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TESTING OF MARKET PRICE DIRECTION DEPENDENCE ON US STOCK MARKET

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Abstract. The correct model of a liquid financial market is very important for all market activities including for example a stock or bond portfolio management or an asset valuation.

Dynamic Financial Market Model is a comprehensive model with a detailed interpretation. The model considers also feedback processes which cause price development direction dependence on the previous development. This is why it is also able to explain departures from normality as leptokurtic deformations with fat tails and sharpness, extreme values or skewness in the returns' probability distributions. These departures are commonly explained using a wide range of models with volatility dependence. The question is then arising, whether the volatility or direction dependence is more in accordance with reality.

Price Inertia Feedback is one of the most important and has a direct impact on probability distribution and also on a price forecasting. Empirical measurement of this feedback is the core of the paper.

Keywords: Dynamic Financial Market Model, departures from normality, leptokurtic returns distribution, sharpness, skewness, feedbacks on financial market, price inertia feedback, dependence, dynamic system, simulation, back testing, S&P 500 Index, price development forecasting, speculative profit.

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JEL classification: G1, G10, G12, G14.

1. Introduction

The main contribution of this article is to measure and quantify the departures from normality of the returns' probability distributions of liquid investment instruments using a market price direction dependence on the previous development.

There is a lot of empirical evidence that in case of a price development of many liquid investment instruments we observe a not normal (Gaussian) returns' probability distribution (S&P500 daily returns' probability distribution, Figure 1). Such a distribution exhibits leptokurtic feature (characterized by fat tails at the borders and sharpness in the central area), extreme values and also skewness. There were many works performed in this area (among many others: Fama (1966: 97): *"Gaussian or normal distribution does not seem to be an adequate representation of distributions of stock price changes.*"; Peiro 1999; Ane, Geman 2000; market wide skewness measurements by Chang, Christoffersen, Jacobs 2010).

There were proposed some other distributions that can better describe departures from normality. Fama (1966) proposed symmetric stable distribution, Blattberg, Gonedes (1974), Student-t distribution.

When the distribution is not of a normal type, it means the process behind the development it is not an independent random process (independent random walk) and we have also reason to expect some price volatility or direction development rule active inside the financial market.

The serious question is how to model departures from normality with a realistic interpretation.

We have basically two ways how to approach it. The first way is to assume price volatility dependence (Campbell, Hentschel 1992; Diebold, Lopez 1995; Engle 1990, 1995; Jondeau, Rockinger 2002), the second way is to assume price direction dependence. The question is then arising, whether the volatility or direction dependence is more in accordance with reality.

The second way how to explain departures from normality is to consider price development direction dependence on the past. Modeling of departures from normality in this way is not so frequent. One rationale behind the studies of directional dependence is that economic patterns may recur in the future. Also commonly used technical analysis trading rules are based on a market price direction forecasting according to the past. We meet many interesting detailed works in area of the development direction dependence but not as a universal model which is explaining observed departures using a general mechanisms (Kendall 1953; Henriksson, Merton 1981; Krolzig 1997; Lo, Mamaysky, Wang 2000; Lillo, Farmer 2004; Tolikas, Brown 2006; Anatolyev, Gerko 2005; Vacha, Barunik, Vosvrda 2009; Huang, Wang 2010), some works are connected to the prediction of business cycles ("Predicting UK Business Cycle Regimes", Birchenhall, Osborn, Sensier (2001); etc), direction of change ideas (Rydberg, Shephard 1999).

Price direction development dependence is also taking place in the basic feedback process according to behavioral finance concept where upward trend is more likely to be followed by another upward movement (Schiller, "From Efficient Markets Theory to Behavioral Finance", 2003).

We have to mention also works of Larrain (1991), states that long term memory exists inside the financial market, other similar works of Hsieh (1991), Peters (1989, 1991, 1994) which focusing mainly on measurement of probability diversions from normality, also using Hurst coefficient, but these theories are not solving in details their explanation using basic processes and elements existed on real liquid financial market.

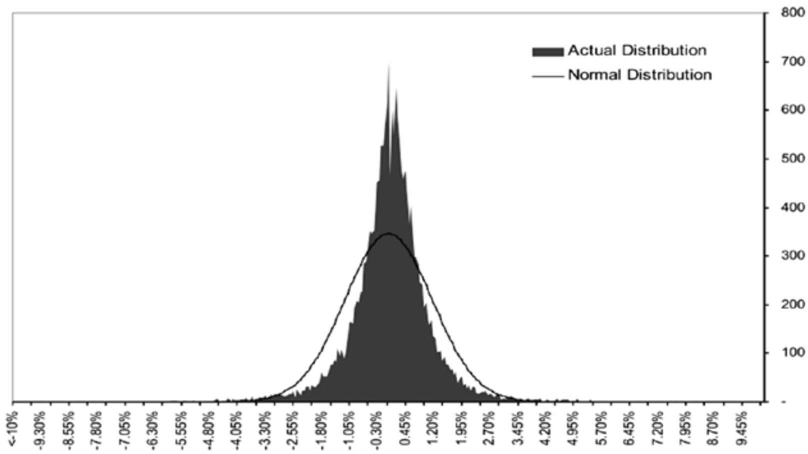

Fig. 1. Daily returns' probability distribution of S&P 500 (1927–2008)
(Source: Cook Pine Capital LLC, Study of Fat Tail Risk)

2. Dynamic Financial Market Model[1] Summary and It's Explanation of Sharpness, Skewness, Fat Tails and Extreme Values in the Probability Distribution

We have already mentioned there are two basic ways how to explain departures from normality in probability distributions. The first way is to consider price volatility dependence and the second way is to consider price development direction dependence.

We have to take into a consideration also the combination of both effects within the real financial market.

Dynamic Financial Model is a comprehensive realistic model putting great emphasis on realistic interpretation which is generally one of the basic rules when constructing a model. Model is based on development direction dependence and it has three basic presumptions:

1. primary random walk presumption,
2. feedback presumption,
3. incoming of economic news presumption (economic news = unexpected economic information which is able to influence a price development).

Primary Random Walk Presumption and Its Impact to a Probability Distribution

Idea of a primary random walk presumption is based on an empirical observation where two the same groups of buyers and sellers accept price range 10160-10340 (by the way of example) price units. Each of investors comes to the market with an order

[1] Stádník, B. 2011. *Dynamic Financial Market Model and Its Consequences* (January 18, 2011). Available at SSRN: http://ssrn.com/abstract=2062511 or http://dx.doi.org/10.2139/ssrn.206251;
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and places it to the book of orders at random during a certain period of time. They can also randomly pick up an order then. The book of orders on liquid financial market works continuously during trading hour's, collects orders and generates a price development. The sequence, frequency and the volume of market orders are generally unpredictable for each investor. If we simulate this situation we get symmetric independent random walk – **"primary random walk",** with step length equals minimum price tick given by certain market rules. There are 4 such simulations in the Figure 2.

The work of book of orders is a stable process if there is still certain amount of orders coming to bid and offer sides. This situation is usual for real financial markets. Stable situation is supported by the fact that investors are interested first of all in a future price development, which is important for their profit or loss, and accept very well current price in a wide price range. For investors is more important their opinion on the future development than on a current price value. It also means that for each price we find enough investors with an opposite opinion on a future price development.

Primary random walk can be influenced by the feedbacks and the whole process is observable also in the time intervals without incoming of economic news which is the case of simulation in the Figure 2.

Primary random walk, when not under influence of a feedback, is an independent pure random process with probability distribution of a Gaussian type.

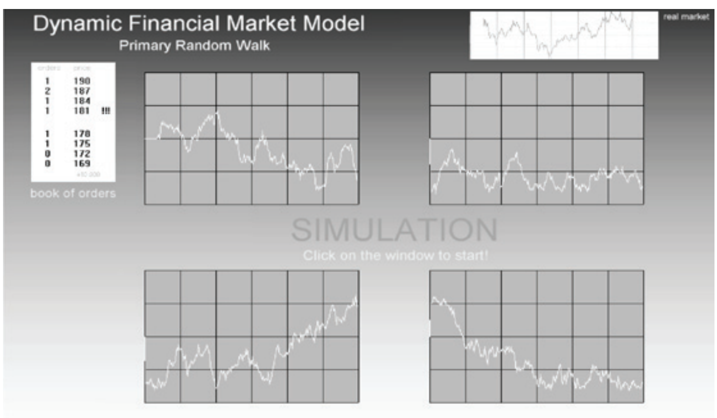

Fig. 2. Outputs from book of orders simulation - 4 independent symmetric random walks

Feedback Presumption and Its Impact to a Probability Distribution

Idea of feedback processes is based on the observations that traders, investors and other market participants don't only watch present or historical data but according to them they are also placing buy or sell orders[2] and thus influence future development. So

[2] There are many studies and empirical evidence that high percentage of investor uses for example technical analysis (75%, according to Arnold, Moizer, Noereen 1984) or other tool which is based on prediction of future development according to the past.

there is a feedback in the financial markets which also influences a future price development. Feedback can increase or decrease a frequency of incoming buy or sell orders and therefore **changes a probability of the next price step direction from 50% (in case of a pure symmetric random walk) to for example 51%**. Feedback is triggered or cancelled according to the past or current circumstances. Feedback can work together or against another feedback. Feedbacks influence and deform the primary random walk.

According to Dynamic Financial Market Model's presumptions we expect more feedbacks[1] (price development limits, technical analysis, trend stabilizing, price inertia, trading techniques, different up/down movements, market price manipulations, market regulations, round numbers, logarithmic correction of a price, etc.)

The most important feedbacks which are observable within the financial markets are:

Price Inertia Feedback

Price inertia is a basic negative feedback which helps to keep price to be unchanged. Feedback works in all periods of time as a minute, hour or day. If there are not any economic news, primary random walk is forced by traders towards the level (as it is shown in the Figure 3) which is adequate to the previous economic news level or to the other levels. Other level can be previous day closing price, day opening price, support or resistance levels given by technical analysis, etc. Especially a closing price is considered to be reflecting all the economic news during a day. Over a long time period traders prefer to close long positions above the level or open long positions below this level. Analogically they prefer close short positions below the level and open short position above the level. Some of them also believe that the price has a tendency to return to the adequate level and support this idea by their own orders. Later in the discussion we will conclude that if only approximately 1% of traders participate on the feedback then the probability distribution is deformed in the same way as we observe in reality.

Trend Stabilizer Feedback

Trend stabilizer feedback is a positive feedback which is stabilizing trends and keeps the price development in a trend direction. Feedback works in all period of time. When a trend formation appears, trend stabilizing is triggered. The principle is shown in the Figure 3. Trend stabilizing (supporting) has an origin in psychology of investors. Trend stabilizer feedback can work against price inertia feedback and try to distribute price from the level. A good example of a price trend stabilizing is a creation of market bubbles.

Trading Techniques Feedback

For example a daily gap trading is a popular trading technique for many liquid investment instruments. Traders believe gaps opened in the morning will be closed during a day trading. They place orders to support this idea. This technique is actually a price inertia feedback on daily basis.

Many techniques are also based on level-level trading. It means if any level is broken, the movement will keep the direction. This technique is actually a trend stabilizer feedback on the daily basis. If level is not broken the market price will return back or keeps the trend. Many levels are represented with round numbers (10, 15 ... 100, 200).

Trading techniques can be recognized not only on daily basis but also during other time periods (on daily basis are the most significant).

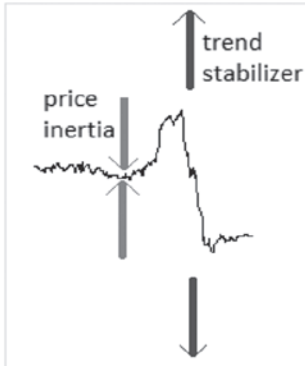

Fig. 3. Price inertia and trend stabilizer feedbacks

Different Up/Down Movements Feedback

Empirical experiences indicate that the dynamics of the market downward movements is different than in case of market upward movements. Downward movements are quicker and less frequent. This effect is probably caused by similar processes as for example a stronger downward movement during a financial crisis or when market price bubble bursts.

Feedbacks´ Impact to a Probability Distribution

In case of a price inertia feedback we expect impact on sharpness of the distribution, in case of trend stabilizer we expect impact mainly on the fat tails and extreme values.

On real financial market we can observe these effects in the chart. There can be differences recognized visually among three charts in the Figure 4a, which can be explained using feedbacks 4b (the most feedbacked price development is the second one in the figure). Charts under the influence of feedbacks are more “staircase-like” than chart of the random walk, Figure 5 (the most feedbacked development is the second one in the figure; the first one is an independent random walk).

An impact to returns’ probability distribution is described in the Figure 6. The most deformed distribution (depends on an intensity of feedbacks) is the black one in the figure. Price inertia feedback keeps price near the center area of a distribution and trend stabilizer feedback distributes the price to the borders of a distribution. Due to the fractal structure it doesn’t depend on the time period in which we measure the distributions (vertical lines in the Figure 5). So for example price inertia feedback causes sharpness in the distribution regardless the starting point of the time interval for the distribution and its length. We can observe described effects in one hour or one day distributions.

From a logic point of view, different up/down movements’ feedback must influence skewness and cause different left and right tail behavior of the distribution.

Incoming of Economic News Presumption and Its Impact to a Probability Distribution

Instead of feedbacks it is empirically evident that the final price development is under the influence of a random incoming of economic news.

As an impact we expect longer steps which are consisted from a certain number of minimum price ticks. Parameters should be set according to the empirical observations for each market. There is possibility to assume the length and frequency of these steps to be independent or we can build news clustering or some other volatility dependence in.

Into this presumption we involve also unexpected steps which length is more than 1 minimum price tick and such steps are not determined by any economic news.

Important economic news can support fat tails and extreme values in the distribution.

An impact to a price development in case of less important economic information is similar to the trend stabilizer feedback. In this case we expect an impact to the probability of a next price step direction.

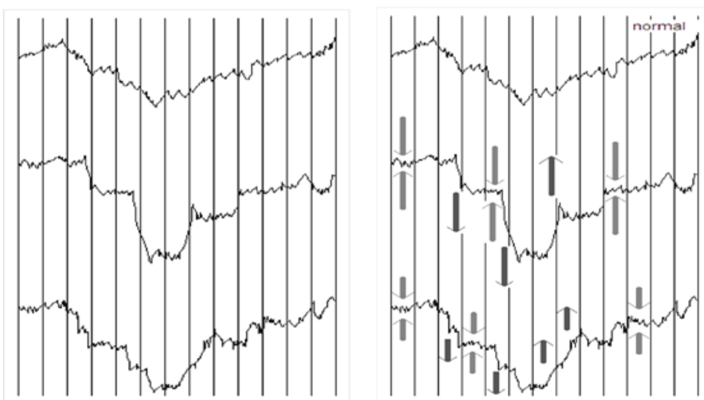

Fig. 4a, 4b. Developments with a different kurtosis

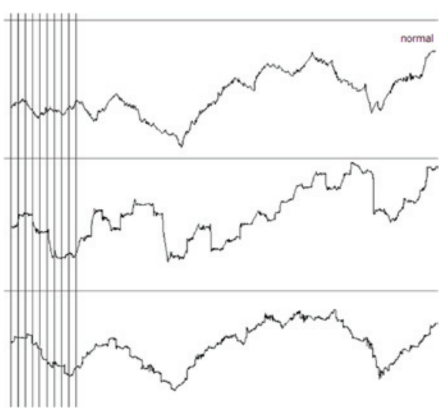

Fig. 5. Developments with a different kurtosis

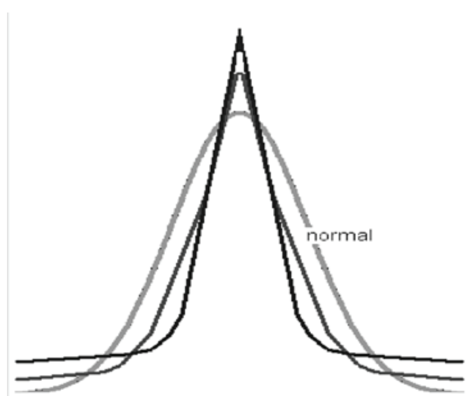

Fig. 6. Different kurtosis

3. Simple Empirical Tests of Price Inertia Feedback

A feedback should increase the value of probability of the future market price development direction from 50% to a higher value. Due to this fact we can use a feedback for speculative investments with a probability of success also higher than 50%. According to Dynamic Financial Market Model price inertia feedback is pushing the market price toward the level which is adequate to the previous economic news level or to other important level. This cause sharpness in the distribution regardless the starting point of the time interval for the distribution and its length.

The principle of empirical testing are back tests on US stocks. Back tests are based on speculative buys/sells using a price inertia feedback. As a price inertia feedback level was used the closing price of the previous day and speculative positions were opened always for 1, 2 and 3 days and then have been closed. The short positions were opened if the opening price was above the level; long positions if the opening price was below the level. For 2 and 3 days speculations we recognize 2 and 3 different days when the first speculation starts (start-1, start-2, and start-3 in the Tables 1, 2).

Back testing results of Price Inertia Feedback on 2553 US stocks over 10 and 5 years time periods are in the Tables 1 and 2. The most important result is that the win ratio is always more then 50% (50.04–51.99). The value is in accordance with a software simulation which gives value approximately around 51%.

Table 1. Results of empirical tests of price inertia feedback on 2553 US stocks over 10 years time period

May 2012, B.Stádník
Back Testing - Inertia Feedback (source: 2553 US stocks, time period: 2001-2011)

time period	intraday	2 days (start-1)	2 days (start-2)	3 days (start-1)	3 days (start-2)	3 days (start-3)
n. of stocks all	2 553	2 553	2 553	2 553	2 553	2 553
n. of stocks tested	2 196	2 196	2 196	2 196	2 196	2 196
n. of B	2 899 445	1 622 926	1 643 180	1 095 378	1 081 266	1 093 327
n. of S	3 154 594	1 675 002	1 652 304	1 104 856	1 118 871	1 106 904
n. of B+S	6 054 039	3 297 928	3 295 484	2 200 234	2 200 137	2 200 231
win ratio	**0.51992**	**0.51132**	**0.50672**	**0.50552**	**0.51126**	**0.50944**
P/L	49 935	40 961	8 116	27 946	35 875	38 293
stocks in plus	0.63964	0.65257	0.54485	0.60282	0.65648	0.65962
average deviation+/-	0.00020	0.00028	0.00028	0.00034	0.00034	0.00034

Table 2. Results of empirical tests of price inertia feedback on 2553 US stocks over 5 years time period

May 2012, B.Stádník
Back Testing - Inertia Feedback (source: 2553 US stocks, time period: 2006-2011)

time period	intraday	2 days (start-1)	2 days (start-2)	3 days (start-1)	3 days (start-2)	3 days (start-3)
n. of stocks all	2 553	2 553	2 553	2 553	2 553	2 553
n. of stocks tested	2 553	2 553	2 553	2 553	2 553	2 553
n. of B	1 303 941	710 046	707 570	467 798	475 179	475 302
n. of S	1 391 948	711 042	711 446	480 102	472 631	470 352
n. of B+S	2 695 889	1 421 088	1 419 016	947 900	947 810	645 654
win ratio	**0.50914**	**0.50968**	**0.50677**	**0.50481**	**0.51108**	**0.50704**
P/L	4 064	22 670	13 674	13 948	28 909	21 571
stocks in plus	0.64055	0.64936	0.59062	0.58561	0.68944	0.64026
average deviation+/-	0.000305	0.000419	0.00042	0.000514	0.00051	0.00062

Higher fluctuation of win ratio is caused by the correlation between US stocks. Other items in the table mean: “n. of stock all” – number of stocks available for testing, “number of stocks tested” – number of stocks under the test (some of them has no 10 years history), “n. of B” – number of buys, “n. of S.” – number of sells, “P/L” – profit/loss (each speculation with 1 stock), “stocks in plus” – number of stocks in profit, “average deviation” – standard deviation for a case of an independent binomial process.

The win ratio result in this test is not under the influence by upward/downward long-term trend.

4. Conclusion

Dynamic Financial Market Model is the model of liquid financial markets putting an emphasis on a realistic interpretation. Model is based mainly on the precise description of an internal structure and internal mechanism on the lowest level of the system.

Dynamic Financial Market Model considers feedback processes within financial markets which cause the price step direction dependence on the previous development.

Idea of feedback processes is based on the observations that investors and other market participants don’t only watch present or historical data but according to them they are placing buy or sell orders and thus influence future development. Feedback can increase or decrease a frequency of incoming buy or sell orders and therefore changes the probability of the next price step direction from 50% (in case of pure symmetric random walk) to another value. System of feedbacks includes price development limits, technical analysis, trend stabilizing, price inertia, trading techniques, different up/down movements, market price manipulations, market regulations, round numbers, logarithmic correction of a price, etc.

In this study we were also trying to confirm the existence of a price inertia feedback within the liquid financial market.

Empirical tests of the price inertia feedback according to the model have supported its existence. Empirically obtained probability of a future price development direction varies approximately from 50% to 52% (50.04–51.99%). Back tests were done on approximately 2500 US stocks over 10 years time period.

Alternatively for modeling of abnormalities we can use models with volatility dependence. We have to take into a consideration also the combination of both effects present in the real financial market.

We can conclude that the empirical measurements support correctness of the Dynamic Financial Market Model with its feedback processes. We can also state that the departures from normality in returns' probability distributions can be caused by the market price development direction dependence on the previous development. According to the empirical results, the intensity of a price inertia feedback, which is active over long time period and cause sharpness in the probability distributions, is very small to significantly help in the predictability of the future price development.

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APPENDIX

Mathematical Description of the Feedback Processes

For the explanation of basic principles of mathematical description we use a simple example named "5 steps" together with an incoming of important economic news. In "5 steps" we consider a specific case of a trend stabilizer feedback when the probability of a future step will increase to 52% (from 50%) in case that previous 5 steps were upwarding and decrease to 48% (from 50%) in case that previous 5 steps were downwarding.

We consider two possible price step directions (up/down) with step length=minimum price tick (given by the certain market rule) in case of primary random process (primary random walk which can be feedbacked). For important economic news we consider then longer steps with step length=integer number of minimum price ticks.

Calculation of a probability distribution of "5 steps" (using two dimensional non-homogenous Markov chain)

For the mathematical description we use two dimensional non-homogenous Markov chain. The first process is connected with the distance from the initial value and the second with the number of steps consecutively with the same direction. The all 5 steps history, which is required, will be included to the certain states and each of the states will determine the probability of the next step. The deepness of the history is 5 steps.

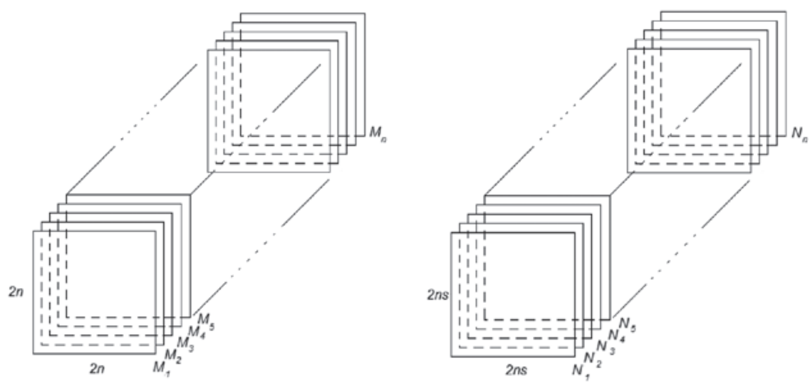

Fig. 7. Two dimensional non-homogenous Markov chain

One dimensional homogenous Markov chain is given by state transitions matrix *P(s, s)*, where *s* is number of states *E* and $p_{a,\,b}$ denotes transition probability from state *a* to state *b*.

$$P = \begin{bmatrix} p_{00} & p_{01} & p_{02} & \cdot \\ p_{10} & p_{11} & p_{12} & \cdot \\ p_{20} & p_{21} & p_{22} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Probability of state transition: $E_0 \rightarrow E_1 \rightarrow ... \rightarrow E_{s-1}$, denoted as $P(E_0, E_1, ..., E_{s-1})$ is given by formula:

$$P\,(E_0, E_1, ..., E_{s-1}) = a_0 \cdot p_{1,2} \cdot p_{2,3} \cdot ... \cdot p_{s-3,\,s-2} \cdot p_{s-2,\,s-1},$$

where a_0 is the probability of an initial state.

$$\sum_{i=0}^{s-1} p_{ji} = 1,$$

$$x_1 = x_0 \cdot P,$$

$$x_n = x_0 \cdot P^n,$$

where x_n is the state probability vector after n steps.

For the two dimensional non-homogenous Markov chain (Fig. 7 – left side) for our example "5 steps" we obtain:

$$x_n = x_0 \cdot M_1 \cdot M_2 \cdot ... \cdot M_n,$$

$$x_n = x_0 \cdot \prod_{k=1}^{n} M_k,$$

where $M_k\ (2n,2n)$ is the state transition matrix for the k^{th} step.

For our example "5 steps" for the first 3 steps we get following matrixes:

$$M_1 = \begin{bmatrix} ... & ... & ... & ... & ... & ... & ... \\ ... & 0 & 0 & 0 & 0 & 0 & ... \\ ... & 0 & 0 & 0 & 0 & 0 & ... \\ ... & 0 & 0.5 & 0 & 0.5 & 0 & ... \\ ... & 0 & 0 & 0 & 0 & 0 & ... \\ ... & 0 & 0 & 0 & 0 & 0 & ... \\ ... & ... & ... & ... & ... & ... & ... \end{bmatrix} \qquad M_2 = \begin{bmatrix} ... & ... & ... & ... & ... & ... & ... \\ ... & 0 & 0 & 0 & 0 & 0 & ... \\ ... & 0.5 & 0 & 0.5 & 0 & 0 & ... \\ .... & 0 & 0 & 0 & 0 & 0 & ... \\ ... & 0 & 0 & 0.5 & 0 & 0.5 & ... \\ ... & 0 & 0 & 0 & 0 & 0 & ... \end{bmatrix}$$

$$M_3 = \begin{bmatrix} ... & ... & ... & ... & ... & ... & ... & ... & ... \\ ... & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ... \\ ... & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & ... \\ ... & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ... \\ ... & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & ... \\ ... & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ... \\ ... & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & ... \\ ... & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ... \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \end{bmatrix}$$

Matrixes are identical to the symmetric random walk's matrixes up to the 5^{th} step.

For our example "5 steps" for the first 3 steps we get:

$$x_0 = \begin{bmatrix} ... & 0 & 1 & 0 & ... \end{bmatrix},$$

$$x_3 = \begin{bmatrix} \dots & 0 & 0.125 & 0 & 0.375 & 0 & 0.375 & 0 & 0.125 & 0 & \dots \end{bmatrix},$$

$$x_3 = x_0 \cdot M_1 \cdot M_2 \cdot M_3, \; x_3 = x_0 \cdot \prod_{q=1}^{3} M_q .$$

For more steps we can calculate transition probabilities according to:

$$m^{(k)}{}_{i,j} = \frac{P(E^{(0)}{}_0 \to E^{(k-1)}{}_i, E^{(k)}{}_j)}{P(E^{(0)}{}_0 \to E^{(k)}{}_i)},$$

where $m^{(k)}{}_{i,j}$ is the transition probability from *i* to *j* for step *k*, $E^{(0)}{}_0$ is the initial state and $E^{(k)}{}_i$ is the state *i* after *k* steps, *i* is of values *(-n)* to *n*, begins from 0 and tells how many steps "down" or "up" we are from initial value.

$$P(E^{(0)}{}_0 \to E^{(k)}{}_i) = \sum_{all\_paths} P_{path}(0 \to i),$$

$\sum_{all\_paths} P_{path}(0 \to i)$ is the sum of probabilities of all paths reach point *i* in *k* steps with the respect of the transition rule from certain state. For the recapitulation, in our example the rule is: probability of a future step direction up will increase to 52% (from 50%) in case that previous 5 steps were upwarding and decrease in to 48% (from 50%) in case that previous 5 steps were downwarding.

$P(E^{(0)}{}_0 \to E^{(k-1)}{}_i, E^{(k)}{}_j)$ is the probability to reach state $E^{(k)}{}_j$ through state $E^{(k-1)}{}_i$.

Numerical calculation

For a numerical calculation we consider another two dimensional non-homogenous Markov chain (Fig. 7 – right side) defined by a transition matrix N_k *(2ns, 2ns)*, which can be better represented by three dimensional matrix of transition probabilities Z_k *(2n, 2n, s)* and $z^{(k)}{}_{i,j,d}$ is the probability of change from *i* to *j* in state *d* in the k^{th} step (Fig. 8).

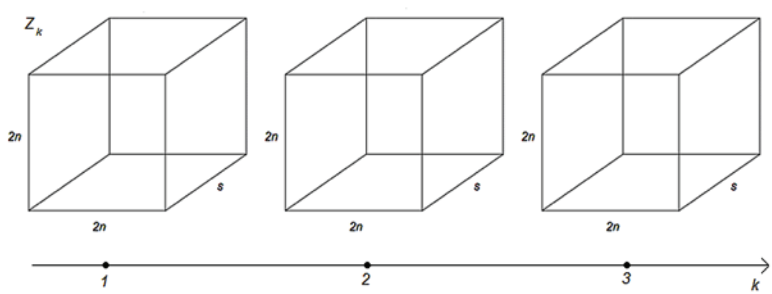

Fig. 8. Two dimensional non-homogenous Markov chain

In our example d is of values *{-5,-4,-3,-2,-1,1,2,3,4,5}*, which express s=*10* states for each *k, i*.

Description of states:

State *1* means, that the previous step was „up“, but a step before the previous one was not „up“.

State *2* means, that the previous 2 steps were „up“, but a step before was not „up“.

State *3* means, that the previous 3 steps were „up“, but a step before was not „up“.

State *4* means, that the previous 4 steps were „up“, but a step before was not „up“.

State *5* means, that the previous 5 steps were „up“, but a step before was not „up“.

Analogically for downward steps.

Modeling of a calculation:

$$y_n = y_0 \cdot \prod_{k=1}^{n} N_k ,$$

where y_k is vector *(2ns)* and gives the probabilities of states for each *i, k* and y_0 an initial vector.

We also obtain:

$$m^{(k)}{}_{i,j} = \sum_d z^{(k)}{}_{i,j,d} \text{ , where } d=-1,-2,-3,-4,-5,\ 1,\ 2,\ 3,\ 4,\ 5.$$

Results of a numerical calculation of our example “5 steps” (with probabilities changed to 70/30) is in the Fig. 9.

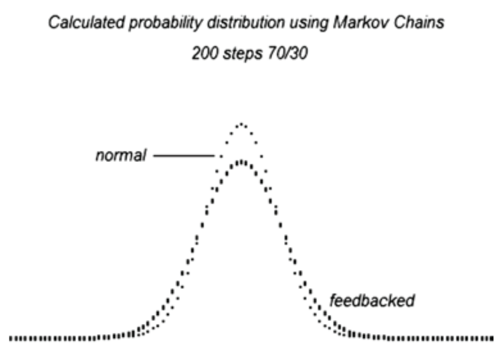

Fig. 9. Calculated departured distributions

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