GEODEZIJA IR KARTOGRAFIJA / GEODESY AND CARTOGRAPHY
2008
34(1): 5–11

UDK 528.14 doi: 10.3846/1392-1541.2008.34.5-11

TWO ALTERNATIVE METHODS FOR HEIGHT TRANSFORMATION

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Received 05 01 2007, accepted 27 11 2007

Abstract. Geodesists have always been dealing with coordinate transformations. There exist various kinds of transformations, like three-dimensional (spatial datum) transformations, two-dimensional (horizontal datum) transformations and one-dimensional (eg, height) transformations. In this article we discuss height transformations.

Height data is usually obtained by levelling. The problematic side of levelling is that this technique is very labour intensive and costly. Nowadays as well GPS measurements can be used, which are much faster and cheaper, but in order to use GPS measurements for height determination, we need a precise geoid model to transform GPS heights to heights above sea level.

In this article two different approaches to this transformation are presented. At first, the affine transformation is discussed. The method is by nature linear, and employs the barycentric coordinates of the point, the height of which is going to be computed. Secondly, the method of fuzzy modelling is used. By these methods, the transformation surface is determined and the heights of desired points can be determined.

As the input data, height information from the precise levelling campaign in Estonia is used. The computed values are tested against height information, gathered from the reference geoid model. The objectives of this research are acquiring insight into using alternative methods for height transformation as well as to statistically characterise the suitability of the proposed methods.

Keywords: height information, coordinate transformations**,** bilinear affine transformation, Fuzzy logic.

1. Introduction

The height is a measure defined from some reference surface. The problem arises with getting the height information. The first choice for obtaining height information is the levelling technique. By its nature this technique is the most precise one, but on the other hand, very time and money intensive. Therefore geodesists have been looking for alternative methods for obtaining height information.

In Estonia there exists a high-precision reference geoid (Jürgenson 2003) which in this work we consider as the reference. Our objective is to study, how well we can predict geoidal heights to be used for height correction in the absence of such a geoid model.

In this article we are using mathematical interpolation techniques for obtaining height corrections when applying the GPS method. Two different methods are tested: the bilinear affine transformation approach and the fuzzy modelling approach.

Bilinear transformation is based on the affined transformation model. The most critical within this approach lies in the assumption of geoid linearity. In reality the geoid has a very complex nature, and the surface of the geoid can be found by using many different and complex methods. In this article we will show, how well the idea of linear transformation can be applied to this.

The input data used are rectangular co-ordinates, ellipsoidal and normal heights for a set of known points. The ellipsoidal heights originate from GPS measurements (Estonian Land Board).

Fuzzy modelling deals with reasoning that is approximate rather than precise. The idea of fuzzy logic theory was invented by Prof Zadeh in the 1960's (Kollo and Sunila 2005).

In the fuzzy method, the representation of the earth physical surface does not really consist of crisp values, but rather of transition zones (Pequet 2002). By using given input values, the fuzzy approach is fitted to the chosen function for the given values. This fitting is done by using neural network programming.

Fuzzy modelling is not very often used with crisp geodetic data. In this article we will show the possibilities to use the fuzzy modelling technique together with geodetic information.

The input data needed for fuzzy modelling are point rectangular co-ordinates and height correction information from levelling and GPS.

2. Theoretical aspects

2.1. Bilinear transformation approach

The basic formula for affined transformation of horizontal point positions is a linear function of x and y coordinates (Vermeer 2003):

$$x^{(2)} = x^{(1)} + a_0 + a_1 x + a_2 y,$$
$$y^{(2)} = y^{(1)} + b_0 + b_1 x + b_2 y, \quad (1)$$

where a_0, a_1, a_2, b_0, b_1 and b_2 are transformation parameters and superscripts (1) and (2) stands for the first and second coordinate systems respectively. This transformation could be written as well for heights (as this resembles one of the coordinates of a point) in a similar way as planar coordinates (Vermeer 2003):

$$H^{(2)} = h^{(1)} + c_0 + c_1 x + c_2 y, \quad (2)$$

where c_0, c_1 and c_2 are transformation parameters and superscripts (1) and (2) stands for the first and second height systems respectively. In this case we need to know heights for three points to determine three unknown transformation parameters.

Applying the idea of barycentric coordinates, we can rewrite the formula:

$$H_i = h_i + p_i^A \left(h_A - H_A\right) + p_i^B \left(h_B - H_B\right) + p_i^C \left(h_C - H_C\right), \quad (3)$$

where p_i^A, p_i^B and p_i^C are barycentric coordinates for the triangle.

For error propagation, we compute the variance of the unknown point as follows (Vermeer 2003):

$$Var\left(H_i\right) = -0{,}5 \times \sigma_0^2 \times \begin{bmatrix} 1 & -p_i^A & -p_i^B & -p_i^C \end{bmatrix} \times \begin{bmatrix} 0 & \|z_{iA}\|^\nu & \|z_{iB}\|^\nu & \|z_{iC}\|^\nu \\ \|z_{iA}\|^\nu & 0 & \|z_{AB}\|^\nu & \|z_{AC}\|^\nu \\ \|z_{iB}\|^\nu & \|z_{AB}\|^\nu & 0 & \|z_{BC}\|^\nu \\ \|z_{iC}\|^\nu & \|z_{AC}\|^\nu & \|z_{BC}\|^\nu & 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ -p_i^A \\ -p_i^B \\ -p_i^C \end{bmatrix}, \quad (4)$$

where σ_0 is the precision, mm/√km, ν is the power law (for levelling networks $\nu = 1$) and z is the location vector difference between points written as a complex number $z = x + iy$, eg $z_{iA} = (x_A - x_i) + i(y_A - y_i)$.

2.2. Fuzzy theory

Fuzziness concerns inherent imprecision and derives directly from two inescapable characteristics of the real world and of human knowledge (Peuquet 2002): first, entities in the real world often do not have sharp boundaries. For example, forests, shorelines, and urbanised areas all tend to be bounded in space by transition zones; second, boundaries are also often fuzzy in the temporal dimension.

Conventional set theory deals with precise entities. Fuzzy set theory can also deal with imprecise entities, but it usually applies standard mathematical tools to this task (Niskanen 2003).

Fuzzy systems reason with multi-valued sets or fuzzy sets (ie the sets of values between 0 and 1) instead of bivalued sets or crisp sets (ie the sets of value of 0 and 1). An advantage of fuzzy classification techniques lies in the fact that they provide a soft decision, a value that describes the degree to which a pattern fits with a class. For comparison, in crisp sets, one makes a hard decision, ie a value to which a pattern matches a class or not (Matlab 2004).

The most popular models of fuzzy systems are the Mamdani models and the Takagi-Sugeno-Kang (TSK) models. More information can be found in Niskanen, 2003 and references therein.

In fuzzy modelling, there are different membership function types available. The formula for linear function could be presented as (Matlab 2004):

$$\mu\left(x; c_1\right) = \left[1 - c_1 \left|x - x_0\right|\right]^+. \quad (5)$$

Because of the linear dependence of each rule on the input variables of a system, the Sugeno method is ideal for acting as an interpolating supervisor of multiple linear controllers that are to be applied, to different operating conditions of a dynamic nonlinear system (Niskanen 2003).

A typical rule in a Sugeno fuzzy model has the form (Matlab 2004): if input 1 is x and input 2 is y, then the output function has the form:

$$z = ax + by + c. \quad (6)$$

3. Overview of data sources

3.1. Input data

In this research three different kinds of data sources were used. First, the rectangular coordinates and ellipsoidal heights of known points from Estonian National Geodetic Network were used. The accuracy of the mentioned coordinates is about 1 cm (Rüdja 1999). Second, the normal heights in BK77 system from the levelling campaign held in 1998 were used. The accuracy of the levelled heights is about 1 cm (Torim *et al.* 1998). In Fig 1 the National

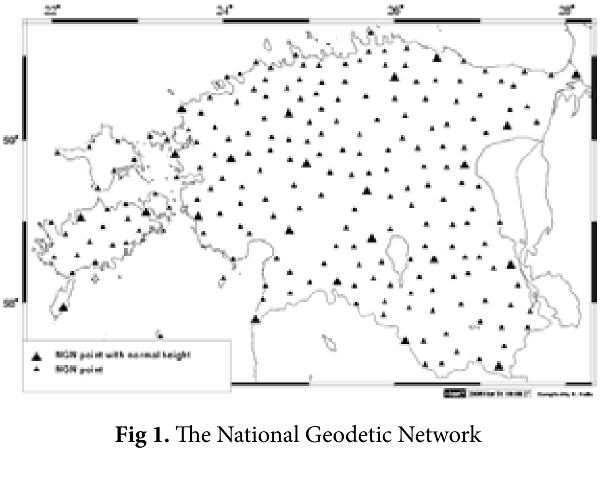

Fig 1. The National Geodetic Network

Geodetic Network is presented. The first two data sources are treated as input data in this research.

Third, the height data from Estonian reference geoid model (cf. Jürgenson 2003). The accuracy of the geoid model is about 1 … 3 cm (Jürgenson 2003). In Fig 2, the reference geoid model is presented.

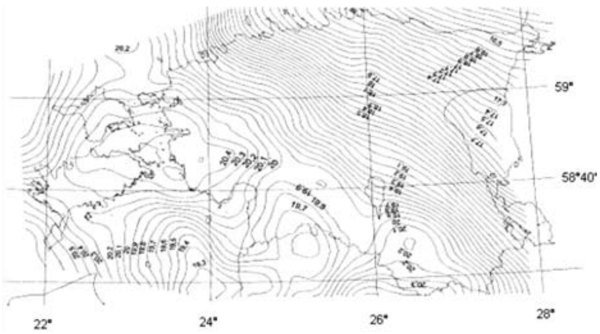

Fig 2. Reference Geoid Model EstGeoid2003 (Jürgenson, 2003)

This dataset is used for comparison with computed heights from two alternative transformation approaches.

3.2. Sample datasets

As input data, 23 points with their rectangular coordinates, ellipsoidal and normal heights were used. In the computational process, the height values for 125 points were obtained, using bilinear transformation approach and fuzzy modelling. For points to be determined, rectangular coordinates and ellipsoidal heights were used as the initial data (Fig 3).

All rectangular coordinates were in L-Est97 coordinate system, based on the ellipsoid GRS-80.

Normal heights of the points were given in BK77 height system, ellipsoidal heights were based on the ellipsoid GRS-80. All mentioned data were obtained from the Geodetic Database in Estonian Land Board.

Altogether three test areas with different triangle side were used. The triangle sides for test areas were:

- triangle side about 60 km – Model 1 (upper right);
- triangle side about 85 km – Model 2 (upper left);
- triangle side about 153 km – Model 3 (lower middle) (Fig 3).

4. Computations

4.1. Bilinear transformation approach

Bilinear transformation was computed by formulas shown in Section 2.1. For the points inside the triangle, height value was computed using point coordinates and ellipsoidal height. As initial data, the rectangular coordinates and height in BK77 system as well ellipsoidal height were used. For the statistical analysis the mean, max, min, standard deviation and root mean square (RMS) values were calculated. In the Table 1 all statistical values for bilinear approach are shown.

4.2. Fuzzy approach

The idea of using fuzzy theory as the tool for creating the elevation surface was presented in (Kollo and Sunila 2005).

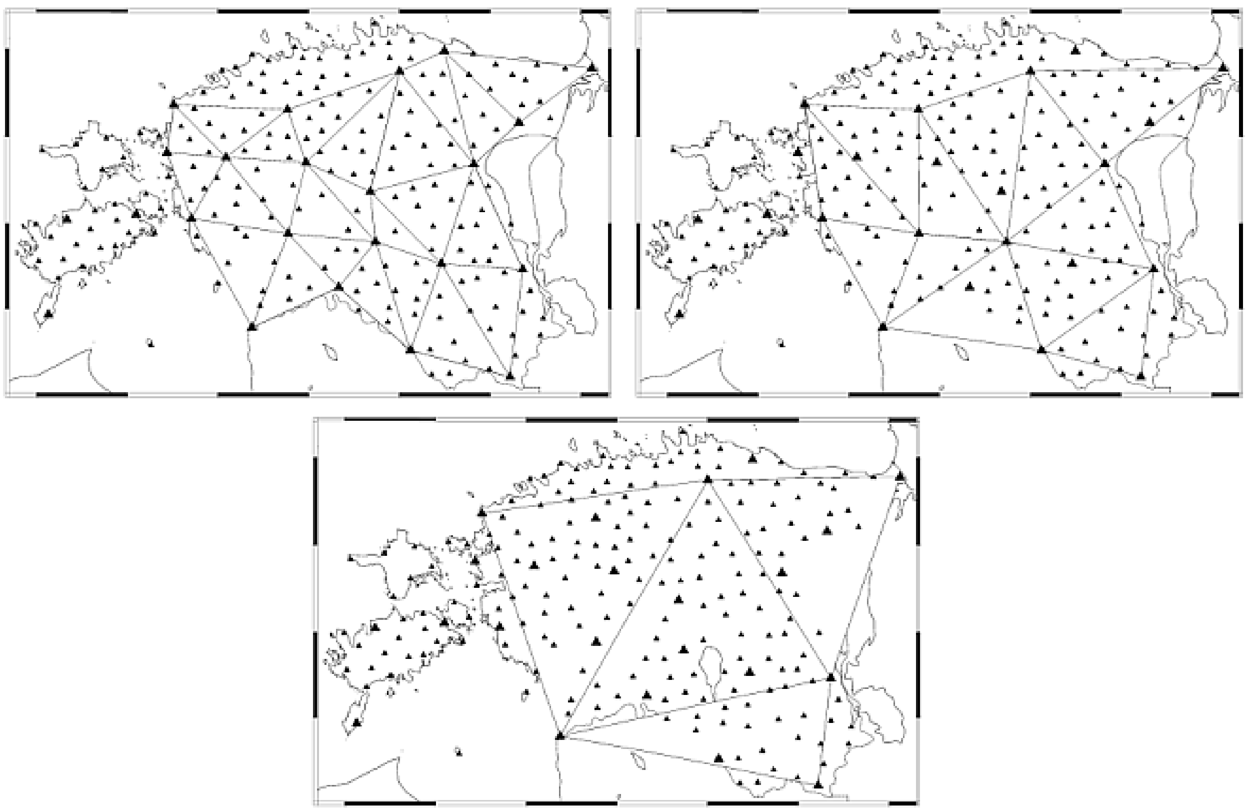

Fig 3. Test areas for transformation approach: Model 1 (upper left), Model 2 (upper right), Model 3 (lower middle)

Table 1. Statistical quantities for bilinear transformation compared to EstGeoid2003

Statistical quantities (cm)	Model 1	Model 2	Model 3
Mean	−4,65	−8,98	−46,46
Minimum	−33,1	−56,89	−110,42
Maximum	24,52	25,19	11,05
Standard deviation	9,96	14,98	31,06
RMS	0,90	1,36	2,81

Table 2. Statistical quantities for fuzzy approach (Gaussian membership function) compared to EstGeoid2003

Statistical quantities (cm)	Model 1	Model 2	Model 3
Mean	−3,16	8,24	31,45
Minimum	−108,75	−61,08	−12,45
Maximum	117,32	63,07	140,92
Standard deviation	40,46	23,34	27,28
RMS	3,65	2,10	2,46

Table 3. Statistical quantities for fuzzy approach (triangular membership function) compared to EstGeoid2003

Statistical quantities (cm)	Model 1	Model 2	Model 3
Mean	−3,22	7,32	28,80
Minimum	−115,26	−51,86	−12,65
Maximum	79,09	69,11	139,88
Standard deviation	33,79	21,76	26,82
RMS	3,05	1,96	2,42

In the computations the Matlab Fuzzy Toolbox was used. In order to test the suitable fuzzy algorithm, different models with different membership functions were created. In testing procedure the following membership functions were used: triangular, trapezoidal, bell-shaped and Gaussian functions with different number of membership function. From the testing procedure the triangular and Gaussian membership functions were chosen as the most suitable ones.

In computation three input datasets were used according to the triangle side length 60 km, 85 km and 153 km for Model 1, Model 2 and Model 3, respectively. During the computations the same input data set was used as in bilinear transformation approach.

The elevation surface was calculated, using Gaussian and triangular membership functions with number of membership functions was 5, 4 and 3 respectively for Models 1, 2 and 3. The predicted formal error for fitting the elevation surface was about 10 cm and less in both cases.

To compute the elevation values for test points, the defuzzification method was used. For statistical analysis the same statistical quantities were used, namely mean, max and min differences, standard deviation and root mean square (RMS) values. In the Table 2 the statistical quantities for the fuzzy approach with Gaussian membership function and in the Table 3 the fuzzy approach with triangular membership function are given.

5. Results

For the comparison of the results, two sets of figures are presented: differences of transformed heights from reference geoid model EstGeoid2003 (Fig 4) and histograms (Fig 5) for both transformation approaches.

For the discussion of results we begin with the statistical quantities from Tables 1, 2 and 3.

From Tables 1, 2 and 3 we could see that the best statistical quantities are for bilinear transformation approach, Model 1. From fuzzy method, the best statistics is for triangular approach, Model 2. The reason is somehow predicted, because Estonia is a flat country and the geoid is changing slowly and nearly linearly (cf. Fig 2), the linear transformation method (from fuzzy – triangular membership function) can easily be used.

For bilinear approach (cf. Table 1) we see that Min and Max values are well balanced for Model 1, for Models 2 and 3 the balance is tilted more to the negative values, moreover, all models have significant negative values for Mean. Standard deviation value is for Model 1 about 10 cm, for Models 2 and 3 this value increases slightly. As RMS value stands for inner quality control, we see that Model 1 has better inner quality as Models 2 or 3 (RMS value for Model 1 is about 1 cm). For Model 3 we see that the RMS value has increased more than 3 times comprising with Model 1 and considering the fact, that most of the differences are negative, which cause worst statistical quantities for that Model.

For fuzzy approach (Tables 2 and 3) one could see that statistical quantities are within the same range with only small differences between two different types of fuzzy approaches (Gaussian and triangulated membership functions). Better statistical quantities stand for fuzzy model, computed with triangular membership function. Comparing different Models in fuzzy approach, we see that better statistical quantities are for Model 2. Models 1 and 2 have well balanced Min and Max values, whereas Model 3 has mainly positive differences. Standard deviation values are about 20 cm for Model 2 for both Gaussian and triangular approaches. Inner quality, presented by RMS is better for Model 2, but the RMS values for Models 1 and 3 do not increase as much as we see it in bilinear transformation approach (cf. Table 1). Standard deviation and RMS values tend to be slightly bigger for Gaussian membership approach (cf. Tables 2 and 3).

To investigate the increase in standard deviation values for fuzzy approach, individual residuals were once more brought into focus. This study did show that for some regions the Gaussian membership function did give smaller residuals (especially in the hilly parts of Estonia). For the triangulated membership function the situation was vice versa, but differences in hilly parts were not as big as for the Gaussian membership function in the flat areas. This phenomenon might cause an increase in standard deviation for the Gaussian membership function. This is due the fact, that in our assumptions we have used linear transformation form, but as in hilly regions the change of geoid is not linear, so the differences between geoid heights and transformed heights are much bigger and thus affecting on the standard deviation values.

From the individual study of residuals the absolute scale for differences was as well studied. It did show that the best distribution was for Model 1, bilinear transfor-

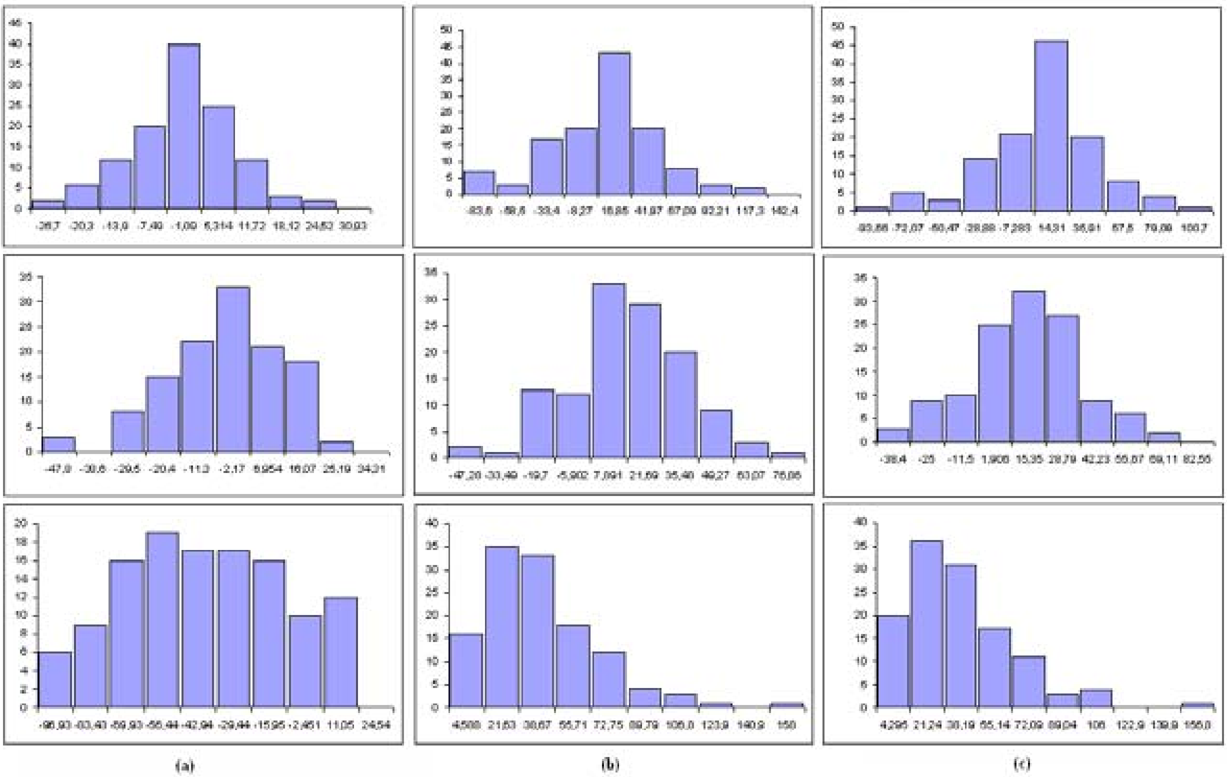

Fig 4. The histograms of computed models: (**a**) bilinear transformation (left); (**b**) fuzzy approach, Gaussian membership function (middle); (**c**) fuzzy approach, triangular membership function (right) for Model 1 (upper) and Model 2 (middle) and Model 3 (lower) respectively. The scale in vertical axis is given in cm

mation approach, while differences being in the range of +30 and –30 cm. For fuzzy approach, in both cases the best distribution was for Model 2 while the scale was in range +80 … –80 cm. The biggest scale appears for Model 3 in all transformation approaches and the differences were mostly negative or positive, which might indicate the presence of systematic errors.

For the next histograms are studied more in detail In the Fig 4 the histograms are presented respectively for bilinear and fuzzy approach (Gaussian and triangular membership functions). As we see from Fig 4a, the best distribution is for bilinear transformation approach, Model 1, and the worst is as well for bilinear approach, Model 3. For fuzzy approach (Fig 4b and Fig 4c), all histogram distributions look mostly the same, with small differences. That is understandable, because in fuzzy approach, we are calculating the height surface, where the error values are distributed equally over the whole transformation surface. Anyhow, the histograms for triangular membership function seem to give slightly better distributions as the ones for Gaussian membership functions.

6. Conclusions

This paper illustrates the possibilities of using two kinds of transformation approach: bilinear transformation and fuzzy modelling in order to predict height corrections from precise measurements of ellipsoidal and normal heights at a set of given points.

Our objective was to study, how well we can predict geoid heights to be used for height correction in the absence of geoid model. From computations we could see, that both used methods have their advantages and disadvantages.

The bilinear transformation approach has a very simple and understandable mathematical structure. The method is linear by its nature and it suits well with Estonian geoid model.

One might argue that the use of fuzzy methods is not appropriate in geodesy. On the other hand, the use of fuzzy methods simplifies the mathematical models used in geodetic applications. These models may be high degree polynomials, which cannot be computed and visualised in an easy way. The use of fuzzy methods gives us a possibility to simplify the mathematical approach.

In Fig 5 the height surfaces for the computed models are given. These models are in effect geoid models but based only on computed GPS and levelled height differences in the known points. The geoid model Est-Geoid2003 is only used for comparison. From Fig 5 we could see, that the height surfaces are mostly the same, but with smoother peaks in fuzzy approach.

To investigate the difference between bilinear approach and fuzzy triangular approach, these differences are visualised in Fig 6. Analysing the Fig 6, we see, that bigger differences are in regions, were Fig 5 shows peaks, and smaller differences are in flat areas. To investigate these differences more in detail, the differences are plotted and histogram is drawn (cf. Fig 7).

From last one we see that most of the differences are between +50 and –50 cm, some of them being over 50 cm. Considering statistical quantities and Fig 4, we could conclude, that the bilinear approach suits for the

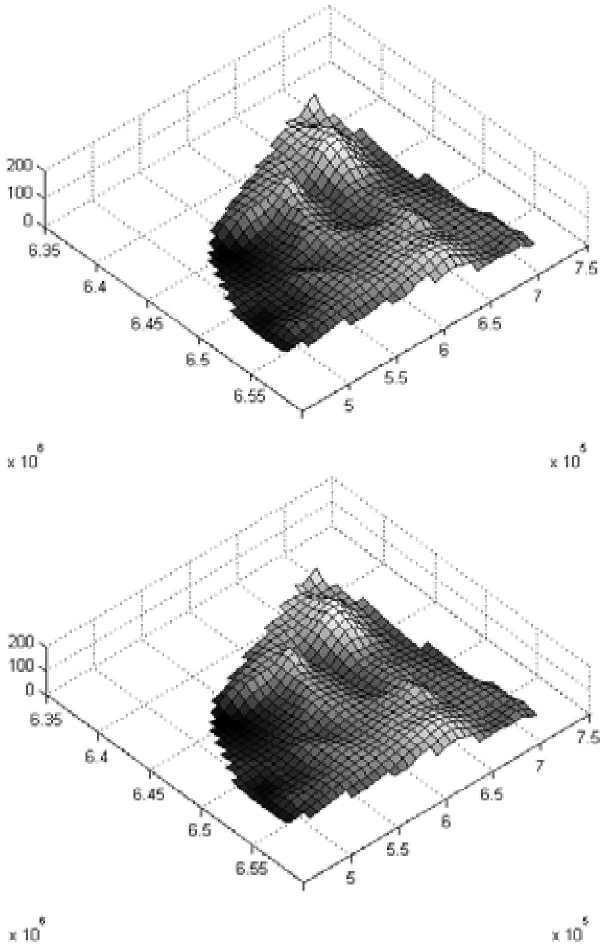

Fig 5. Height surfaces ("geoid models") for bilinear approach (left), and fuzzy approach, triangular membership function (right). The scale is given in m

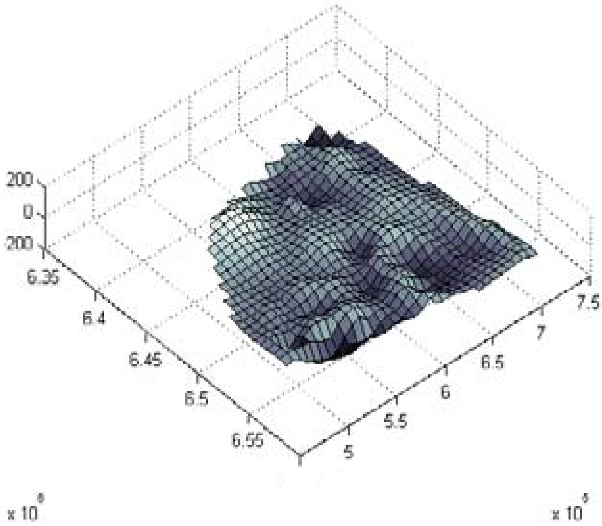

Fig 6. Height surface representing the difference between heights from bilinear transformation and fuzzy approach (triangular membership function). The scale is given in m

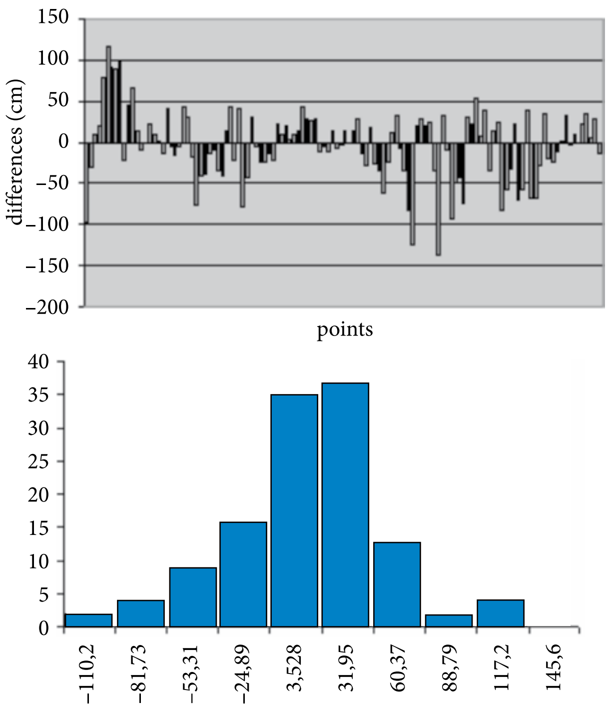

Fig 7. Height differences between bilinear and fuzzy approach (upper) and histogram for these differences. For lower figure the scale for vertical axis is given in cm

given dataset better than fuzzy method. However, fuzzy method is very powerful, and with a denser initial dataset we could achieve better results as we have now.

Within this article, we have shown the possibility of usage of two different kinds of approaches for computing height information. Firstly – bilinear transformation. The weakness of the approach is its linear form (geoid is not linear!). Within our initial dataset we did achieve acceptable results for geodetic works which need accuracy around 10 cm. Secondly – fuzzy modelling. The use of fuzzy modelling is not often used in geodetic computations, but the algorithm did show the possibility to use it in geodetic applications. The strong side of the method is the ability to handle large datasets in order to determine transformation surface. The weakness of the method is in low accuracy measures (standard deviation about 20 cm), so the results could be used only in prediction process, not in geodetic applications.

The imprecision of our results is mainly caused by too large distances between the known points. Unfortunately, there is not available newer and denser dataset for Estonia yet. The standard deviation values in Tables 1–3 show, that the accuracy of the fitting process tends to decrease. We could assume that with triangle side about 15 km, the standard deviation values around 3…5 cm could easily be achieved. For a fuzzy approach, even denser dataset could be needed (triangle side 5 to 10 km), especially in regions, where geoid changes rapidly.

In conclusion we may say that the both presented approaches can be used in height prediction process, while geoid information is not available. As well we have seen that the suitability of algorithms is dependent on the given initial dataset. Within presented study we are sure, that while denser height information is available, the accuracy of 3…5 cm is possible to reach. The accuracy of 5 cm in height values is acceptable for many applications in geodesy, including mapping, GIS, engineering and constructional geodesy.

Acknowledgements. The author would like to thank Prof M. Vermeer for his support and proofreading the early versions of the article. The author's regards goes to Mr Harli Jürgenson from Estonian University of Life Sciences for the original geoid figures and Mr Priit Pihlak from Estonian Land Board for levelling and geoid data used in this research. This study was supported by the Helsinki University of Technology.

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