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ON GEOMETRIC PROPERTIES OF SPHERICAL CONICS AND GENERALIZATION OF π IN NAVIGATION AND MAPPING

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Abstract. First, we cover the conical curves on 2-dimensional modeling sphere S^2 showing their geometric properties affecting the hyperbolic navigation. We place emphasis on the geometric definition of spherical parabola and relate it to the notions of spherical ellipse and hyperbola and give simple geometric proofs for relations between conical curves on the sphere. In the second part of the paper function $\tilde{\pi}$ representing the ratio of the circle's circumference to its diameter has been defined and researched to analyze the potential discrepancies in the spherical and conical projective models on which the navigational computations are based on. We compare some non-Euclidean geometric properties of curved surfaces and its Euclidean plane model in reference to the local and global approximation. As a working tool we use $\tilde{\pi}$ function for geometric comparison analysis in the theory of long-range navigation and cartographic projection. We state the existence of the infinite number of the circles having the same radius but different circumference on the conical surface. Finally, we survey the exemplary proposals of generalization of function $\tilde{\pi}$. In particular, we focus on the geometric structure of applied model treated as a metric space showing the differences in the outputting computations if the changes in a metric are made. We also relate the function $\tilde{\pi}$ to Tissot's indicatrix of distortion.

Keywords: geometry of navigation, mapping, spherical conic, number π.

1. Introduction

The practical importance of the computation of hyperbolic position lines has often been discussed. The lines of altitude difference originally applied to astronomical position lines are hyperbolic position lines, as are also the position lines of equal difference in distance from two transmitters used to be adapted in such electronic fixing systems as Decca, Loran and Omega. For instance, we refer the reader to Freiesleben (1976) which deals with geometric construction of hyperbolae on a spherical surface for navigational application. 2-dimensional differentiable manifolds of positive curvature are mostly used to model the Earth's surface globally. In the paper firstly we consider 2-dimensional sphere S^2 and we find the range of values for the ratio of the circumference of an arbitrary circle to its diameter in general. The spherical model is also used in cartography creating the frame of the navigational charts for instance in the stereographic projection. In particular, the flow of the geodesics and the rhumb line looks different depending on the method of the projections given by the strict formulae. More precisely as the Earth's model an oblate spheroid is widely applied providing the base for the navigational computations. The navigational computations generally mean the computations of two essential notions: the distance and angle. These notions are computed and output directly to the user as the final results or state for the basement for the further calculations e.g. ROT, speed. Thus, we place special emphasis on the geometric structure of models applied in navigation. The spherical conics can be defined in natural terms as the locus of the set of points having certain properties which depend only on the notions of angle and distance. The geometric structure implies the way of calculating the distances, angles and finally the form of formulae applied in the navigational software. Obviously, the knowledge and understanding the geometric properties coming from the foundations of used geometric modeling structure implies the proper understanding of limitations and boundary conditions of safe using the navigational systems, software, methods, algorithmic procedures based on it as well as their reliability, indications, approximations and potential errors.

Geometrically we can treat the modeling surfaces e.g. sphere, spheroid or triaxial ellipsoid as the special subset of more general manifolds. We recall that the manifold must be Riemannian to measure distances and angles on it. Briefly, a Riemannian manifold is an analytic manifold in which each tangent space is equipped with an inner product in a manner which varies smoothly from point to point. This allows one to define various notions such as length, angles, areas, curvature, gradients of functions and divergence of vector fields.

Locally in the physical (Euclidean) 3-dimensional space we approximate curved surface by the Euclidean plane tangent in a given position. That is satisfactory if we do not exceed the required accuracy of provided computations. However in geodesy, navigation, cartography or astronomy there are many examples when non-Euclidean notions differ essentially from their flat analogues. In particular, the range of defined in the further part of the paper function $\tilde{\pi}$ may differ essentially from the constant number π representing the ratio of an arbitrary circle's circumference to its diameter on the plane. That means the plane locally approximating model may not satisfy the accuracy requirements for the precise calculations in the real world. Hence, the boundary conditions of the models e. g. Euclidean plane or spherical geometry ought to be strictly determined in the potential applications. For curved or more complicated surfaces the notion of metric can be used to compute the distance between two points by integration. The distance generally means the shortest distance between two points. Most often the research and calculus in the navigational literature are considered on the spherical or spheroidal models of Earth because of the practical reasons. The flow of geodesics on the spheroid differs from the geodesics on the sphere. There are known different geodesics on the same surface with the same metric considered. However geodesic refers to the metric. There are different flows of geodesics on the same surface when different metrics are applied. That means we can obtain very interesting results in the navigational aspect if we change researched object with its geometrical and physical features. The examples of the differences in the forms of the metrics implying the different computational results of $\tilde{\pi}$ are shown in the further part of the paper.

2. Spherical conics

The conical curves (circle, ellipse, hyperbola, parabola) considered on the Euclidean plane are widely known and can also be found in the navigational applications. Discussing the geometry of ellipses and hyperbolae on the sphere we recall here these are the curves which correspond to position lines based on the constant sum or the absolute value of difference of distances from two fixed points. For instance, the net of hyperbolic lines of positions used to be added to the navigational charts in hyperbolic navigation, in particular in system Decca or still in Loran. The Euclidean plane is used to approximate the Earth surface locally meeting required accuracy; however the limits of approximation must be determined clearly. In the local geodesic terrain modeling the area of the Earth can be treated flat if it is inside the circle of the radius ca. 15.5 km. Then its area does not exceed 760 km^2. This corresponds to the spherical area of the circle of the diameter ca. 17' of the great circle (Kopacz 2010). Practically, the results of the direct geodetic measurements and the calculations provided in such an area neglecting the curvature of the Earth can be provided on the plane in a proper scale. Then for the simplicity the limitations of cartographic projections can be omitted and the results are satisfactory in many applications. The surface of the Earth may be taken mathematically as a sphere instead of ellipsoid for maps at smaller scales. In practice, maps at scale 1:5000000 or smaller can use the mathematically simpler sphere without the risk of large distortions.

In global modeling the situation is essentially different. For instance, the radionavigation bearing cannot be considered only as the straight line on the plane but the geodesic. If we use the spherical model then such a bearing line means the arc of a great circle. Approximating the area of spherical equidistant triangle which side length equals 400 km as a flat Euclidean triangle makes a difference of ca. 232 km^2 at the accuracy of ⅔" for angle measurement.

2.1. Spherical ellipse and hyperbola

Let us transform now the definition of the conical curves from the Euclidean plane to the "spherical plane", i.e. sphere S^2. We look for the corresponding notions of the flat conical curves on the 2-dimensional sphere. To compute a curve of equal sums of distances from two points on the sphere (which corresponds to an ellipse on the plane) the simplest method is to follow the graphical construction, points on the curve being defined by the intersection of circles round the two foci, so that the sum of the distances remains constant while the radii change by unitary steps.

Let F_1, F_2 be the points on the 2-dimensional sphere S^2, $F_1 \neq F_2$ and the spherical distance $\delta(F_1, F_2) := 2c < \pi$. We define the geometric locus for which the sum of the distances to foci F_1, F_2 is constant and equals $2a$. So the spherical ellipse is a set of positions P:

$$\left\{P \in S^2 : \delta(P, F_1) + \delta(P, F_2) = 2a\right\}. \tag{1}$$

In the Euclidean plane there exists one restriction: $c < a$. As the shortest spherical distance cannot be larger than π there is additional restriction in spherical geometry: $c < a < \pi - c$. The graphical presentation of the family of ellipses with given focal points F_1, F_2 as the parameter a varies from its minimum to maximum value is presented in http://3d-xplormath.org.

Let F' state for the antipodal point of the corresponding focal point F. The spherical hyperbolae include one of these foci and the opposite point in the other hemisphere, in each of two separate families of curves. It would therefore seem likely that the spherical hyperbolae and ellipses are related, and indeed it is easy to demonstrate this. Thus, $\delta(P, F) = \pi - \delta(P, F')$. On the spherical surface we observe the relation between the ellipses and hyperbolae as follows:

$$\left\{P \in S^2 : \delta(P, F_1) + \delta(P, F_2) = 2a\right\} =$$
$$\left\{P \in S^2 : \left|\delta(P, F_1) - \delta(P, F_2')\right| = \left|2a - \pi\right|\right\}. \tag{2}$$

The relations between geometric properties of conical curves state for the crucial foundations of models used in practical applications. For the equation of the spherical ellipse in terms of the inner products and as a homogenous quadratic equation we refer the reader to Poodiack (2004). Thus we observe that the spherical ellipse is the intersection of the unit sphere with a quadratic cone whose vertex is positioned at the midpoint of the sphere. So, if one projects a spherical ellipse from the midpoint of the sphere onto some plane then one obtains a (planar) conic section. We research the connections between the circles in the conical and spherical models in next paragraph using as a tool the ratio of the circle's circumference to its diameter representing by defined further function $\tilde{\pi}$.

The navigational charts were prepared in such a way to show the curves of the constant absolute value of the difference of the distances to the two transmitting stations. In areas of the sea where the sets of curves (for at least two pairs of transmitters) intersect reasonably transversal it is sufficient to measure two time differences, then a look on the sea chart shows the ships position as the intersection point of two spherical hyperbolae. Following Griffiths and Culpin (1975) we recall since the lines of equal sums of distances from two points on the sphere are related to the lines of equal absolute value of differences of distances it would seem likely that the ellipses as well as the hyperbolae should be divided into sub-families separated by a great circle. The color-coded family of spherical conics and its corresponding domain is presented in Figure 1. The ellipses for foci F_1 and F_2 may be regarded as hyperbolae for F_1 and the antipodal point to F_2, named F_2' (or for F_2 and the antipodal point to F_1, named F_1'). Thus, part of the family of hyperbolae which relate to focus F_2 and the point diametrically opposite to focus F_1 is identical with the family of ellipses which include the same two points. However the hyperbolae and ellipses denned in relation to the same pair of points are not identical. As in the plane, they are related by the fact that they intersect orthogonally. The example of the conical (elliptical) coordinates grid on plane chart is presented in Figure 2. The elliptic coordinate system is a two-dimensional orthogonal coordinate system in which the coordinate lines are confocal ellipses and hyperbolae (Kopacz 2013).

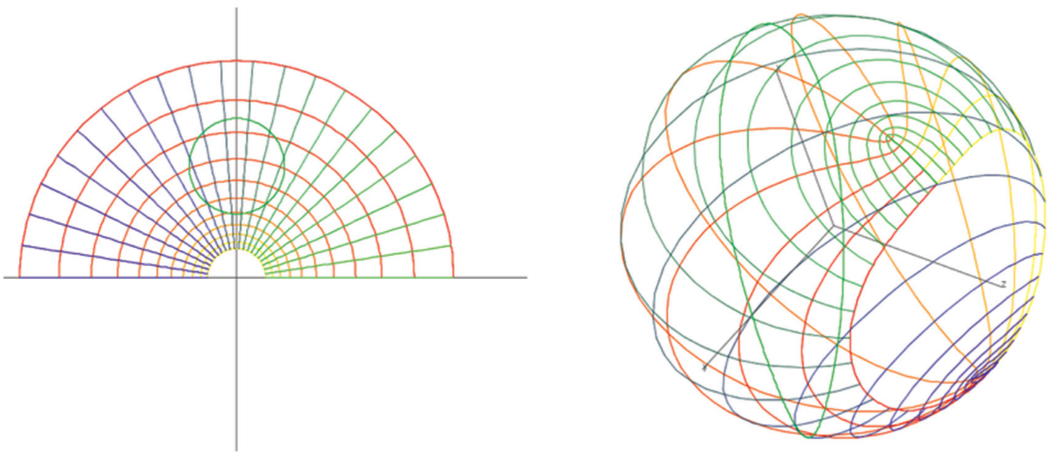

Fig. 1. The color-coded family of spherical conics and its corresponding domain

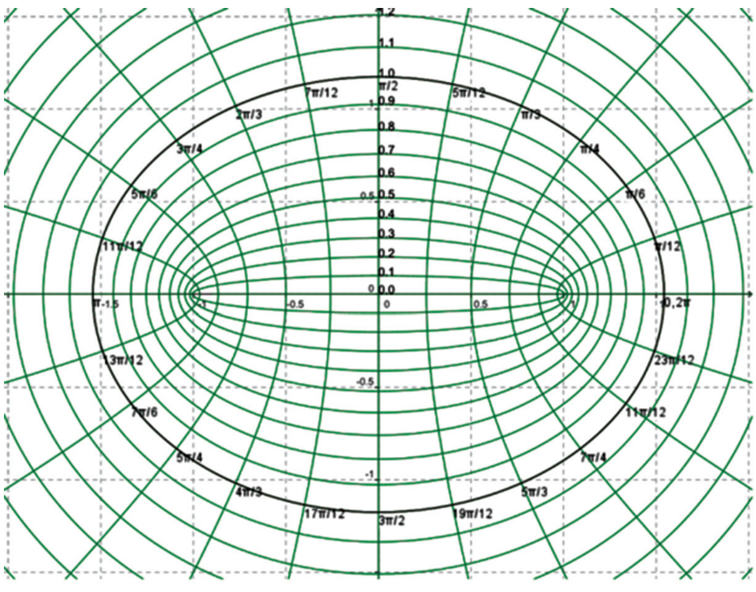

Fig. 2. The conical coordinates grid on the plane chart

Spherical ellipses and hyperbolae having the same foci intersect orthogonally. For an ellipse, one of those foci is a focus of the hyperbola and the other the point opposite to the second focus of that hyperbola, the ellipse and hyperbola in general are not orthogonal. We also consider Tissot's indicatrix (ellipse) referring to distortions of spherical circles in projection on the plane chart in paragraph 3.4.

2.2. Spherical parabola

We now prove geometrically the relation between spherical ellipses and hyperbolae complementing another spherical conic, i.e. spherical parabola. We aim to show the geometric relationship between spherical ellipses, hyperbolae and parabolas. We follow the same notations as above. F_2 determines uniquely its antipodal point F_2' and the referring spherical straight line f which is the great circle at spherical distance 0,5π from both points F_2 and F_2'. Thus, the spherical circle f have two centers F_2 and F_2' like the equator and the poles what is shown in Figure 3.

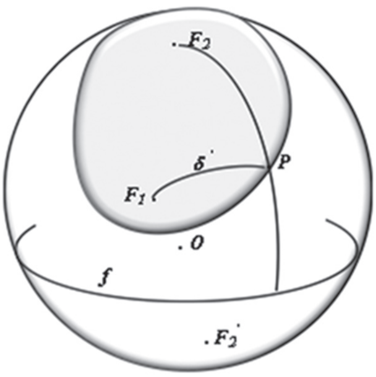

Fig. 3. The spherical conic "3 in 1" (parabola, ellipse, hyperbola)

In the Euclidean plane geometry parabola is the locus of points equidistant from a fixed line (directrix) and a fixed point (a focus) not on the line. Let us consider the set γ of all points on the sphere equidistant from a given point F_1 and the spherical circle f. This is a spherical parabola of the focus F_1 and the directrix f. Let F_2 be in the hemisphere determined by f which includes the parabola. Let P be the point of the parabola and denote PF_1 by δ'. Then the spherical distance from P to f equals δ' according to the definition of parabola and $PF_2 = 0{,}5\pi - \delta'$. Thus,

$$PF_1 + PF_2 = 0{,}5\pi = \text{const}. \tag{3}$$

That means above mentioned spherical conic meets the definition of the spherical ellipse. Precisely, the same spherical conic which is the parabola of the focus F_1 and directix f is also the ellipse of two foci F_1 and F_2. Moreover, if we consider the point F_2' and calculate the difference $|PF_1 - PF_2'|$ as follows:

$$\left|PF_1 - PF_2'\right| = \left|\delta' - \left(0{,}5\pi + \delta'\right)\right| = 0{,}5\pi = \text{const}., \tag{4}$$

then we obtain the constant value 0,5π. Thus, above mentioned spherical conic also fulfills the definition of the hyperbola. So the same spherical curve satisfies the definition of parabola, ellipse and hyperbola on the sphere S^2 what is presented geometrically in Figure 3 (Kopacz 2013).

We showed the spherical curve γ is the parabola of the focus F_1 and directix f, the ellipse of two foci F_1 and F_2 and the hyperbola of two foci F_1 and F_2'. More generally, it can be shown that on the sphere each ellipse is a hyperbola and vice versa. Additionally assuming that f covers the spherical equator and if F_1 is positioned in the center F_2 of the directrix f, i.e. $F_1 = F_2$ then the spherical parabola is a parallel of latitude of 0.25π (45°) so it is also the particular case of the spherical circle.

The spherical conics have various applications in navigation from the geometrical point of view. They often create the foundations for the methods and the following computations based on them. We recall the contributions presented in Freiesleben (1976) which also affects only the spherical model so included equations are valid only for the sphere and not for the ellipsoid. In this article our aim is to place the special emphasis on the geometric approach to the subject instead of the analytic computations. The geometry of modelling structures implies the calculus essentially, in particular the mathematical formulae in the algorithms applied in the navigational electronic device and systems. For instance, the navigation based on geodesic lines and connected software of the ship's devices (electronic chart, positioning and steering systems) gives a strong argument to research and use geodesic-based methods for calculations instead of the loxodromic trajectories in general. Next step is the research of the conical curves on the locally and globally modelling differentiable manifolds of the Earth surface having differing curvatures, including the spheroid and triaxial ellipsoid. The theory on geodesics is developing as well what may be found in the wide literature on geometry and topology. This motivates to discuss the subject in more general geometric structures and research its properties in the navigational context.

3. Spherical and conical circles $\tilde{\pi}$ function

First, we consider the notion of set's diameter applied in the further reading.

Definition. The diameter d of a non-empty set $A \subset X$ in a metric space (X, δ) is the supremum of the distances δ between pairs of points in the set A:

$$d(A) = \sup_{x,y \in A} \delta(x, y).$$

Briefly, the generalized diameter is the greatest distance between any two points on the boundary of a closed figure.

3.1. Sphere S^2

The great circle is the equivalent of the Euclidean straight line, it has the finite distance and it is closed. To obtain the spherical distance between two points on the two dimensional sphere S^2 the notion of the latitude is used for our calculations. We aim to consider the whole family of the spherical circles (great and small). It is always possible to orient our sphere in such a way that considered circle becomes a parallel of latitude. We use this observation to obtain the diameter of the circles on curved surfaces (2-dimensional differentiable manifolds), in particular the spherical and conical circles. The notion of circle's diameter is rather intuitive in the spherical case however it is not so in a conical one considered in paragraph 3.2. The mathematical formulae used in approximation of the navigational computations are being studied and mostly based on the spherical (spheroidal) global model. However if we research a different "shape" of the modeling surface the formulae change considerably. Let us imagine that the vessels do not sail on spheroidal Earth but locally torus - shaped planet. In that case the flow of geodesics, the rhumb line or projected charts are based on other mathematical expressions due to the fact the different geometrical structure is applied. The torus $T^2 = S^1 \, x \, S^1$ is topologically more simple than the sphere, yet geometrically it is a very complicated manifold indeed. The round torus metric is most easily constructed via its embedding in a Euclidean space of one higher dimension.

Let $R > 0$ be the radius of the sphere. Then the length of its equator equals $2\pi R$ and the length of the diameter d equals two spherical radii and it is a half of the equator, that is $d = \pi R$. Thus, the ratio of the circumference of the great circle to its diameter equals 2. Figure 4 shows the diameter d of spherical circle l at the latitude x passing through the spherical circle's center positioned in S.

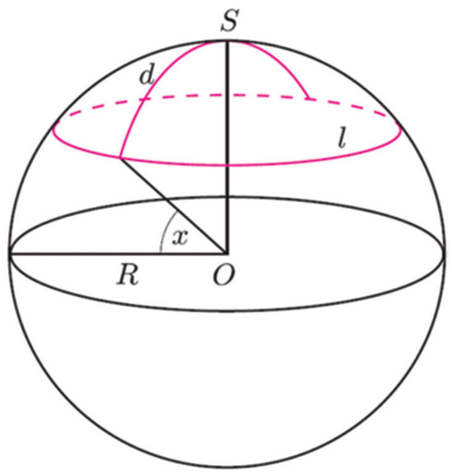

Fig. 4. The diameter of the spherical circle (Kopacz 2010)

Without loss of generality the length L of the spherical circle (great and small) can be obtained from:

$$L = 2\pi R \cos x, \tag{5}$$

where x denotes the latitude in radians,

$$x \in \left[0; \frac{\pi}{2}\right).$$

For the latitude $\frac{\pi}{3}$ the length of the corresponding circle is a half of the length of the equator i.e. πR and the diameter equals $\frac{1}{6} \cdot 2\pi R$. Hence, the ratio of the circumference of the circle to its diameter equals 3. It becomes clear that the ratio of circumference to the corresponding diameter is not constant and does not equal $\pi = 3.1415926535\ldots$ as in the flat Euclidean geometry. We ask here what is the range of the ratio in general? We define the function $\tilde{\pi}: \mathbb{R} \to \mathbb{R}$ representing the ratio of the circumference of the circle to its diameter and determined in the following general way:

$$\tilde{\pi} = \frac{L}{d}. \tag{6}$$

In the spherical case the latitude x as the variable is applied so $\tilde{\pi}(x) = \frac{L(x)}{d(x)}$. Then we obtain the formula for the diameter of a spherical circle d:

$$d = 2R\left(\frac{\pi}{2} - x\right). \tag{7}$$

Thus, for $x \in \left[0; \frac{\pi}{2}\right)$

$$\tilde{\pi}(x) = \frac{2\pi R \cos x}{2R\left(\frac{\pi}{2} - x\right)} = \frac{2\pi}{\pi - 2x}\cos x. \tag{8}$$

Function $\tilde{\pi}$ is increasing in considered domain and its range $\tilde{\pi}(x) \in [2; \pi)$. Without the loss of generality we limited the domain of $\tilde{\pi}$ down to $\left[0, \frac{\pi}{2}\right)$ what corresponds the northern hemisphere without a pole where the latitude is positive and takes into account all the circles of the whole sphere. Obviously, there is analogous situation in the southern hemisphere where the latitude is negative. Thus, we obtain $\tilde{\pi}$ for the spherical circles at arbitrary latitude as follows:

$$\tilde{\pi}(x) = \frac{2\pi}{\pi - 2|x|}\cos x. \tag{9}$$

Corollary: On 2-dimensional sphere the ratio of the circumference of an arbitrary non-degenerated ($r > 0$) circle to its diameter is not constant and ranges $[2; \pi)$.

Figure 5 presents the graph of y-axe symmetric $\tilde{\pi}$ function for $x \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$.

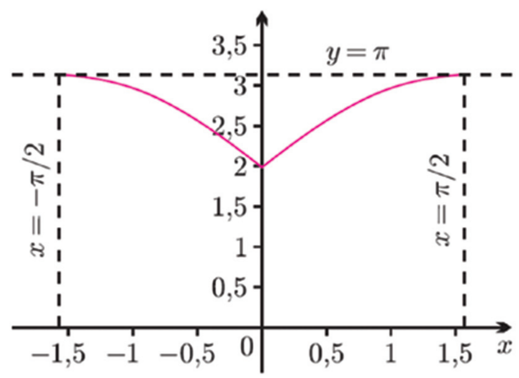

Fig. 5. The graph of function $y = \tilde{\pi}(x)$ for the spherical circles (Kopacz 2010)

In the boundary case when the radius r of the small spherical circle becomes shorter and shorter $(r \to 0)$ then $\tilde{\pi}(x) \to \pi$. As the Earth's model the sphere S^2 or the spheroid is most often used and locally their surface is approximated by the Euclidean plane tangent in a given position. For some applications such approximation is practically allowed and sufficient for researched problem. However in the situation when the strict boundary accuracy is required and the global model of the Earth's surface is considered (long-range navigation, cartographic projections in small scales, astronomy) the Euclidean geometry becomes not sufficient for the geometric description and the analytic calculations coming from it. Then the value of function $\tilde{\pi}$ differs from π essentially $(\tilde{\pi}(x) \neq \pi)$ and the limits of the approximations based on the flat Euclidean geometry ought to be clearly determined. The formulae based on the plane model and Euclidean geometry one may find, for instance, in the navigational software applied on the Dynamic Positioning (DP) vessels.

If S^2 is considered for global modeling then as a calculating tool the spherical trigonometry is also used which states the basis for the comparison analysis and algorithms implemented in the software of navigational aids. If the spheroids of differing eccentricity model the Earth surface then the corresponding tools are applied including great ellipse or generally geodesic-based trajectories. The Earth is not an exact ellipsoid, and deviations from this shape are continually evaluated. Although the basic solutions for navigational purposes have already been known and widely used there are still the new spherical and spheroidal approaches made to the subject. The main efforts affect the optimization and approximation methods which potentially may give the practical benefits for the navigators. We conclude here stating the curvature affects the geometry of the modeling surface essentially, in particular the flow of geodesic trajectories and the conical curves.

3.2. Cone of revolution

We consider the modeling surface of revolution generated by the straight line of equation $z = ay$ $(a > 0)$ sloped at angle $\alpha \in \left(0, \frac{\pi}{2}\right)$ to the axe y around axe z in the Cartesian coordinate system $OXYZ$. For instance, the function $z(y) = y$ generates the double cone of revolution defined by $z^2 = x^2 + y^2$ in $\mathbb{R}^3$. Let $z \leq 0$, we obtain the single cone of revolution which vertex is positioned in $O = S$ (Fig. 6). We aim to find the range of $\tilde{\pi}$ function.

Let us consider the set of the conical circles which are generated by the planes parallel to the base of the cone. These are the conical parallels of latitudes and the point S is the center of all such conical circles. The conical surface is developed on the plane unlike the sphere. Our calculations are based on two parameters – distance l of the circle from the vertex of a cone and the angle β determining the conical surface. Flattening it in the 2-dimesional Euclidean space $\mathbb{R}^2$ one obtains the flat disc without the central angle β what is shown in Figure 7. The length L of the conical circle equals

$$L = l(2\pi - \beta). \tag{10}$$

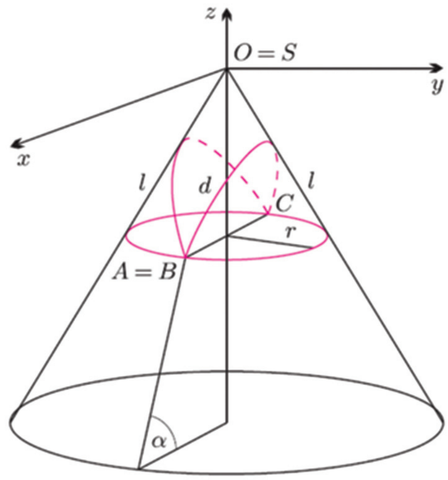

Fig. 6. The conical parallel of latitude circle and its diameters (Kopacz 2010)

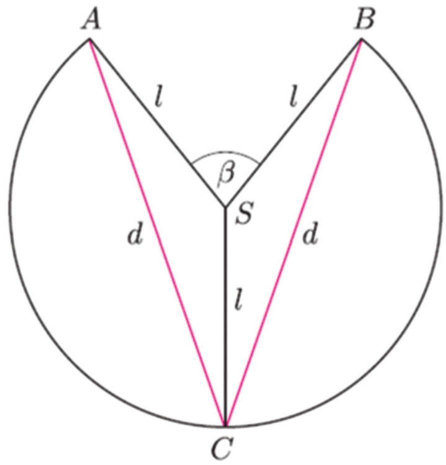

Fig. 7. The conical circle and its diameters in 2-dimensional Euclidean space $\mathbb{R}^2$ (Kopacz 2010)

Note that the diameter of a circle on the plane or sphere agrees to its 'traditional' definition followed by our Euclidean intuition. Nevertheless to find the diameters of the conical circles we refer to above mentioned formal definition of set's diameter and treat the circle as a particular subset. Let us observe that the diameter of the conical parallel of latitude circle does not pass through its center what is illustrated in Figure 7 and its length does not equal double length of circle's radius. This fact is not so intuitive. Generally speaking, we note that the form of a modeling space, for example a metric space in which an approximation and the computations for navigational purposes, i.e. the distances and the angles, must be taken into consideration basically. Thus, it is necessary to research how the distance between two positions in considered geometrical structure is determined, where the centre of the circle is positioned and how the diameter passes. The diameter of the conical circles fulfills the definition of geodesic on the conical surface. Changing the metric causes the differences in the obtained distances. The notion of local metric is required to define geodesics locally. We observe here that the conical circle has got two diameters AC and BC which length equals d in both cases as presented in Figure 6 and Figure 7. Applying the cosine formula in the plane triangle ΔCSA we obtain:

$$d(l,\beta)^2 = AC^2 = AS^2 + CS^2 - 2\cdot AS\cdot CS\cdot\cos\left(\pi - \frac{\beta}{2}\right) = 2l^2\left(1+\cos\frac{\beta}{2}\right). \quad (11)$$

Substituting $AC = d$ and $AS = CS = l$ and recalling the length of the conical circle (10) the value of function $\tilde{\pi}: \mathbb{R}^2 \to \mathbb{R}$ equals:

$$\tilde{\pi}(l,\beta) = \frac{L(l,\beta)}{d(l,\beta)} = \frac{2\pi - \beta}{\sqrt{2\left(1+\cos\frac{\beta}{2}\right)}}. \quad (12)$$

We observe that the value of $\tilde{\pi}$ does not depend on the distance l of the conical circle from its center S (the vertex of a cone) so on each cone all the parallels of latitude have the same ratio of the circumference to the diameter. However, this value is different for different cones – it depends on the central angle β. Thus, $\tilde{\pi}(\beta) \in (2;\pi)$ for every $\beta \in (0;2\pi)$. The resulting function is continuous, decreasing and takes all values from $(2;\pi)$. For example, if $\beta = \pi$ then $\tilde{\pi}(\beta) = \frac{\pi}{\sqrt{2}}$. The graph of $\tilde{\pi}(\beta)$ for $\beta \in (0;2\pi)$ is presented in Figure 8.

Corollary. *On the cone of revolution of given central angle* $\beta \in (0;2\pi)$ *the ratio of the circumference of the parallel of latitude circle to its diameter is constant and the value is given by* (12).

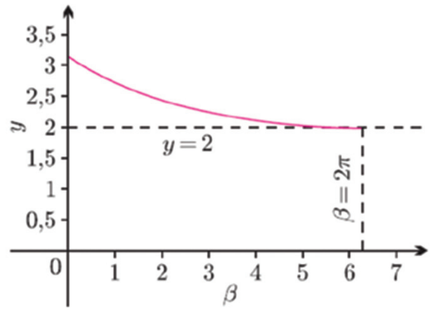

Fig. 8. The graph of function $y = \tilde{\pi}(\beta)$ for the conical parallel of latitude circles (Kopacz 2010)

In the paper we consider the conical parallel of latitude circles. This family of the circles is not the only subset of the circles on the cone. For the conical circles which does not contain the vertex of a cone in their interior $\tilde{\pi}(x) = \pi$ as they are equivalent to the common flat Euclidean circles. That can be easily shown after flattening the conical surface on the Euclidean plane $\mathbb{R}^2$. The third subset of the conical circles is created by the circles which are not centered in the vertex of a cone but contain the vertex in their interior. Interestingly, this family has the property which does not follow the Euclidean intuition. Hence, a careful approach is to be taken into consideration when providing, in particular long-range navigational calculations and presenting the results in the conical cartographic projections of smaller scale. As an illustrative example we ask the following question: *Given a circle of radius r' > 0 and center S' which is not the vertex of a cone of revolution of given central angle* $\beta \in (0;2\pi)$. *Does the circle of the same radius r' and centered in the vertex of a cone have the same circumference?* The answer is negative and differs from the obvious result obtained on the plane Euclidean model. The conical circle of the same radius *r'* but centered in the vertex of a cone (parallel of latitude) does not have the same circumference. Moreover, we state the following

Statement. *On an arbitrary cone of revolution for each circle exists the infinite number of the circles of the same radius but different circumference.*

Thus, we observe the geometry affects essentially the calculus based on it, in particular the range of $\tilde{\pi}$ function or the computations of notions crucial for navigation – the distances and angles. Although the proofs of above mentioned statements do not require the technical calculations we do not give them because of the limited length of the article. Our research may state for the starting point of such research on generalized cone. Then the cone of revolution is treated as a particular subset of the set of cones.

Let us consider the cone which is tangent or secant to the spherical surface. The intersection of these two sets is a single circle or a doubleton of circles, respectively. Because of the fact the intersection is both the spherical and conical locus we can indicate the identically equivalent circle of the spherical one on the flattened conical surface but of different properties e.g. radius, diameter or centre. Obviously, the results of comparing the geometric properties of an arbitrary circle and its image depend on applied projective transformation. For the simplicity the idea of transformation of oblique spherical circles onto conical surface is presented graphically in Figures 9 and 10.

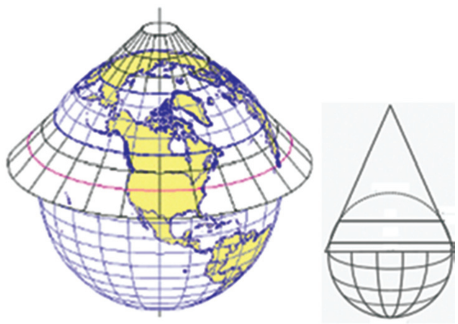

Fig. 9. Transformation of oblique spherical circle to the secant conical surface (Bunch 2004)

One may find the conical model as a base in some cartographic projections applied in constructing the charts.

Comparing the value of $\tilde{\pi}$ function for both (the conical and spherical) intersecting surfaces it allows to determine the discrepancies between some geometric properties of modeling and modeled surface. Generalizing, the circles of modelling surfaces of differing curvature can be transformed to another geometric structure. Such a transformation sometimes enables to know the properties of the original structure what is not possible

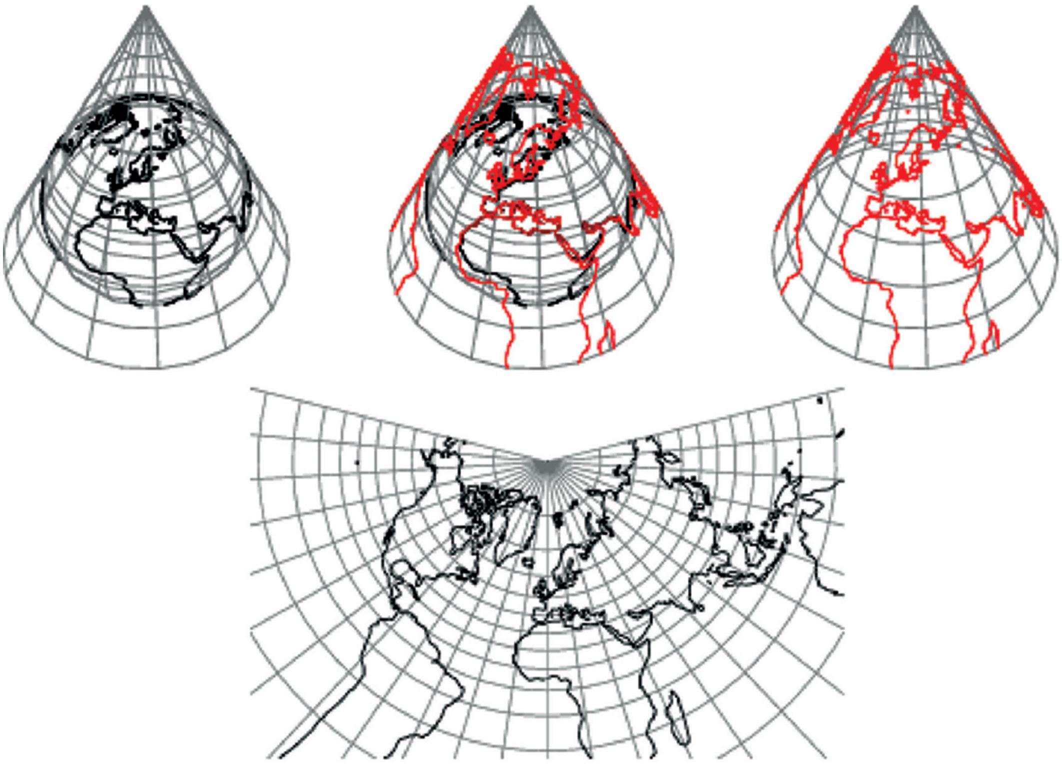

Fig. 10. Transformation of oblique spherical circle to tangent conical surface cut along the solid line and flattened out (http://mathworld.wolfram.com/ConicProjection.html)

directly or is more complicated. We recall the shape of the circles may depend on the position of its center as well as the length of its radius. More generally, the trajectories of the geodesic lines are determined by the metric which works in the geometrical structure used as a modeling metric space. That affects the computations regarding the distances and angles, in particular the outputting formulae.

3.3. Exemplary generalization of $\tilde{\pi}$

By now we have considered the circular objects. However in the layers of vector electronic navigational charts, for instance, there are the triangle or polygon-shaped objects. We can generalize the $\tilde{\pi}$ function on the Euclidean plane locally approximating curved surface.

Let F be an arbitrary plane, convex and bounded figure having non-empty interior. Then $\tilde{\pi}(F)$ we define as follows:

$$\tilde{\pi}(F) = \frac{L(F)}{d(F)}, \tag{13}$$

where $L(F)$ and $d(F)$ denote the circumference and the diameter of the figure F, respectively. Then, basing on the definition of the set's diameter mentioned above, for the equilateral triangle $\tilde{\pi}(F) = 3$, for the square $\tilde{\pi}(F) = 2\sqrt{2}$, for an arbitrary triangle $2 < \tilde{\pi}(F) \le 3$, for an arbitrary rectangular of two sides a, b $\tilde{\pi}(F) = \frac{2(a+b)}{\sqrt{a^2+b^2}}$ what implies $2 < \tilde{\pi}(F) \le 2\sqrt{2}$. From Barbier's theorem stating that all figures of constant width $\bar{d}$ have the same perimeter $\pi\bar{d}$ we conclude that for an arbitrary plane, convex, bounded and non-empty figure F we obtain $2 < \tilde{\pi}(F) \le \pi$. The equality occurs only for the figures of the constant width. The way of measuring the distance implies the changes not only in the shape of the circle but also its length. Thus, the ratio of the circumference to the diameter of the circle depends on the metric function as we have already mentioned in paragraph 3.2. For example, if we consider in $\mathbb{R}^2$ the distance (more generic Minkowski l^p metrics) defined in the following way:

$$\delta_p(P_1, P_2) = \begin{cases} \left(|x_1 - x_2|^p + |y_1 - y_2|^p\right)^{\frac{1}{p}}, & 1 \le p < \infty \\ \max\left\{|x_1 - x_2|, |y_1 - y_2|\right\}, & p = \infty \end{cases}, \tag{14}$$

where $P_i = (x_i, y_i)$ denotes the i-position in considered structure, then function $\tilde{\pi}_p = \frac{L_p}{d_p}$ depends basically on the above metric, where L_p, d_p denote the circumference and the diameter of generalized circle in the metric defined by (14), respectively. Thus, for $p = 2$ it holds $\tilde{\pi}_2 = \pi$. If $p = 1$ or $p = \infty$ then $\tilde{\pi}_1 = \tilde{\pi}_\infty = 4$. The attempt of assessment the value of $\tilde{\pi}_p$ for $1 < p < \infty \wedge p \ne 2$ meets the intrivial integrals, series and open questions, too. In Adler and Tanton (2000), Euler and Sadek (1999), Keller and Vakil (2009), Poodiack (2004) the authors also investigate the value of the numerical integration of $\tilde{\pi}$ as a function of p what yields to $\tilde{\pi} \in [\pi, 4]$. The circle's (sphere's in general) form is determined by the metric, the position of its center and the radius. In many analytic fields of research and applications this is the metric which has essential influence on the geometric properties of the modeling structure. We conclude here without going into technical details there are still made theoretical contributions to the generalization of number π. For the exemplary results we refer the reader to the literature on the geometry and functional analysis, in particular Adler and Tanton (2000), Euler and Sadek (1999), Keller and Vakil (2009), Poodiack (2004), Richter (2008).

3.4. Tissot's ellipse of distortion (indicatrix) and $\tilde{\pi}$ function

Tissot's ellipse of distortion is used to characterize the distortions of original circles of mapped Earth's surface in different cartographic projections. Although the image of the original circle seems to be deformed it is the result of applied mapping function. Tissot's indicatrices illustrate linear, angular and area distortions of maps. It is the geometry that results from projecting a circle of infinitesimal radius from curved geometric model onto a map. Tissot proved that the resulting diagram is an ellipse whose axes indicate the two principal directions along which scale is maximal and minimal at that point on the map. A single ellipse describes the distortion at a single point. Because distortion varies across a map, generally Tissot's indicatrices are placed across a map to illustrate the spatial change in distortion. A common scheme places them at each intersection of displayed meridians and parallels. These schematics are important in the study of map projections used for navigational purposes, both to illustrate distortion and to provide the basis for the computations that represent the magnitude of distortion precisely at each point.

Tissot's indicatrix is based on a set of equally sized circles on curved surface of the globe. Different projecting functions distort them in different ways, either changing their size or their shape, or both. The example of the spherical circles set out at 30 degrees intervals (in latitude and longitude) with diameters of 15 degrees of latitude is presented in Figure 11.

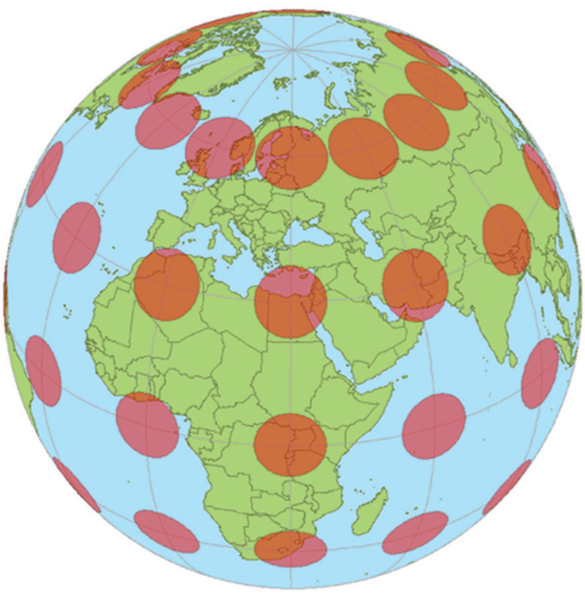

Fig. 11. The Tissot's indicatrices based on a set of equally sized spherical circles (http://en.wikipedia.org/wiki/Tissot's_indicatrix)

The circles defined in a spherical or ellipsoidal model of the Earth are imaged by the Tissot's ellipses of distortion that result from their projection on the plane chart. The example of plane map of the world in an equirectangular projection with Tissot's indicatrix of distortion of radius of 500 km is presented in Figure 12.

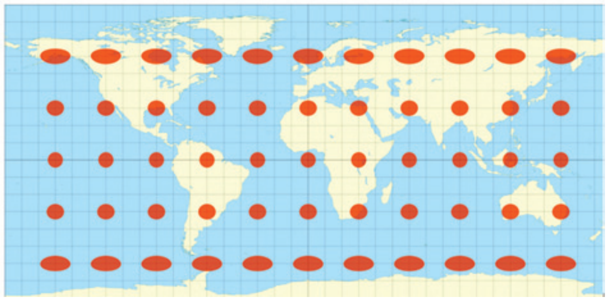

Fig. 12. Map of the world in an equirectangular projection with Tissot's ellipses of distortion (http://en.wikipedia.org/wiki/Tissot's_indicatrix)

Finding the circumference of a plane ellipse of semi-major axis a and semi-minor axis b involves the complete elliptic integrals of second kind. Applying the power series we calculate the circumference with required accuracy and then we obtain the formula for $\tilde{\pi}$ function in case of a plane ellipse. Substituting $e=\frac{\sqrt{a^2-b^2}}{a}$ yields (Bronstein, Semendjajew, Musiol, Muhlig 2001):

$$L = 2\pi a\left[1-\left(\frac{1}{2}\right)^2 e^2-\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{e^4}{3}-\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\frac{e^6}{5}-\cdots\right]. \quad (15)$$

Following the definition of set's diameter mentioned above the diameter of the plane ellipse equals $d = 2a$. Hence:

$$\tilde{\pi} = \frac{L}{d} = \pi\left[1-\left(\frac{1}{2}\right)^2 e^2-\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{e^4}{3}-\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\frac{e^6}{5}-\cdots\right]. \quad (16)$$

We observe for the plane ellipse $\tilde{\pi}$ ranges $(2,\pi]$. It is possible to calculate the value of $\tilde{\pi}$ function for hyperbola and parabola on the sphere (ellipsoid) which we considered in paragraph 2.1 and paragraph 2.2. This property differs from the same conical curves considered on the plane. The circular Tissot's indicatrices of distortion in the Lambert conical conformal projection on conical surface is presented in Figure 13.

In the paper we have focused on the spherical and conical circles. The circle as the intersection of two surfaces have in general the same circumferences but different radii, diameters and the positions of the center if we consider it as the particular subsets of the intersecting structures separately. The value of $\tilde{\pi}$ function for the same circle generally ranges differing values in both cases. We may use $\tilde{\pi}$ function as a parameter showing the differences in the geometric properties of the circles embedded in the original geometric structure (modeled space, surface) as a basic object and presenting its geometric property in comparison to its image (modeling structure), for example projective curved surface.

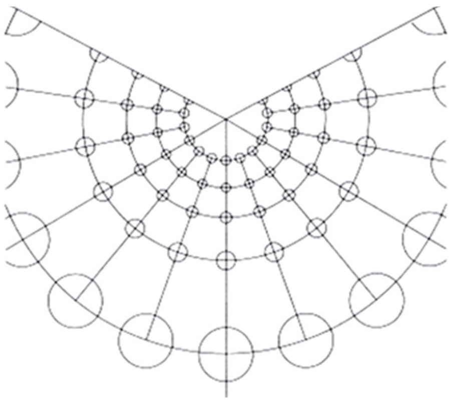

Fig. 13. The circular Tissot's indicatrices of distortion in the Lambert conical conformal projection (Snyder, Voxland 1989)

Both modeled and modeling one may be connected by the projective function. It represents one of the fundamental geometric quantities which is the ratio of the circle's circumference to its diameter. However, the image of transformed circle should not be treated as distortion like Tissot's indicatrix but the natural form of the circle existing in modeling structure. As we showed above the possible generalizations of $\tilde{\pi}$ function can also be used for other than circle-shaped objects. Additionally, we notice that the theoretical research affects the n-dimensional objects, mapping and the abstract mathematical fields where the $\tilde{\pi}$ function and its equivalents are used fruitfully.

The bijective function $\tilde{\pi}$ gives a unique solution for an arbitrary spherical circle of radius $r \in \left[0, \frac{\pi}{2}\right]$. That means in the inverse problem assuming the boundary condition for the value of $\tilde{\pi}$ we obtain the unique spherical length of the radius r. This determines the coverage area centered at the spherical circle's center (e.g. the position of navigational transmitting station) where $\tilde{\pi}$ function does not exceed required value. Approximating locally the spherical surface (curved space in general) by the plane model we cannot omit the differences in the geometric properties of both structures. In particular, we can define the relative error $\varepsilon(r)$ for function $\tilde{\pi}$ in case of the spherical circle is approximated by the plane circle of the same radius on the tangent plane and centered in the same position. Recalling (9) and substituting $r = \frac{\pi}{2} - x$ we obtain for $r \in \left[0, \frac{\pi}{2}\right]$:

$$\varepsilon(x) = \left| \frac{\tilde{\pi}(x) - \pi}{\tilde{\pi}(x)} \right| 100\% \overset{r=\frac{\pi}{2}-x}{\rightarrow} \varepsilon(r) = \left| 1 - \frac{r}{\sin r} \right| 100\%. \quad (17)$$

The graph of increasing and convex function $\varepsilon(r)$ is presented in Figure 14.

The relative error $\varepsilon(r)$ may be defined for other curved surfaces and show the discrepancies in computations due to differences in the geometries of connected modeling and modeled structures. The geometries of the structures which naturally differ affect the navigational computations made in them.

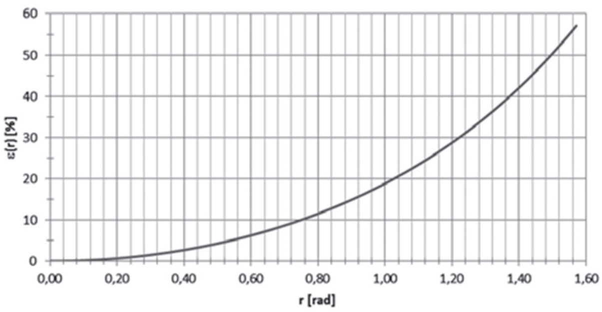

Fig. 14. The graph of relative error $\varepsilon(r)$ for the spherical circle of radius $r \in \left[0, \frac{\pi}{2}\right]$

4. Conclusions

In calculus it is of crucial importance that we state on which geometry we place our truths.We aim to recall the importance, usefulness and otherness of some geometric properties coming from non-Euclidean geometries which affect the base of computations referring the angle and distance measurements. These two notions are fundamental for navigational computations. If space is curved then Euclidean geometry, which is one of many axiomatic systems, does not apply. The flow of geodesic trajectories depends on the type of metric we use in modelling geometric structure, in particular 2-dimensional surface of positive curvature like sphere or spheroid. Our research on conical curves presented here in the spherical case may state for the starting point of such research on surfaces of differing curvature, in particular spheroid and triaxial ellipsoid. As a working tool function $\tilde{\pi}$ was used which researches the ratio of the circle's circumference to its diameter. The circle is one of the fundamental geometric objects and the diameter depends on the geodesic flow if there exists. Researching the ratio it is necessary to answer first how the circle looks like in considered geometric structure, how the distance between two points is determined, where the center of the circle is positioned and how the diameter passes. As comparing Euclidean example we presented the diameter of the conical parallel of latitude which does not pass through its centre. That differs from both the plane and spherical model. The Euclidean intuition insists on looking at the diameter as a part of geodesic on given surface passing through the centre of a circle. However, the diameter depends on applied metric as the flow of geodesics does. Therefore, the shape of the circles researched in the metric spaces depends on the position of the center and the radius. The navigable trajectories as great ellipse or great circle are the examples of geodesic lines on the spheroid and sphere, respectively. The geodesics may look different even on the same surface if different metrics are applied. The notion of local metric is required to define geodesics locally. Thus, changing the metric causes the differences in obtained distances. For precise navigational computations it is of high importance to know the geometric description of applied model which states the basis for the navigational quantities which are generally the distances and angles. For instance, π as a number is constant and has the same value in each geometry (Euclidean, elliptic, hyperbolic) it is used in the computations. However, "π" considered as a ratio of the circle's circumference to its diameter, represented by above defined function $\tilde{\pi}$, may range different values, in particular $\pi = 3.1415926535\ldots$ (Kopacz 2010). In non-Euclidean geometry the ratio of a circle's circumference to its diameter may also differ from π. This does not change the definition of π but it does affect many formulae in which the ratio appears. The present research also refers to generalization of the ratio in the different geometric structures and compares the discrepancies between applied models. There are over a dozen principal ellipsoids which are still used by one or more countries. The different dimensions do not only result from varying accuracy in the geodetic measurements but the curvature of geoid is not uniform due to irregularities in the gravity field. Thus, it motivates to discuss the problem locally on other models of differing curvature as well as globally, in particular on spheroid and triaxial ellipsoid which guarantee the better accuracy of measurement and approximations in the navigational applications. Such research may also affect other than circle-shaped geometric objects as we showed in the Euclidean plane in paragraph 3.3. Hence, the boundary conditions of applied geometry ought to be strictly determined. This goal may be achieved by investigating the geometric properties to see how the basic axioms and definitions lead to quite different and often contradictory results. Our research may also state for the starting point of such research on surfaces of differing curvature, more general Riemannian manifolds, the metric and topological spaces.

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