second sample contains observations of land prices and the associated attributes on 793 unbuilt land parcels without a local detailed plan sold in the period spanning from January, 1985 to March, 2004 in the municipality of Nurmijärvi, which lies just outside the Helsinki metropolitan area with approximately 36 000 habitants and three distinctive population centres (the parish village, Klaukkala and Rajamäki). Nurmijärvi has recently also witnessed several years of rapid expansion. In terms of quantity, this data set offers more opportunities to flexible modeling than the Espoo case, albeit it is more pronounced to any errors (e.g. recording errors), since it has not been pre-checked for hedonic modeling purposes. In Table 2 are documented some standard sample statistics for the study variables in the Nurmijärvi case.

Figures 1 and 2 depict the time-amplitude representations for the disaggregated and untransformed unit prices of the municipalities of Espoo and Nurmijärvi, respectively. They clearly show that the determination of appropriate signal is notoriously difficult visually, at least for the Espoo case (for the Nurmijärvi case there is some structure discernable in the Figure 2). That is why wavelet transforms are needed; they give a complete multiresolution representation of the underlying signal in a more interpretable and insightful form. In this paper the focus is on the low-frequency (trends and cycles) characteristics of the temporal variation.

Choice of the Wavelet Basis Functions

There exist several different criteria of how to choose an appropriate class of wavelet functions to represent data (see e.g. Ramsey, 2002). Symmetry of the wavelet basis function is useful for describing signals, which exhibit local symmetries. Orthogonality is highly desirable property of any transform as it significantly simplifies calculations associated with the transform in question. Smoothness of the wavelet is yet another important characteristic that is measured by the number of continuous derivatives of the basis function. In this empirical section orthogonal, compactly supported (with finite energy) and reasonable symmetric wavelet basis functions called *Symlets*

Table 2. Some common sample statistics for the municipality of Nurmijärvi

Variable (unit)	**Arithmetic mean**	**Minimum**	**Maximum**	**Std. Deviation**
Total price (€)	22 019.03	673.00	479 336.00	21 262.25
Square price (€/m^2) (unit price)	3.80	0.34	22.83	3.27
Parcel size (m^2)	7387.36	1 000.00	30 000.00	4 651.66
Distance to CBD of Helsinki (km)	33.11	22.28	44.91	5.22
Distance to parish village of Nurmijärvi (km)	8.55	0.32	16.06	3.37
Quarterly price index of single-family houses	154.30	100.00	226.00	32.00
Parcel type (=0 if whole site; 1 otherwise)	-	0	1	-

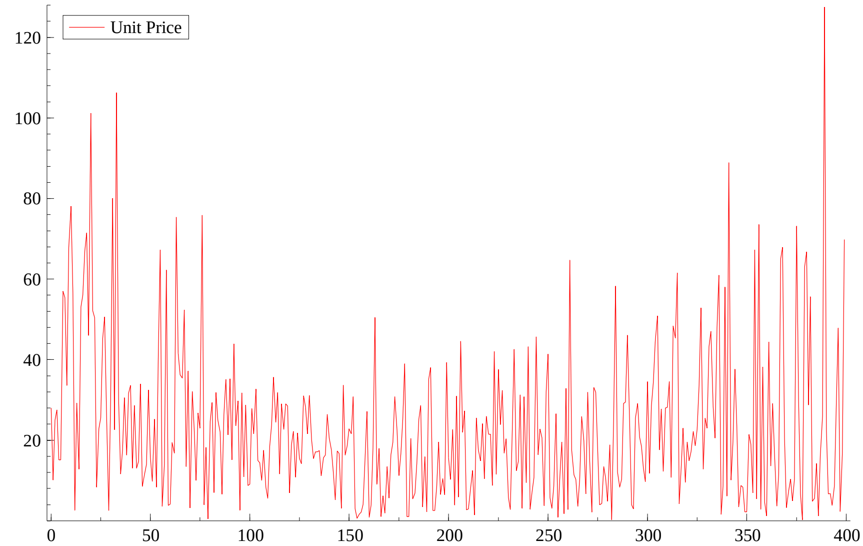

Figure 1. The time-amplitude representation of unit prices (case Espoo)

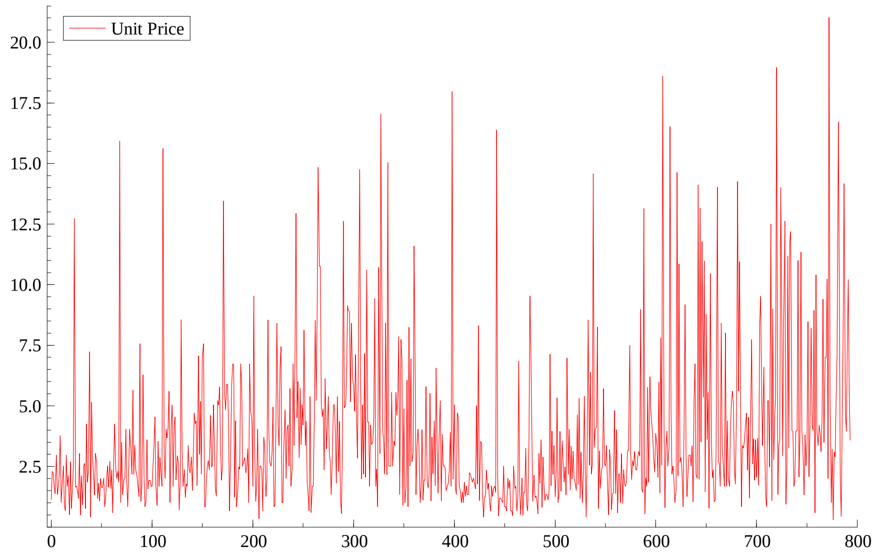

Figure 2. The time-amplitude representation of unit prices (case Nurmijärvi)

are applied that are modifications of the standard Daubechies family of wavelets. The approximation order is denoted by the number of vanishing moments, which is, by some empirical experimentation, chosen to be eight in the study. In general, increasing the number of vanishing moments makes wavelets more symmetric. Figure 3 depicts the chosen father and mother basis functions.

In the following sections unit prices of land parcels are empirically analysed with wavelets transforms under two different scenarios in

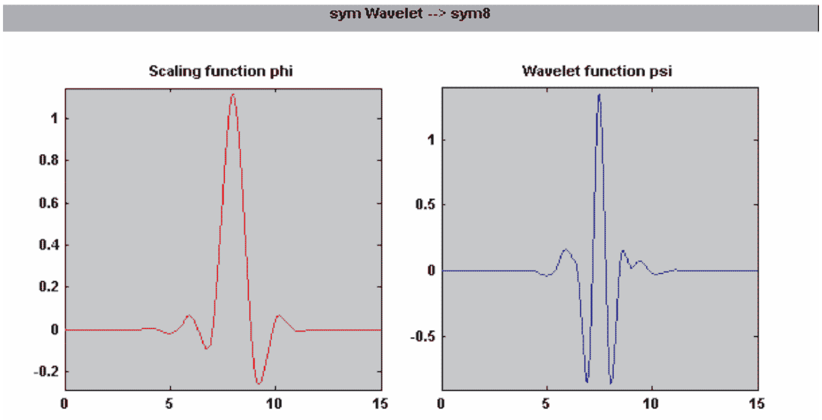

Figure 3. Father and mother wavelets of the study

both the Espoo submarket and Nurmijärvi submarket. In the first scenario unit prices of land are investigated by wavelets without any adjustments made with respect to differences in the attributes such as location or parcel size, i.e. unconditional low-frequency variation of land price series is examined. Under the second scheme unit prices are first quality corrected in an adaptive fashion by local hedonic modeling tools to account for any differences in the attribute space; after the series are homogenized they are fed as an input to the wavelet estimator to account for the unexplained low-frequency time variation of land prices. The low-frequency time variability of unit land prices is further divided into the cyclical variation and into the trend component by suitable adjusting the scale. The choice of proper scale involves subjective assessment and different formulations for the cycle and trend can be obtained by changing the scale of the analysis.

Case Espoo

Unadjusted Unit Price Series

Figure 4 represents the estimated price trend of the one-dimensional inverse discrete wavelet transform for the raw and the disaggregate form of time-series observations (i.e. no quality adjustments nor price aggregation are made) on land prices in the municipality of Espoo during a twelve-year period of 1990-2001. As can be seen from that figure, the untransformed series witness clear downward and upward movements over time, which a conventional linear time trend cannot properly reproduce. The unit land price has fallen from approximately 33 €/m^2 to circa 16 €/m^2 in the period spanning from the 1st of January 1990 to the turn of the years 1995 and 1996. In other words, the trend of unit prices has decreased quite steadily and almost linearly by circa 52 % in the period of six years. The five-year time period of 1996 to 2001 evidence a steady increase in the low-frequency price level, which goes from circa 16 €/m^2 to approximately 26 €/m^2 with the rise of circa 63%. This sounds intuitively plausible. The Figure 4 is suggestive that a piecewise linear approximation might be sufficient for the trend in the unconditional price series. The average high-scale level of unit prices of land in the period of 1990-2001 is approximately 22.5 €/m^2 (see Table 3), which parallels surprisingly closely with the common arithmetic average value (of circa 23 €/m^2) in that period (see Table 1). However, one should be cautious in interpreting average values, as the low-frequency levels and the unit prices per se are not normally distributed. Tables 1 and 3 also reveal that the standard deviation is almost 4 times higher in the original unit price series (Table 1) when compared to the corresponding variability measure of estimated levels (Table 3) - a significant improvement in the reliability of results.

Figure 5 shows the estimated cyclical variation of unit land prices in the time period of 1990-2001 for the unadjusted and the disaggregate time-series observations. The overall movements in the price level are naturally much more volatile; there are several peaks and valleys present in that figure. The pronounced dip in the series corresponds to the turn of the years of 1995-1996 as with the previous figure of the trend of unit prices. The other smaller troughs relate to the last quarters of 1998 and 2000 and the peaks to the first quarters of 1998 and 2000 and to the last quarter of 2001. This locally fluctuating behavior of unit prices might, in part, be a reflection of the high seasonal variability. There is a clear boundary problem in the beginning of the price series, which slightly distorts the interpretation (the dominant peak in the left cannot be seen at all). In general, the reconstructed approximation implies that price movements are intense over time, i.e. the local slopes (tangents) at any given point in time are high in absolute value at the given scale.

Quality-Adjusted Unit Price Series

A two-stage procedure is applied here. At the first phase, the unconditional unit price series are quality-adjusted using local regression methodology with adaptive bandwidth selection rule (see Appendix)[6], which is, at the

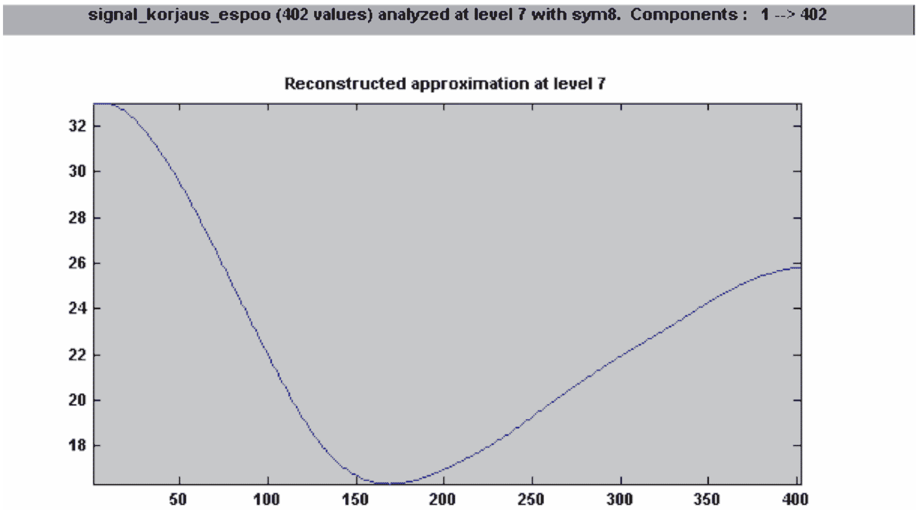

Figure 4. Estimated long-term movement of the untransformed unit price series

Table 3. Summary statistics of the estimated trend component

Mean	22.52	**Maximum**	33.01	**Standard Deviation**	4.86
Median	22.00	**Minimum**	16.34	**Median Abs. Deviation**	3.57
Mode	16.62	**Range**	16.67	**Mean Abs. Deviation**	4.01

[6] The house price index variable was dropped from the final hedonic model, as it appeared to distort the analysis here. The explanatory variables used in the hedonic regression to achieve the necessary quality correction are parcel size, parcel type and distance to the Helsinki CBD.

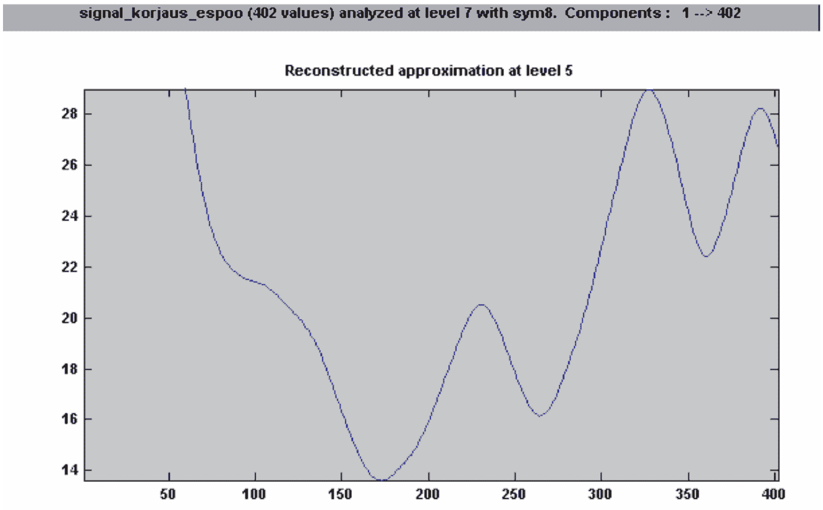

Figure 5. Estimated cyclical patterns for the raw unit price series

second stage, fed as an input to the wavelet estimator to account for the low-frequency time variation of land prices that is left unexplained at the first stage. Figure 6 depicts the estimated price trend of the one-dimensional inverse discrete wavelet transform for the quality-adjusted and the disaggregate form of time-series observations - i.e. no price aggregation is made - on land prices in the municipality of Espoo during a twelve-year period of 1990-2001. This figure ought to be contrasted with the Figure 4; the case of unadjusted, disaggregate time series. The overall shape is remarkable different implying that quality-adjustment does matter.

There are one dominant peak and trough observable in the estimated level component. This peak, perhaps surprisingly, occurs in the turn of the years 1992 and 1993, which is three years after the corresponding peak in the Figure 4. Also the pronounced drop occurs later; this time at the first quarter of 1999, lagging

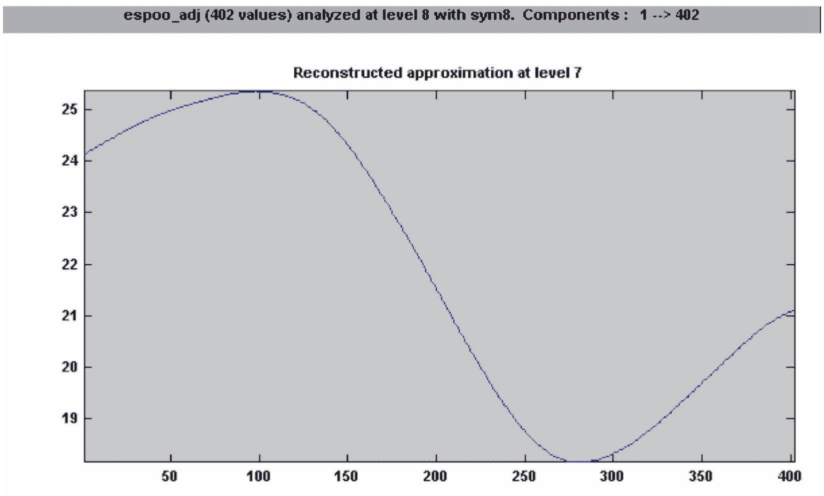

Figure 6. Estimated long-term movement of the transformed unit price series

again three years, as compared to the level in the Figure 4. The trend of unit prices has decreased almost linearly about six years (as in the Figure 4) from the value of 25 €/m^2 to 18 €/m^2, i.e. the decrease has been about 28 % in that time frame, which is, however, 24 % units smaller decrease than in the case of unadjusted series. The increases are also lower here; in the period of 1990-1992 the level has increased by about 8 % and in period of 1999-2001 by about 17 %, which both are significantly lower than in the Figure 4. The mean level value is circa 22 €/m^2, which is very close to the arithmetic average value of the series (see Table 1). The standard deviation of the estimated trend component is about (see Table 4) seven times smaller than in the case of original series in the Table 1; and almost 1.6 smaller than in the case of unadjusted levels in the Table 3. The original observations and level values are not normally distributed, so one has to be careful in the inferences.

Figure 7 shows the estimated cyclical behavior of unit land prices in the time period of 1990-2001 for the quality-adjusted and the disaggregate time-series observations. The overall movement in the price level is naturally much more volatile than in the previous case of Figure 6; there are several peaks and valleys present in the Figure 7. The estimated pattern of level component is clearly transient or non-stationary. The major peak occurs, surprisingly, at middle of the year 1994 (the Finnish economy as a whole were in recession), and the major valley in the last quarter of 1998. Figures 6 and 7 together indicate quite strongly, given the boundary problem, that the *low-frequency and quality-adjusted level values of unit prices seem to lag the untransformed price values about three years*. Table 4 shows

Table 4. Summary statistics of the estimated trend component

Mean	21.94	**Maximum**	25.36	**Standard Deviation**	2.70
Median	21.43	**Minimum**	18.15	**Median Abs. Deviation**	2.85
Mode	25.24	**Range**	7.21	**Mean Abs. Deviation**	2.51

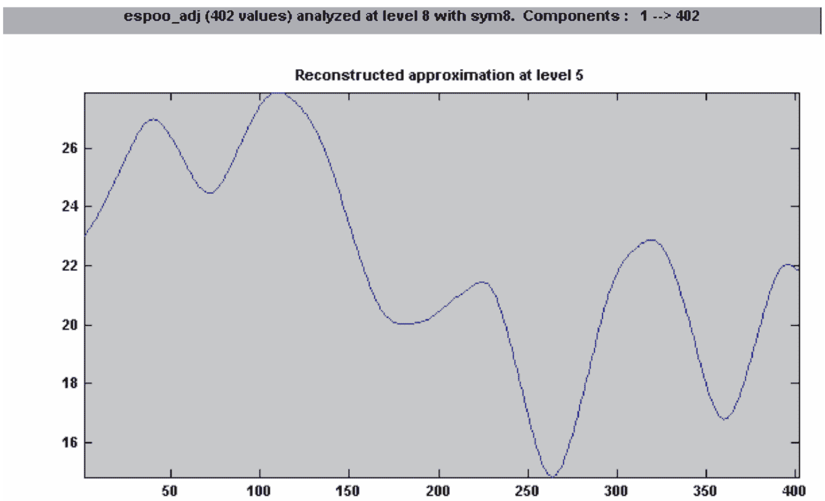

Figure 7. Estimated cyclical patterns for the transformed unit price series

that deviation-based measures for the estimated levels of trend are reduced significantly, if quality-adjustments are made, which increases the reliability that we can place on our inferences (The same applies to estimated cyclical movements, which are not tabulated here).

Case Nurmijärvi

Unadjusted Unit Price Series

Figure 8 represents the estimated price trend of the one-dimensional inverse discrete wavelet transform for the raw and disaggregated time-series observations (i.e. no quality adjustments nor price aggregation are made) on land prices in the municipality of Nurmijärvi during circa a twenty-year period of the 1st of January 1985 to the early March of 2004. The reconstructed signal shows a clearly discernible dominant peak and trough in the middle of the series. This dominant peak corresponds quite closely to the turn of the years 1989 and 1990 after which the level of unit prices has fallen systematically about eight years (the time period of 1st of January, 1990 to 31st of December, 1997) from 4.5 €/m^2 to almost 2.5 €/m^2 (i.e. by circa 44 %). The dominant valley thus occurs in the turn of the years 1997 and 1998. Since that point, the low-frequency component of unit price fluctuations has steadily increased over six years by an impressive 160 %. This should be contrasted with the five-year period of 1985-1989 in the same figure, where the increase in the high-scale level of the series has been about 125 %. The arithmetic average value of estimated level component (see Table 5) is identical to the arithmetic average value of original series (see Table 2); however, the variability in the estimated levels is almost three times smaller than

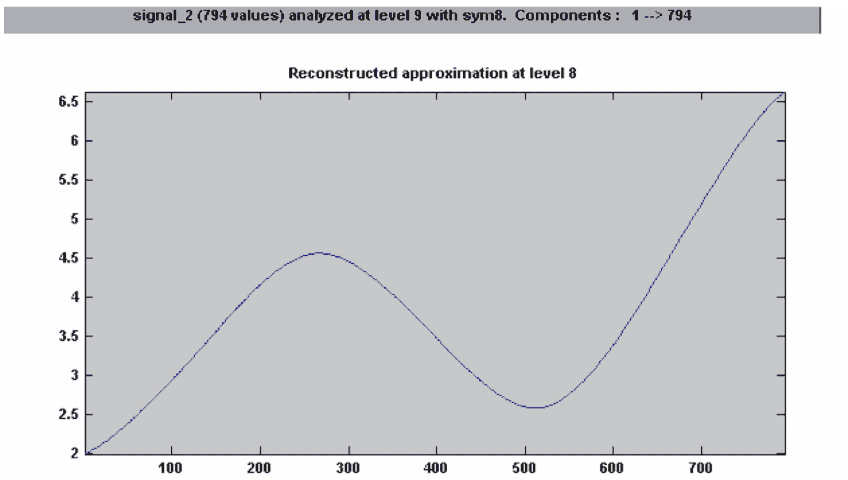

Figure 8. Estimated long-term movement of the untransformed unit price series

Table 5. Summary statistics of the estimated trend component

Mean	3.80	**Maximum**	6.63	**Standard Deviation**	1.11
Median	3.69	**Minimum**	1.99	**Median Abs. Deviation**	0.79
Mode	4.54	**Range**	4.64	**Mean Abs. Deviation**	0.90

in the original series of the Table 2. In other words, the reliability of results improves significantly if wavelet estimator is applied.

Figure 9 shows the estimated cyclical variation of unit land prices in the time period of 1990-2001 for the unadjusted and the disaggregate time-series observations. The overall movement in the price level is significantly more variable than in the previous graph; several peaks and valleys occur that are hidden in the original data. The pronounced dip corresponds to the last quarter of 1996, which differs somewhat from the previous figure of the trend of unit prices. The other smaller valleys correspond to the turn of the years 1988 and 1989 (the first valley is, in fact, so minor that it is not reported here); to the last quarter of 1998; to the first quarter of 2000 and to the last quarter of 2002. The dominant peak occurs in the 3rd quarter of 1990, which slightly differs from the information given by the estimated trend component. Scale seems to matter. The other peaks occur in the 3rd quarter of 1988; in the first quarter of 1998; in the second quarter of 2000 and in the first quarter of 2002. These minor changes can be, partly, an indication of strong seasonal variability that is present in the data.

Quality-Adjusted Unit Price Series

Figure 10 depicts the estimated price trend of the one-dimensional inverse discrete wavelet transform for the quality-adjusted[7] and disaggregated time-series observations on land prices in the municipality of Nurmijärvi during circa a twenty-year period spanning from the 1st of January 1985 to the early March of 2004. There exist a clearly discernible dominant peak and valley in the figure; the dominant peak relates to the turn of the years 1988 and 1989 after which the quality-adjusted unit price has fallen steadily over nine years; the pronounced dip in the this series occurs in the first half of the year 1998. The unit price level has decreased steadily from 3.5 €/m^2 to 2.9 €/m^2 with a relative drop of circa 17 %, which is surprisingly low, if compared to the

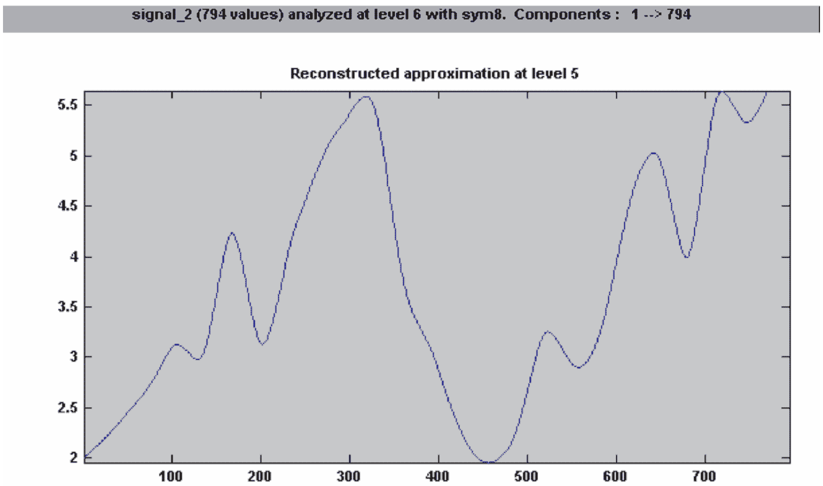

Figure 9. Estimated cyclical patterns for the untransformed unit price series

[7] The house price index variable was dropped from the final hedonic model, as it appeared to distort the analysis. The explanatory variables used in the hedonic regression to achieve the necessary quality correction are parcel size, parcel type, distance to the Helsinki CBD and distance to the parish village of Nurmijärvi.

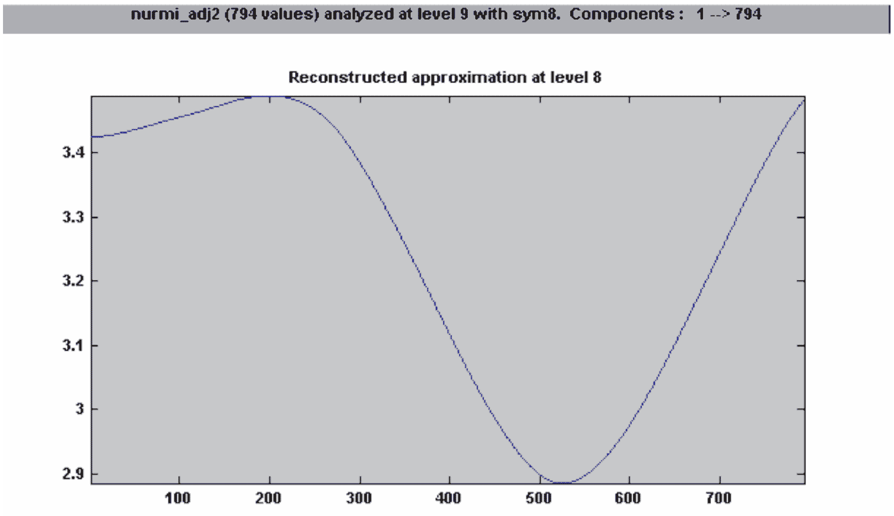

Figure 10. Estimated long-term movement of the transformed unit price series

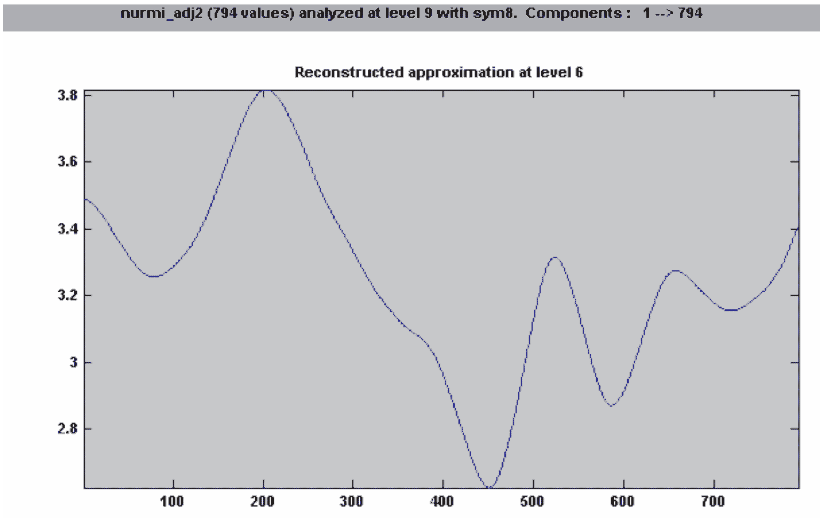

Figure 11. Estimated cyclical patterns for the transformed unit price series

Table 6. Summary statistics of the estimated trend component

Mean	3.25	**Maximum**	3.49	**Standard Deviation**	0.22
Median	3.32	**Minimum**	2.89	**Median Abs. Deviation**	0.16
Mode	3.48	**Range**	0.60	**Mean Abs. Deviation**	0.20

unconditional series. Since that point, the quality-adjusted unit price has increased steadily and linearly almost six years reaching the high unit price level of the turn of years 1988 and 1989. The quality-adjusted high-scale mean level of unit price series is about 3.25 €/m^2, which is circa 15 % smaller than the arithmetic average value of the Table 2. This sounds intuitively plausible. It is important to notice that standard deviation of quality-adjusted level values of unit price is fifteen times (!) smaller than the corresponding value of original series in the Table 2. This implies significant improvements in the reliability of results.

Figure 11 represents the estimated cyclical behavior of unit land prices in the time period of 1st of January, 1985 to early March of 2004 for the quality-adjusted and the disaggregate time-series observations. The estimated level component is naturally more variable now, as compared to the previous graph, with several local peaks and valleys. The striking feature of this fluctuation is that the volatility of the series is decreased in the period of middle 1996 to early March, 2004. Now the pronounced dip of the unit price series occurs 1½ to 2 years before (middle of 1996) than in the figure 10. The dominant peak is, however, the same relating to the turn of years 1988 and 1989.

4. CONCLUSIONS

This paper has analysed the low-frequency or high-scale temporal variability of unit land prices in two single localities, the municipalities of Espoo and Nurmijärvi, of the Finnish real estate markets by the use of relatively new modeling tools: the wavelet transforms. These transforms are nonparametric orthogonal series estimators, which are capable of providing the necessary time and frequency information of land prices simultaneously in a highly flexible fashion. They are particularly suitable for the multiresolution analysis of complex, non-stationary signals that are plagued with different kinds of regime shifts; a typical case in the analysis of unit prices of land parcels.

In the empirical section of this paper both the raw and quality-adjusted unit price series were modeled using the one-dimensional discrete inverse wavelet transform to account for cyclical patterns and trends that were hidden in the original time-amplitude representation of the unit prices. The Symlets, with eight vanishing moments, were chosen by some empirical experimentation as the proper class of wavelet basis functions to represent data. It turned out that wavelet estimators yielded, in all cases, to meaningful and plausible representations for the low-frequency unit price fluctuations. The estimated cycles and trends were all nonlinear and, in particular, the behavior of cyclical component was highly curvelinear and transient in time. These findings strongly suggest that the sub-markets in question are not in a steady state, but are continuously evolving in time.

The modeling of the raw and the quality-adjusted unit price series resulted to different kinds of descriptions for the high-scale temporal variability. It seems that much of the temporal variation present in the untransformed series is, in fact, explained by the quality differences in the attribute variables. As a result the use of unadjusted price series tended to exaggerate the salient features of low-frequency fluctuations. The use of quality corrections produced seven to fifteen times smaller internal variability when compared to the original, untransformed series; and also significantly reduced internal variability when contrasted to the unadjusted wavelet-based series. Quality-adjustments seem to matter, which were generated in the study by the local regression tools. An important empirical finding in the Espoo case was that the low-frequency and quality-adjusted level values of unit prices seem to lag the untransformed price values about three years. The same phenomenon was not observed in the Nurmijärvi case.

In contrast to conventional methods (e.g. dummy time variable technique or Fourier analysis) the wavelets estimators can, in principle, handle all sorts of non-stationary price behavior in the land market, i.e. it can well manage complex, non-periodic, noisy, intermittent and transient signals, whose valid measurement would otherwise prove to be highly difficult. Empirical analysis supported this claim; wavelet estimators provided insightful descriptions for the trend and cyclical component of the raw and quality-adjusted price series. The choice of the proper scale was found be an important modeling issue, where subjective assessment is needed.

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SANTRAUKA

ŽEMĖS SKLYPŲ KAINŲ TENDENCIJŲ IR CIKLŲ ANALIZĖ TAIKANT WAVELET TRANSFORMACIJĄ

Marko HANNONEN

Šiame darbe, naudojant Wavelet transformaciją, analizuojamas mažo dažnio laikinas žemės sklypų kainų svyravimas dviejose skirtingose Suomijos nekilnojamojo turto rinkos dalyse: Espoo ir Nurmijärvi savivaldybėse. Šios transformacijos yra beparametrės stačiakampės sekų vertintojos, kurios padeda vienu metu ir labai lanksčiai pateikti reikiamą informaciją apie laiką bei dažnumą, kai kalbama apie sklypų kainas. Empirinėje šio darbo dalyje analizuojamos tiek neapdorotos, tiek kokybiškai pakoreguotos vieneto kainų sekos. Visi apskaičiuoti ciklai ir tendencijos yra netiesiniai: labai svyruoja ciklinio komponento elgsenos kreivė, o pokyčių dažnis labai didelis. Išvadose teigiama, kad aptariamosios rinkos dalys nėra stabilios, o nuolat kinta. Panašu, kad didžioji laikinų variacijų, esančių netransformuotose sekose, dalis realiai gali būti paaiškinta kokybiniais kintamųjų skirtumais. Panaudojus kokybines korekcijas, gerokai išaugo vidinis rezultatų patikimumas.

APPENDIX: A Local regression

The local regression problem can be formalized by using locally weighted least squares (e.g. Ruppert and Wand, 1994; Loader, 2004):

$$\textit{Minimize} \quad \sum_{i=1}^{n} \mathbf{W_H}(\mathbf{x}_i - \mathbf{x})\left(p_i - \langle \boldsymbol{\theta}, \mathbf{F}(\mathbf{x}_i - \mathbf{x}) \rangle\right)^2, \tag{1A}$$

where $\boldsymbol{\theta}$ is the $d+1$ vector of unknown coefficients and $\mathbf{F}(\cdot)$ is a vector of basis polynomials. $\mathbf{W_H}$ is a multivariate *weight function* and $\mathbf{H}^{1/2}$ is a *bandwidth matrix*. The local least squares estimate of the unknown regression function $f(\mathbf{x})$ is then[8]:

$$\hat{f}(\mathbf{x}) = \mathbf{e}_1'(\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}\mathbf{p}. \tag{2A}$$

For local cubic regression $\mathbf{e}_1$ is a $\left\{1+d+\frac{1}{2}d(d+1)+\frac{1}{6}d(d+1)(d+1)\right\}\times 1$ vector having 1 in the first entry and all other entries 0. For local quadratic and linear model the dimension of $\mathbf{e}_1$ is, respectively, $\left\{1+d+\frac{1}{2}d(d+1)\right\}\times 1$ and $\{1+d\}\times 1$. $\mathbf{p} = [p_1, \ldots, p_n]'$ is a vector of observed land prices and the *data matrix* $\mathbf{X}$ for local cubic model is:

$$\mathbf{X} = \begin{bmatrix} 1 & (\mathbf{x}_1 - \mathbf{x})' & vec'\{(\mathbf{x}_1 - \mathbf{x})(\mathbf{x}_1 - \mathbf{x})'\} & (\mathbf{x}_{(1)} \otimes \mathbf{x}_{(1)} \otimes \mathbf{x}_{(1)})' \\ \vdots & \vdots & \vdots & \vdots \\ 1 & (\mathbf{x}_n - \mathbf{x})' & vec'\{(\mathbf{x}_n - \mathbf{x})(\mathbf{x}_n - \mathbf{x})'\} & (\mathbf{x}_{(n)} \otimes \mathbf{x}_{(n)} \otimes \mathbf{x}_{(n)})' \end{bmatrix}, \tag{3A}$$

where $\mathbf{x}_{(i)} = (\mathbf{x}_i - \mathbf{x})$ and vec-operator stacks the columns of $(\mathbf{x}_i - \mathbf{x})(\mathbf{x}_i - \mathbf{x})'$, each below the previous, but with entries above main diagonal omitted; $\otimes$ is the Knonecker (tensor) product. The local linear and quadratic model uses only the first two and three, respectively, columns of the data matrix. The weight matrix is $\mathbf{W} = diag\{\mathbf{W_H}(\mathbf{x}_1 - \mathbf{x}), \ldots, \mathbf{W_H}(\mathbf{x}_n - \mathbf{x})\}$ $\equiv diag\{w_1(\mathbf{x}), \ldots, w_n(\mathbf{x})\}$.

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[8] Assuming, as usual, that $\mathbf{X}'\mathbf{W}\mathbf{X}$ is non-singular.