BUSINESS, MANAGEMENT AND EDUCATION
ISSN 2029-7491 / eISSN 2029-6169
2014, 12(2): 245–265
doi:10.3846/bme.2014.235

CHARACTERISTICS OF OMEGA-OPTIMIZED PORTFOLIOS AT DIFFERENT LEVELS OF THRESHOLD RETURNS

Renaldas VILKANCAS

Vilnius Gediminas Technical University, Saulėtekio al. 11, LT-10223 Vilnius, Lithuania
E-mail: renaldas.vilkancas@vgtu.lt

Received 31 October 2014; accepted 07 November 2014

Abstract. There is little literature considering effects that the loss-gain threshold used for dividing good and bad outcomes by all downside (upside) risk measures has on portfolio optimization and performance. The purpose of this study is to assess the performance of portfolios optimized with respect to the Omega function developed by Keating and Shadwick at different levels of the threshold returns. The most common choices of the threshold values used in various Omega studies cover the risk-free rate and the average market return or simply a zero return, even though the inventors of this measure for risk warn that "using the values of the Omega function at particular points can be critically misleading" and that "only the entire Omega function contains information on distribution". The obtained results demonstrate the importance of the selected values of the threshold return on portfolio performance – higher levels of the threshold lead to an increase in portfolio returns, albeit at the expense of a higher risk. In fact, within a certain threshold interval, Omega-optimized portfolios achieved the highest net return, compared with all other strategies for portfolio optimization using three different test datasets. However, beyond a certain limit, high threshold values will actually start hurting portfolio performance while meta-heuristic optimizers typically are able to produce a solution at any level of the threshold, and the obtained results would most likely be financially meaningless.

Keywords: downside risk, Omega function, portfolio optimization, threshold return, differential evolution (DE).

JEL Classification: D81, G11, C61.

1. Introduction

Since Markowitz (1952) has first introduced his Portfolio Theory, the search for an optimal return-risk ratio is concerned as the "Holy Grail" of investment management. In order to measure risk, Markowitz used variance, i.e. a symmetric measure of risk that equally assesses both negative and positive risk deviations. If returns and losses were distributed according to the Gaussian law, we could construct efficient portfolios based solely on a Mean-Variance Model proposed by Markowitz. The fact that variations in price are not normally distributed and asymmetric was observed long ago by Mandelbrot (1967) and Fama (1965). Today, there is no doubt, that extreme changes in security prices are much more common than one could expect with reference to the Gaussian random process, which means that the actual risk of the investment portfolio, faced by the portfolio manager, is significantly higher than that represented by variance historically widely used for risk measurement. In addition, variance, along with other symmetrical risk measures, both positive and negative risk deviations from the average rate is considered as a source of risk, i.e. "does not discriminate" the risk of losses, although investors are concerned about losses rather than about an opportunity to earn higher returns. There is no doubt that investors differently assess "downside" and "upside" risks thus claiming priority to positive asymmetry. After all, a successful investment is the one that brings in gains rather than losses. Therefore, an appropriate risk measure should also differently treat downside and upside risk.

One of such measures are the Omega function proposed by Keating and Shadwick (2002a, 2002b). This indicator is a ratio of the expected excess returns over a threshold to the expected loss below the same threshold level and allowing an objective assessment of the investment when returns are characterized by asymmetry and heavy tails. Unlike Sortino, the Upside-Potential or Kappa ratios, the Omega ratio is the first-order ratio of upper and lower partial moments. The first order means that the investor's preference in positive and negative deviations from the target (threshold) return is symmetrical (Farinelli, Tibiletti 2008), and therefore, at first glance, the use of the Omega ratio may seem paradoxical: after all, undesirable negative deviations must be "severely punished" and the desired positive ones should be encouraged. However, the author of this article emphasizes that the ratio has several advantages. First, although the avoidance of loss is often equated to the avoidance of risk, the two concepts are not necessarily identical. The Prospect Theory proposed by Kahneman and Tversky (1979) hypothesized that, in order to avoid loss, investors were more inclined to take risks in the field of loss and less risk for the gain. Such a conclusion seems surprising but explains the behaviour of investors when they sell securities, the price of which is going up too quickly or keep securities, the price of which is going down, for too long, experiencing higher and higher losses but believing that until securities are not sold, which is a "paper" loss only and they will still manage to "win back". Objective and "undistorted" information provided by the first-order partial moments of the Omega ratio allows assessing return asymmetry without assumptions about investors' risk preferences that are subjective, depend on the selected benchmark and may change depending on the situation. Second, when selecting a threshold rate of return, the investor expresses the objectives of investment and risk tolerance. Although researchers and practitioners observe this fact (e.g. the term "target return" at the beginning used by Sortino was later replaced by the term "minimum acceptable return", as investment fund managers started seeking for high returns despite the growing risk), still there is no elaborate discussion or specific recommendations on how this threshold level should be set. In practice, the threshold rate of return is generally set to the risk-free rate, the average expected return or simply a zero. The author believes that controlling investors' "risk appetite" for changing the threshold rate of return is much "more natural" than changing the orders of partial moments. Third, no less important reason is that the results of empirical studies show that the Omega ratio can be successfully used for the management and optimization of investment portfolios, although these studies are not numerous.

This study complements scientific literature regarding three important aspects. First, it provides theoretical and empirical investigation on the characteristics of Omega-optimized portfolios at different levels of threshold returns. Second, after back testing of the Omega model using historical stock data on DJIA, EURO STOXX 50 and FTSE 100 indices, the main characteristics of Omega-optimized stock portfolios were identified. Third, the received results were directly compared with those obtained using classical portfolio optimization methods, including most widely applied modifications as well as the portfolios constructed using alternative techniques for heuristic portfolio optimization.

After model backtesting with historical data, the results suggest that increasing the Omega ratio threshold level leads to higher returns, and absolute net (i.e. after the deduction of turnover costs) returns of Omega-optimized portfolios exceed all other tested portfolio returns. Moreover, the provided results are relatively stable within a certain interval of threshold returns. Above this interval, the portfolio return usually starts declining – meta-heuristic optimization algorithms allow portfolio optimization at any level of the threshold; however, financially, the achieved results are almost meaningless.

2. Previous research

The idea of an asymmetric risk measure is not a novelty. At the time Markowitz announced his portfolio theory, Roy (1952) proposed the concept of a portfolio based on the "safety first" principle, which means imposing constraints on portfolio positions of reducing probability that, within the next period, the gain will be lower than the critical level set in advance. Expanding his portfolio theory, Markowitz acknowledged deficiencies of variance, and alternatively, considered the use of semi-variance (Markowitz 1959). According to Markowitz, semi-variance is a better measure for risk, since it allows limiting undesirable losses only, as opposed to ordinary variance which, being limited, reduces both undesirable downside and desirable upside risks. Although mathematical convenience has resulted into that Markowitz finally gave priority to the ordinary variance, still, the search for alternative measures for risk had started to accelerate.

A general downside or the loss risk theory was developed by Bawa (1975) and Fishburn (1977) who recommended to measure risk using a lower partial moment or LPM:

$$LPM_n(\tau) = \int_{-\infty}^{\tau} (\tau - x)^{\lambda} \, dF(x). \tag{1}$$

With the help of the LPM, the risk of losses is described using two parameters: a specified target return or reference point τ, in respect of which the loss ratio is measured, and an order of LPM λ, which expresses an investor's risk tolerance. It is easy to understand intuitively that a higher order of the LPM means lower risk tolerance, i.e. negative deviations from the target return are "punished more severely". The LPM allows describing not only the quadratic utility function, i.e. semi-variance, which is a special case of the LPM, when $\lambda = 2$, but also the most of well-known von Neumann-Morgenstern utility functions. The main deficiency of the LPM recognized by Fishburn (1977) himself in his thesis is that the upside deviations of returns are measured linearly, which means that investors become risk neutral as soon as the return exceeds the specified target value. Due to this deficiency, the entire LPM theory was later severely criticized (Kaplan, Siegel 1994).

The most common downside risk indicators include the Sortino ratio, Value-at-Risk (VaR) proposed by JP Morgan Investment Group in 1994, Conditional Value-at-Risk, Maximum Drawdown, etc. While calculating the Sortino ratio (Sortino, Van Der Meer 1991), a standard deviation included into the denominator of a widely used Sharpe's ratio is substituted with a lower partial standard deviation (i.e. second order LPM) taking into account only negative undesirable volatility. Value-at-risk (VaR) and conditional value-at-risk (CVaR) are percentile measures for downside risk. Value-at-risk represents the maximum expected loss over the given period at the given confidence level and, however, underestimates the risks resulting from the excess of the selected VaR level. Conditional value-at-risk (CvaR) introduced by Rockafellar and Uryasev (2000) estimates losses exceeding the VaR level. In addition, this indicator meets the characteristics of a coherent measure (Artzner *et al*. 1999; Pflug 2000). Thus, it can be easily optimized – the efficient portfolio plot is developed by taking different values of the expected return and minimizing CVaR. The maximum drawdown proposed by Grossman and Zhou (1993) represents the difference between the minimum and maximum value of the portfolio or a position resulting over the selected period of time. Since the maximum drawdown value depends on a single estimate – the highest price, this measure is not appropriate for comparing the performance of different investment strategies for different periods of time. In this case, a much more practical measure is Conditional Drawdown-at-Risk, (CDaR) that aggregates the total number and extent of drawdowns over the selected period (Chekhlov *et al*. 2005).

Despite the common attitude that by limiting the risk of loss, albeit indirectly, but still the maximization of upside risk is ensured, the main criticizing aspect of the "risk of loss theory" is that these methods are too focused on the avoidance of losses and little attention is paid to the provision of returns (Avouyi-Dovi *et al*. 2004). A natural solution to this problem is measurements allowing independent modelling of investors' behaviour in respect of both negative and positive variations in returns: a regret-reward measure (Dembo, Rosen 1999; Dembo, Mausser 2000), the Upside-Potential Ratio, UPR (Sortino *et al*. 1999), the Omega ratio (Keating, Shadwick 2002a, 2002b), the Kappa ratio (Kaplan, Knowles 2004), etc. While using these indicators, not only downside but also upside or excess returns are measured, i.e. they allow identifying investments characterized by a relatively large excess return ascribed per unit of drawdown risk. Excess returns are generally defined as the difference between a mean and target returns, or as an Upper Partial Moment (UPM), i.e. an upside deviation from target return τ:

$$UPM_n(\tau) = \int_{\tau}^{\infty} (x-\tau)^p \, dF(x). \qquad (2)$$

Farinelli, Tibiletti (2008) introduced a generalized ratio of upper and lower partial moments – Φ, which allows expressing the favour (disfavour) of upside (downside) deviations of various investors:

$$\Phi_{(p,q,\tau)} = \frac{\sqrt[p]{UPM_p(\tau)}}{\sqrt[q]{LPM_q(\tau)}}. \qquad (3)$$

By changing parameters p and q, we respectively obtain different risk indicators: if p = 1 and q = 2, we obtain the upside-potential ratio; if p = q = 1, we obtain the Omega ratio. Thus, the Farinelli–Tibiletti Φ ratio allows expressing flexibly investor preferences in respect of returns and associated risks.

3. Advantages and disadvantages of the Omega ratio

Looking back at the evolution of return and risk measures, the Omega ratio suggested by Keating and Shadwick (2002a, 2002b) is not an entirely new idea. Kazemi *et al*. (2004) also note that the Omega can be defined as a ratio of a European call option to a European put option when the threshold return is equal to an exercise price of the option. Therefore, the Omega is not new in this relation either.

The Omega ratio is equivalent to the total distribution as it evaluates all higher-order moments. Thus, it is not necessary to rely on assumptions about investors' risk tolerance and their utility functions when using it, and hence, according to researchers, this is a "universal ratio" that helps with an objective assessment of the performance of investments.

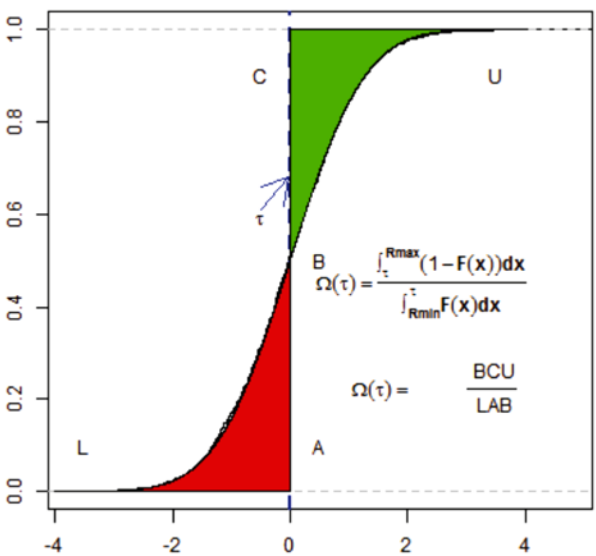

Fig. 1. The total distribution of returns and the Omega function

The Omega ratio is described as:

$$\Omega(\tau) = \frac{\int_{\tau}^{R_{\max}} \left(1 - F(x)\right) dx}{\int_{R_{\min}}^{\tau} F(x) dx} = \frac{\int_{\tau}^{R_{\max}} BCU}{\int_{R_{\min}}^{\tau} LAB}, \tag{4}$$

where τ – the threshold; R_{min} and R_{max} – the minimum and maximum values of returns respectively. When τ is closer to the R_{min} value, the BCU area is larger than that of the LAB and the Omega value is high, and vice versa. While calculating the Omega, the threshold level of return is taken into account, in respect of which the result is considered as gain or loss; thus, if τ is seen as the required rate of return, the Omega ratio shows at what extent the obtained result exceeds the expectations of the investor. Accordingly, a higher Omega ratio means higher performance, i.e. return.

Although Keating and Shadwick introduced the Omega as a "universal measure of efficiency", which fully characterizes return-risk distribution and is intuitive, easy to understand and calculate, they soon recognized themselves, that in order to get full information about return-risk distribution, the Omega function should be assessed not at a single point τ of the threshold return but within the whole range. Later, the authors' position has become even more critical: according to them, "a function estimated at only one point can be completely misleading". Although an interpretation of an estimate of the Omega function obtained at one point of the threshold return – "more is better" – is really very simple, interpreting the estimates obtained within the range of the threshold return is far from simple.

The Omega function is strictly descending: where τ is lower than the mean of distribution μ, the Omega is higher than one (i.e. $\Omega > 1$ where $\tau < \mu$); where τ is higher than the mean of distribution μ, the Omega is lower than one (i.e. $\Omega < 1$, where $\tau > \mu$) and it is equal to one when $\tau = \mu$. It is intuitively understandable that the higher is the threshold return, the lower is the opportunity to achieve it, and therefore, an increasing threshold results into the value of the Omega coming to 0. Furthermore, the situation becomes complicated.

The level of investment risk depends on the characteristics of the Omega function (plot): the steeper is the plot, the lower is the risk, i.e. a lower probability of "extreme" return variations, accordingly, the flatter is the plot, the higher is the risk.

Figure 2 represents Omega plots for United Health Group (UNH), Exxon Mobil Corporation (XOM) and Verizon Communications Inc. (VZ) where threshold returns ranges from 0 percent up to 5 percent. The figure shows that the UNH is more attractive than the XOM or VZ not taking into account the selected threshold; however, the attractiveness of the XOM, compared to VZ, will depend on the selected marginal return. Therefore, assessing the attractiveness of assets relying on a single selected threshold

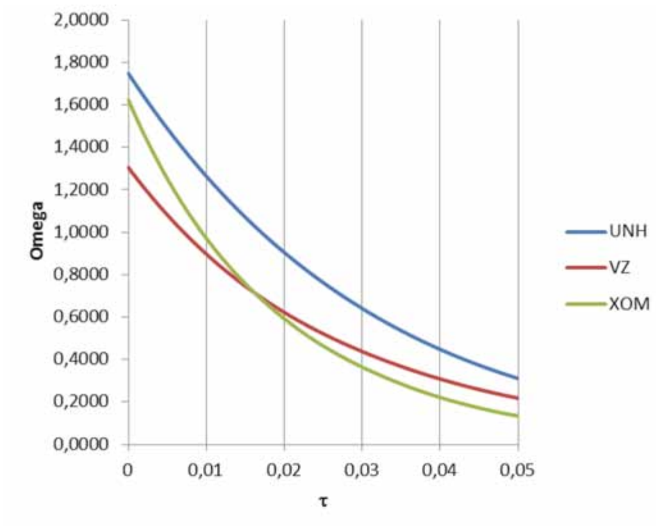

Fig. 2. Omega plots for UNH, XOM and VZ

value is dangerous – assessment within the whole range is required. The plot of the Omega function obtained by changing threshold values actually allows a more efficient assessment of investment attractiveness, but there is a legitimate question in what way the Omega ratio is better than a direct comparison of the distributions of returns, i.e. investment assessment using stochastic dominance criterias (Frey 2009).

Despite the warnings of Omega deficiencies while taking "point" estimates of investments with respect to this function, the issue of threshold selection is not analyzed more explicitly in literature thus often simply recognizing it is not clear how this threshold should be specified, or indicating that the threshold should be specified depending on the risk-return preferences of each investor (Avouyi-Dovi *et al*. 2004; Mausser *et al*. 2006). In practice, the threshold value is generally equated to the rate of risk-free investments, the average expected return or simply a zero (Gilli *et al*. 2011).

4. Portfolio optimization with respect to the Omega function

Performing the Omega-optimization of a portfolio, each threshold return value is attributed with positional weight, which maximizes the ratio of the expected gain and loss. Formally the problem of Omega optimization can be written as:

$$\max \Omega_s (x,\tau); \sum_{i=1}^{N} x_i = 1; \ x_l \le x_i \le x_u \,, \tag{5}$$

where x_l and x_u are lower and upper bounds on weights. When choosing an interval of the threshold return [minτ, maxτ], within which the Omega function will be opti-

Table 1. Review of studies on portfolio optimization with respect to the Omega function (Source: created by the author)

Authors	Brief description of the study, main results
Avouyi-Dovi *et al.* 2004	Data: index weekly returns of US, British and German stock markets over the period 1974–2003. Optimization method: threshold acceptance. Results: general considerations that the Omega can be used for optimizing investment portfolios.
Kane *et al.* 2009	Data: artificial data – three series of stock returns comprising the 50-day period. Optimization method: the Nelder-Mead method and MCS global minimum algorithm. Results: general considerations that Omega portfolios differ from minimum risk and minimum loss portfolios.
Gilli, Schumann 2010	Data: one-year data covering several hundred of European companies. Optimization method: threshold acceptance. Results: highlights that the main objective is to evaluate the optimization algorithm rather than portfolio construction strategies.
Gilli *et al.* 2011	Data: data on stock return from a few hundred largest European companies covering the period 1998–2008. Optimization method: threshold acceptance. Description: the optimization of the 130/30 portfolio (i.e. the portfolio allowing selling borrowed securities) and the portfolio that prohibits "borrowed positions" performing classical mean, variance and Omega-optimization. Results: classical and Omega optimization results are not directly compared by the authors, indirectly – the Omega-optimized portfolio was not superior.
Gilli, Schumann 2011	Data: Dow Jones Euro STOXX index companies Optimization method: threshold acceptance. Results: risk minimization, in contrast to return maximization, leads to good results obtained beyond the boundaries of the prediction sample. An additional parameter of the reward function makes only marginal improvements.
Hentati-Kaffel, Prigent 2012	Description: Omega and Omega-Sharpe optimization of plain vanilla structured products (stocks and the option portfolio as well as risk-free investment and the option portfolio). Results: the payment function of the structured products is non-convex.

mized, the maximum threshold of return should not exceed the maximum historical or simulated yield of portfolio securities. Obviously, ex post portfolio returns can never be higher than the returns of the component securities.. In case of a higher threshold of return, meta-heuristic optimization algorithms, as opposed to traditional exact optimization algorithms, can actually give the solution having no logical sense (Shaw 2011).

The Omega ratio is easy to use in assessing prior performance, but is non-convex and may have plenty of local minimums; thus, portfolio optimization using this function is quite tricky. Mausser, Saunders and Seco (2006) proposed a method that, under certain conditions, allows solving the problem of the Omega-optimized portfolio using linear programming techniques; however, generally this method is not appropriate. In other cases, the portfolio is optimized using heuristic optimization (Gilli, Schumann 2010) or other techniques of global optimization (Kane *et al*. 2009).

Return series to be used in portfolio optimization can be obtained in two ways: by taking historical data on the previous periods or applying simulation techniques. As various studies show, relying solely on historical data leads to encountering of the so-called model over-fitting, i.e., the model perfectly fits to the period of construction or testing sampling but has little predictive power beyond this period. A theoretically better method is to "develop" returns using simulation techniques, but in fact, a process generating returns is not known (or even does not exist); thus, historical data are often used hoping that the past scenarios will remain relevant in the future or at least for some time.

For the purposes of this paper, the Omega-optimized portfolio has been carried out using historical data and applying a genetic algorithm for differential evolution implemented in the R package DEoptim (Ardia *et al*. 2011).

Despite the fact that the Omega ratio has considerably interested both academic and financial sectors, there are not many studies that could answer to the question of whether this ratio is somehow better than classic techniques for portfolio optimization or other strategies based on portfolio construction methods. One of the reasons – Omega-optimization are quite complex, and is itself an object of a number of studies. To the best of author's knowledge, there are no empirical studies determining the characteristics of Omega-optimized portfolios at different levels of threshold returns. A review of the conducted studies is presented in Table 1.

5. Classic portfolio theory, its modifications and techniques for heuristic portfolio optimization

The creation of the classic portfolio theory can be dated back to 1952 when Markowitz published portfolio selection theory (Markowitz 1952). Based on the portfolio theory by Markowitz, investors selecting a portfolio refer only to two characteristics, including the expected return and risk measured by variance, i.e. a multidimensional problem of investment selection, where due to indetermination, the investment portfolio must be composed of many differently characterized types of assets, was simplified by Markowitz to two dimensions. Therefore, this technique for portfolio optimization is often called a Mean-Variance method (MV). The portfolio is considered optimal in respect of MV, if, at given fixed average yields, risk is minimized or, at certain fixed risks, the expected returns are maximized. A set of optimal portfolios constructed taking into account different risk tolerance of investors is called an efficient frontier.

Despite great academic success, a practical application of this model has not been very successful, because it has been quickly noticed that MV portfolios can neither be characterized by good investment diversification nor by stability. The MV portfolio is *ex ante* optimal when input parameters are “known”. Since in practice we do not know the “true” parameters, and their estimates used generally overestimate the “true” values, or duly underestimate them, the “optimized” portfolio can be much worse than the unoptimized portfolio constructed using “naive” risk diversification techniques, e.g. distributing portfolio funds into equal parts. While using moment estimates, in order to construct MV portfolios, we have to deal with the risk of estimates, the source of which is the difference between the estimates and the actual values of parameters. Therefore, while optimizing the portfolio, we no longer deal with one but already two sources of risk: i) the risk of markets and security price volatility; ii) the risk of estimates or false expectations. The risk of parameter estimates and its significant negative impact on the optimization of the MV portfolio is quite well researched and documented.

In the process of MV optimization, disproportionately large weights are assigned to securities with high expected yield, negative correlation and low variance, and disproportionately small weight coefficients are assigned to securities with low expected yield, positive correlation and high variance. Such weight distribution is understandable, but it is most likely that these securities will have the biggest estimation errors. For this reason, Michaud designated MV optimization as the “maximization of estimation errors” (Michaud 1989).

Asset distribution within the portfolio is remarkably affected by the estimation errors of the expected returns (Merton 1980). The MV portfolio is particularly sensitive to changes in return in case of the ban on short sale – even a very small increase in the mean of only one security returns may result into a large part of securities being “expelled” from the portfolio (Best, Grauer 1991).

While studying the relative influence of mean, variance and covariance estimation errors on MV portfolios, Chopra and Ziemba (1993) found out that the errors of means were up to ten times more significant than those of variances; meanwhile, the errors of variances are up to two times more significant than those of covariances. In this context, an interest in minimum variance portfolios has greatly increased, the optimization of which does not require the prognosis of the expected returns and is limited to the covariance matrix only. Unfortunately, minimum variance portfolios also do not avoid the high concentration of low variance securities (Clarke *et al*. 2011).

Most of the proposed MV optimization solution techniques, in one way or another, are related to constraints on model input parameters or portfolio weights (Frost, Savarino 1988; Jagannathan, Ma 2003). It is obvious, that by limiting maximum weights, illogically, high stock concentration is prevented and better risk diversification is ensured; however, constraints on weight mean there is less reliance on optimization procedures and market “signal” reliability but more on “naive” risk management.

Another solution to the problem was proposed by Ledoit and Wolf (2003, 2004). They claimed that, due to the resulting major errors of estimates, for portfolio optimization, no one should use covariance matrix S obtained from the sample of stock returns. Instead, scientists have proposed the use of the covariance matrix "shrunk towards" a structured covariance matrix and obtained using a well-known capital asset pricing model (CAPM). Their argument is that structured covariance matrix F received using the Sharpe's simplified one-factor model has far less parameters and can therefore be estimated with much less error. The proposed "shrunk" covariance matrix $\hat{\Sigma}$ is calculated as follows:

$$\hat{\Sigma} = \hat{\delta}F + (1-\hat{\delta})S, \tag{6}$$

where $\hat{\delta}$ – optimal "shrinkage" constant obtained by minimizing $\hat{\Sigma}$ mean quadratic error. Using a structured covariance matrix reduces estimation error, but model specification or a "wrong" model error (if the chosen model does not or badly conforms to reality) appears. As an alternative to the structured covariance matrix, Ledoit and Wolf suggested the use of a uniform correlation model, in which all pair correlations of securities are replaced by a uniform correlation equal to the mean of security correlations (Ledoit, Wolf 2004).

Due to the enlisted disadvantages of the MV optimization algorithm, "precise" optimization algorithms are often generally refused instead of using the so called "heuristic optimization techniques", i.e. various portfolio construction rules based on empirical facts of security or investor's behaviour. One of such techniques are the portfolio of equal weights or simply a 1/N portfolio. While comparing various strategies for portfolio optimization, this portfolio is often used as a benchmark portfolio. Naturally, the 1/N strategy is not based on any input assumptions but depends on a sample of stocks. The carried out studies have shown that naive 1/N portfolios often surpass MV-optimized portfolios when the results of strategies are assessed using new data outside the data sample used for optimization (DeMiguel *et al*. 2009a; Duchin, Levy 2009). Although DeMiguel *et al*. (2009b) recognized the advantage of the Ledoit and Wolf's method in their later work, the 1/N strategy still remains a favourite "standard" to which various portfolio construction and optimization techniques are compared.

6. Portfolio turnover and transaction costs

The portfolio of securities can be actively or passively managed. Due to lower turnover, passive portfolio management costs (management costs – a broader concept that includes investment management fee and other expenses, but for this paper, only the costs associated with the purchase and sale of securities were assessed) are much lower than those of the active ones. To cover additional transaction costs and ensure their superiority, actively managed portfolios have to earn additional returns.

The implementation of the portfolio strategy requires the reallocation of portfolio weights for two reasons: 1) while the parameters of the model are changing, optimal portfolio weights also vary; 2) while the prices of the stock market portfolio are changing, the actual weights deviate from the theoretical ones; and in order to eliminate this difference, we have to reallocate the positions of the portfolio. For the second reason, we must regularly reallocate weights, even when the weights of the optimal model remain unchanged, e.g. in order to maintain a portfolio of equal weights.

As portfolio turnover and transaction costs directly affect the net portfolio return, portfolio optimization strategies characterized by stability and low turnovers are significantly superior compared to other strategies, e.g. the 1/N strategy characterized by low turnovers, as its weights need to be adjusted only due to changes in the price.

Within this paper, the portfolio turnover is calculated according to standard practice (DeMiguel *et al*. 2009a; Gilli, Schumann 2011). The average portfolio turnover is derived from:

$$Turnover = \frac{1}{T}\sum_{t=1}^{T}\sum_{n=1}^{N}\left|x_{n,t} - x_{n,t-1}\right|, \tag{7}$$

where $x_{n,t}$ – the weight of the i-th position of the portfolio after reallocation; $x_{n,t-1}$ – the weight of the i-th position of the portfolio before reallocation. As it is common to present return and risk using annual data, the turnover rate is accordingly converted multiplying it by 12.

The calculation of the net portfolio return obtained following the deduction of portfolio reallocation costs is more complex. Marketing securities imposes direct and indirect costs. Direct costs include commissions and similar expenses while the indirect ones – the difference between purchasing and selling prices associated with the liquidity of securities and the costs associated with the "impact on the market", i.e. impact on prices. A theoretical assumption that one can purchase or sell securities without restrictions making no impact on their prices is unlikely in the real world, especially when selling portfolios. Costs can also be proportional, i.e. depending on the transaction amount and fixed, i.e. independent of the transaction amount.

While modelling transaction costs, it is easier to assume that costs are proportional to the amounts of money traded. If the "rate" of proportional transaction cost is equal to c, the dynamics of the portfolio value, after deducting costs, can be described as follows:

$$W_{t+1} = W_t(1+R_{t+1})\left(1 - c\sum_{n-1}^{N}\left|x_{n,t+1} - x_{n,t}\right|\right), \tag{8}$$

where R – general portfolio returns, W_t – initial wealth at time t, and W_{t+1} – wealth at time t+1. Then, the net return of the portfolio is equal to $W_t / W_{t-1} - 1$.

While assessing the net result of the investigated strategies based on the study by Carhart (1997), a rate of 1 percent proportional turnover was "determined", i.e. it is modelled that a portfolio turnover of 100 percent reduces the net income of the portfolio by 1 percent, or, in other words, the portfolio must achieve a 1 percent alpha to cover the turnover increased by 100 percent (a detailed review of studies relating to transaction costs was done by Kasten (2007)). While the selected proportional turnover rate of 1 percent is high enough, it is two times higher than the rate of 0.5 percent used in the study by DeMiguel, Garlappi and Uppal (2009a) where actual portfolio turnover costs can be significantly higher due to an indirect "impact on the market" costs; therefore, dramatic variations in the portfolio are undesirable.

7. Results, discussion and limitations

The characteristics of the Omega portfolio were assessed with threshold return value τ varying in the range from 0 to 5 percent. Range step is 0.001%. The tables of the obtained results include concise data ranging from 0 to 4 percent, the range step of 0.005 percent and the maximum value of a hypothetical portfolio.

In order to objectively evaluate Omega portfolios, their performance are compared with the performance of the portfolios optimized using other techniques. A total of nine benchmark portfolios are compared within this paper: the classic minimum variance portfolio, the tangency or maximum Sharpe ratio portfolio (respectively C.MV and C.TG), tangency portfolios optimized using uniform correlation and "shrunk" covariance matrices proposed by Ledoit and Wolf (respectively LWCC.TG and LW1F.TG), minimum conditional value-at-risk and highest return-risk CVaR portfolios (minCVaR and maxCVaR), the equal weighted portfolio (EQW), the equal risk portfolio (ERC) proposed by Maillard, Roncalli and Teïletche (2010) and the minimum correlation portfolio (MinCor) suggested by Varadi, Kapler and Rittenhouse.

The study refers to three data sets of the global stock market: Dow Jones Industrial Average index composed of data on monthly industry stock returns for the period from 30/01/1998 to 31/12/2013 (a total of 30 stocks and 192 periods), FTSE index composed of 30 randomly selected monthly stock returns for the period from 31/01/2001 to 30/05/2014 (in order to calculate the weights of classical MV portfolios, the inverse covariance matrix that can be obtained only when the number of observations M is greater than the number of stocks N is required; since the selected duration for one backtesting period is equal to 36 months, i.e. M = 36 (see below), to estimate the weights of the classic portfolio, it was necessary to reduce the number of stocks or use weekly or daily data on stock returns; thus, in order the FTSE index portfolio could be directly compared to DJIA stocks, a random set of 30 stocks was selected) and EURO STOXX 50 index composed of data on weekly industry stock returns for the period from 04/01/2002 to 31/12/2013 (a total of 50 stocks and 627 periods).

The performance of the studied strategies for portfolio optimization, including return and other indicators, were evaluated using a moving sample window method often applied in scientific studies (DeMiguel *et al*. 2009a; Gilli *et al*. 2011). The choice of the method is usually based on a well-known heteroskedasticity feature of financial data series. Primarily, the duration of one testing period is selected; this paper accepts M = 36 months, or 104 weeks. Based on the return series of the first testing period, the parameters required for implementing a particular strategy are obtained and then used for calculating optimal portfolio weights that are received and used for calculating portfolio returns for the next period, i.e. M + 1. The process is continued with an addition of a new period and the exclusion of one of the earliest periods until the end of the entire data period is reached. As a result of this backtesting using a moving window approach, a series of T-M monthly (weekly) out-of-sample portfolio returns is obtained, i.e. calculated using data that was not included into a data sample during portfolio optimization. The procedure applies to each testing strategy and each stock data set.

Another decision to be made in the management of investment portfolios is to choose how often the investment portfolio will be reallocated. Fund managers usually reallocate portfolio positions either in accordance with a specified frequency or when portfolio weights "deviate" from the specified allowable threshold or, more commonly, over a certain period if, at that time, weights are above the specified "threshold". Such portfolio management can be called tactical, as for certain tactical objectives, e.g. reducing transaction costs it is allowed to deviate from the basic strategy – optimal weights. The frequency of portfolio reallocation can be also changed for other reasons such as reducing the taxes paid. Based on the results of the author's previous study on the Omega portfolio, a half-year frequency for reallocating weights was selected for the present study, which is empirically proven as a good compromise in order not to deviate significantly from the selected strategy and the desire to reduce turnover costs (Vilkancas 2014).

The series of derived portfolio returns were assessed considering various aspects, including the overall return, risk and portfolio turnover required for a particular strategy, portfolio concentration and net return received after the deduction of costs and incurred in the reallocation of portfolio weights. In order to assess portfolio performance, a total of 14 different indicators were used. In addition to conventional risk indicators – standard variance (AnnSD) and the Sharpe ratio (Ann.SR), the study presents the maximum drawdown and the average drawdown rate (Max.Draw and Avg.Draw) as well as maximum and minimum annual returns received over the period. While assessing portfolio turnover, the average annual turnover and the total, i.e. covering the whole period, turnover are given (Ann.Turn and Tot.Turn). In order to assess portfolio concentration, the Gini coefficient calculated according to the method and developed by the Italian statistician Corrado Gini is employed. The coefficient ranges from 0 to 1. If the value of the Gini coefficient equals to 1, this shows complete portfolio concentration (portfolio consists of only one position), and the Gini coefficient for a well-diversified portfolio of equal weights equals to 0. The table of the obtained results includes the average values of the Gini coefficient obtained during the entire period of study. Finally, it provides the net annual returns of the portfolio and the net value (NetCumRet) received after the deduction of proportional turnover charges as well as the Sharpe ratio estimated using net returns (NetAnn.SR).

The main obtained results are presented in Tables 2–5, a net value of hypothetical \$ 1 (£ or €) portfolio obtained by changing the threshold limit – in Figures 3–4 (detailed results of FTSE 100 and EURO STOXX 50 benchmark portfolios are omitted).

Table 2. The performance of the Omega portfolios of DJIA index at different levels of threshold returns where weights are reallocated every six months (Source: created by the author)

Threshold, τ	0	0.005	0.01	0.014	0.015	0.02	0.025	0.03	0.04
CumRET	3.370	3.068	3.794	4.340	3.948	3.143	2.008	1.633	1.090
AnnRET	9.80	9.01	10.80	11.95	11.14	9.21	5.51	3.84	0.66
AnnSD	13.23	13.90	14.64	15.61	15.80	16.79	20.25	22.13	23.52
Max.Draw	36.23	36.49	33.08	30.62	34.29	29.89	45.59	54.51	55.78
Avg.Draw	6.08	6.75	7.66	7.29	7.25	11.31	15.63	17.75	55.78
MaxAnnRet	35.49	37.80	36.99	36.55	34.34	35.02	59.18	49.48	52.22
MinAnnRet	–22.44	–22.09	–17.21	–13.81	–18.00	–19.14	–28.46	–37.20	–37.86
Ann.SR	74.06	64.79	73.79	76.57	70.52	54.84	27.20	17.37	2.82
Ann.Turn	161	159	159	170	174	200	227	211	177
Tot.Turn	2088	2071	2072	2207	2259	2600	2947	2745	2297
Gini.Avg	89	90	92	92	92	90	89	88	84
NetAnnRet	8.19	7.41	9.21	10.26	9.40	7.21	3.24	1.73	–1.10
NetAnn.SR	61.91	53.33	62.90	65.70	59.52	42.93	16.00	7.83	–4.69
NetCumRet	2.727	2.491	3.084	3.481	3.151	2.425	1.496	1.240	0.864

Note: Here and below in the tables, the indexes are shown as percentages, except CumRet and NetCumRet, the values of which are specified in the units of currency.

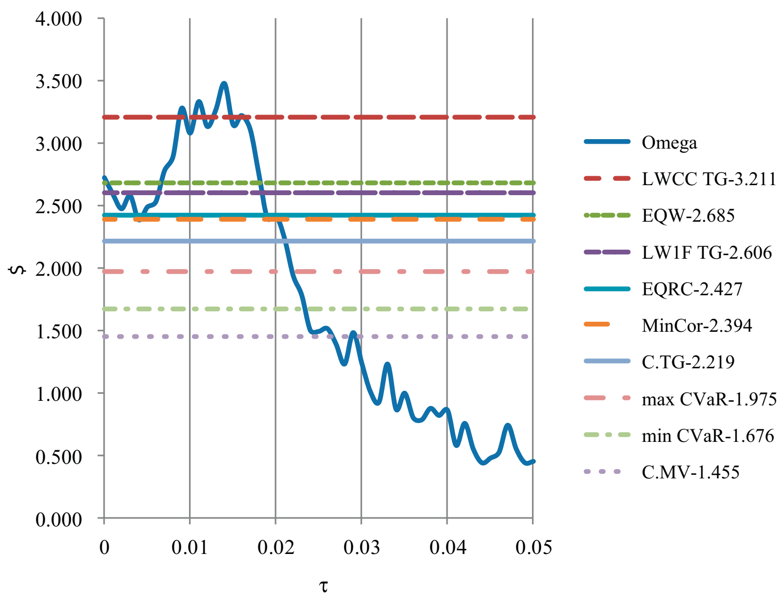

Fig. 3. Value growth of a hypothetical $1 investment in stocks from the DJIA index universe where weights are re-allocated every six months (Source: created by the author)

Table 3. The performance of the benchmark portfolios of DJIA index where weights are reallocated every six months (Source: created by the author)

	EQW	C.MV	C.TG	LWCC TG	LW1F TG	min CVaR	max CVaR	ERC	MinCor
CumRet	2.753	1.684	2.677	3.841	3.116	2.148	2.459	2.527	2.511
AnnRet	8.10	4.09	7.87	10.91	9.14	6.06	7.17	7.39	7.34
AnnSD	15.27	11.99	13.33	13.25	12.85	13.23	14.25	13.84	13.55
Max.Draw	45.71	37.72	41.13	34.36	36.87	38.89	38.90	43.89	43.24
Avg.Draw	7.25	5.92	7.50	5.92	6.08	6.82	7.70	6.62	6.01
MaxAnnRet	34.32	21.96	34.81	36.06	34.39	22.40	36.23	30.24	27.76
MinAnnRet	–28.52	–24.57	–29.68	–21.38	–24.94	–25.14	–26.81	–28.10	–27.45
Ann.SR	53.07	34.10	59.04	82.30	71.09	45.77	50.30	53.43	54.16
Ann.Turn	19	111	143	138	137	189	166	31	37
Tot.Turn	248	1442	1857	1795	1778	2452	2155	404	478
Gini.Avg	4	85	90	90	87	87	91	23	32
NetAnnRet	7.91	2.98	6.44	9.53	7.77	4.17	5.51	7.08	6.97
NetAnn.SHR	51.82	24.85	48.32	71.88	60.45	31.52	38.66	51.18	51.45
NetCumRet	2.685	1.455	2.219	3.211	2.606	1.676	1.975	2.427	2.394

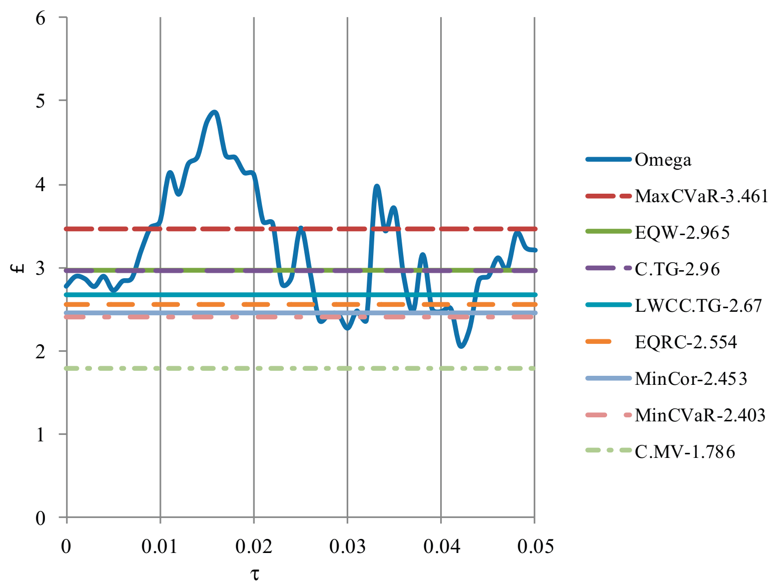

Fig. 4. Value growth of a hypothetical 1£ investment in stocks from the FTSE 100 index universe where weights are reallocated every six months (Source: created by the author)

Table 4. The performance of the Omega portfolios of FTSE 100 index at different levels of threshold returns where weights are reallocated every six months (Source: created by the author)

Threshold, τ	0	0.005	0.01	0.015	0.016	0.02	0.025	0.03	0.04
CumRET	3.306	3.286	4.332	5.738	5.920	4.995	4.266	2.797	2.924
AnnRET	12.06	12.00	14.98	18.10	18.46	16.55	14.82	10.29	10.76
AnnSD	16.40	17.04	19.96	23.44	24.36	25.79	25.94	26.00	28.31
Max.Draw	42.10	40.69	40.41	45.11	48.07	50.44	47.88	40.64	50.57
Avg.Draw	7.00	7.45	7.00	7.78	7.76	8.91	9.75	11.95	16.03
MaxAnnRet	37.11	42.54	62.23	81.09	83.44	76.46	75.69	48.17	115.57
MinAnnRet	–39.55	–37.68	–34.95	–35.34	–34.21	–33.82	–29.79	–24.32	–35.96
Ann.SE	73.57	70.39	75.06	77.22	75.75	64.19	57.13	39.58	38.01
Ann.Turn	159	170	180	171	181	174	185	187	151
Tot.Turn	1752	1875	1982	1886	1995	1913	2036	2059	1665
Gini.Avg	88	88	90	91	91	91	91	91	89
NetAnnRet	10.47	10.29	13.18	16.39	16.64	14.82	12.97	8.42	9.25
NetAnn.SR	63.85	60.38	66.03	69.91	68.31	57.45	49.99	32.38	32.67
NetCumRet	2.773	2.723	3.551	4.747	4.849	4.115	3.473	2.269	2.468

Table 5. The performance of the Omega portfolios of EURO STOXX 50 at different levels of threshold returns where weights are reallocated every six months (Source: created by the author)

Threshold, t	0	0.003	0.005	0.01	0.015	0.02	0.025	0.03	0.04
CumRET	2.592	2.748	2.614	1.814	1.673	1.205	1.173	1.058	1.186
AnnRET	9.99	10.64	10.08	6.14	5.28	1.88	1.61	0.56	1.72
AnnSD	14.80	15.89	18.13	21.68	24.60	25.53	26.11	26.00	25.15
Max.Draw	52.51	51.10	57.43	58.42	61.07	64.15	66.19	68.10	63.36
Avg.Draw	9.14	8.44	8.47	11.82	10.13	13.27	13.43	13.20	13.97
MaxAnnRet	33.61	31.95	33.27	36.44	38.39	35.54	37.22	35.01	34.34
MinAnnRet	–41.39	–39.19	–39.85	–45.04	–44.63	–46.46	–49.58	–50.11	–46.15
Ann.SR	67.52	66.96	55.61	28.30	21.47	7.37	6.17	2.16	6.84
Ann.Turn	177	176	184	167	168	161	156	143	143
Tot.Turn	1772	1762	1838	1673	1677	1608	1563	1427	1430
Gini.Avg	86	83	82	78	78	77	76	74	71
NetAnnRet	8.22	8.88	8.25	4.46	3.60	0.27	0.05	–0.87	0.29
NetAnn.SR	55.55	55.87	45.47	20.59	14.65	1.07	0.18	–3.33	1.15
NetCumRet	2.175	2.308	2.178	1.537	1.416	1.027	1.004	0.917	1.028

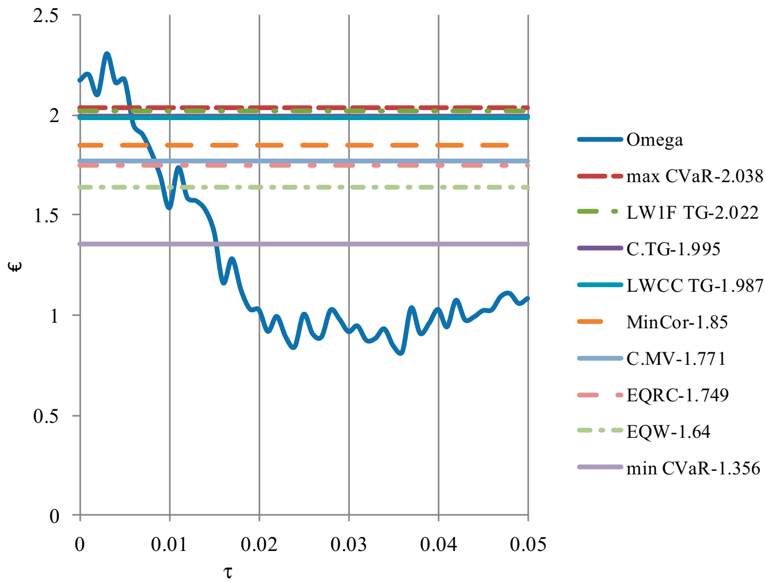

Fig. 5. Value growth of a hypothetical 1€ investment in stocks from the EURO STOXX 50 index universe where weights are reallocated every six months
(Source: created by the author)

8. Conclusions

The results presented in Tables 1–6 show that the return and risk of Omega-optimized portfolios depend on the threshold return selected by the investor: a growth in the threshold return leads to an increase in the performance of the net portfolio; however, at the expense of a growing portfolio risk. In fact, the result is quite impressive as Omega-optimized portfolios tested applying all three sets of stocks and using data obtained at three different stock markets, managed to surpass 9 competing strategies for portfolio optimization.

In addition, Omega-optimized portfolios were distinguished by stability with threshold return value τ varying in the range from 1 to 2%. The optimal threshold return of EURO STOXX 50 stocks was lower, but in this case, weekly data were investigated – the highest portfolio return was achieved selecting a weekly threshold return of 0.003 percent, which is equivalent to 1.2 percent of monthly returns.

Alternative portfolio optimization strategies did not demonstrate stability: in the US market, LWCC.TG was characterized as the best one, but, when surveyed in other markets, it was not even in the top three of the best strategies. As for the Eurozone and London stock markets, the max CVaR strategy was the best player; however, its advantage in the British market was minimal. A look at the second and the third places points to a similar situation: EQW and LW1F.TG portfolios take the 2nd and 3rd places in the US market, whereas – EQW and C.TG in the Eurozone and – LW1F.TG and C.TG in London markets.

When the threshold return of the Omega portfolio exceeds a certain limit, the portfolio return starts declining but risk continues growing. In addition, unlike algorithms for classical deterministic optimization where setting too high optimization parameters makes a solution impossible, metaheuristic algorithms usually find a solution even in such cases, which makes clear that a similar option is not financially meaningful.

References

Ardia, D.; Boudt, K.; Carl, P.; Mullen, K.; Peterson, B. 2011. Differential evolution with DEoptim: an application to non-convex portfolio optimization, *The R Journal* 3(1): 27–34.

Artzner, P.; Delbaen, F.; Eber, J.-M.; Heath, D. 1999. *Coherent measures of risk, mathematical Finance* 9(3): 203–228. Blackwell Publishers Inc.

Avouyi-Dovi, S.; Morin, S.; Neto, D. 2004. *Optimal asset allocation with Omega function*. Technical report, Banque de France.

Bawa, V. S. 1975. Optimal rules for ordering uncertain prospects, *Journal of Financial Economics* 2: 95–121. Elsevier. http://dx.doi.org/10.1016/0304-405X(75)90025-2

Best, M. J.; Grauer, R. R. 1991. On the sensitivity of mean-variance-efficient portfolios to changes in asset means: some analytical and computational results, *Review of Financial Studies* 4(2): 315–342. Soc. Financial Studies. http://dx.doi.org/10.1093/rfs/4.2.315

Carhart, M. M. 1997. On persistence in mutual fund performance, *The Journal of Finance* 52(1): 57–82. Blackwell Publishing Ltd. http://dx.doi.org/10.1111/j.1540-6261.1997.tb03808.x

Chekhlov, A.; Uryasev, S.; Zabarankin, M. 2005. Drawdown measure in portfolio optimization, *International Journal of Theoretical and Applied Finance* 8(1): 13–58. http://dx.doi.org/10.1142/S0219024905002767

Chopra, V. K.; Ziemba, W. T. 1993. The effect of errors in means, variances, and covariances on optimal portfolio choice, *The Journal of Portfolio Management* 19(2): 6–11. http://dx.doi.org/10.3905/jpm.1993.409440

Clarke, R.; De Silva, H.; Thorley, S. 2011. Minimum variance portfolio composition, *The Journal of Portfolio Management* 37(2): 31–45. Institutional Investor Journals. http://dx.doi.org/10.3905/jpm.2011.37.2.031

Dembo, R.; Mausser, H. 2000. The put/call efficient frontier, *Algo Research Quarterly* 3: 13–25.

Dembo, R.; Rosen, D. 1999. The practice of portfolio replication: a practical overview of forward and inverse problems, *Annals of Operations Research* 85: 267–284. Springer. http://dx.doi.org/10.1023/A:1018977929028

DeMiguel, V.; Garlappi, L.; Uppal, R. 2009a. Optimal versus naive diversification: how inefficient is the 1/N portfolio strategy?, *Review of Financial Studies* 22(5): 1915–1953. Soc Financial Studies. http://dx.doi.org/10.1093/rfs/hhm075

DeMiguel, V.; Garlappi, L.; Nogales, F. J.; Uppal, R. 2009b. A generalized approach to portfolio optimization: improving performance by constraining portfolio norms, *Management Science* 55(5): 798–812. INFORMS. http://dx.doi.org/10.1287/mnsc.1080.0986

Duchin, R.; Levy, H. 2009. Markowitz versus the Talmudic portfolio diversification strategies, *The Journal of Portfolio Management* 35(2): 71–74. http://dx.doi.org/10.3905/JPM.2009.35.2.071

Fama, E. F. 1965. The behavior of stock-market prices, *Journal of Business:* 34–105. http://dx.doi.org/10.1086/294743

Farinelli, S.; Tibiletti, L. 2008. Sharpe thinking in asset ranking with one-sided measures, *European Journal of Operational Research* 185(3): 1542–1547. http://dx.doi.org/10.1016/j.ejor.2006.08.020

Fishburn, P. C. 1977. Mean-risk analysis with risk associated with below-target returns, *The American Economic Review* 67(2): 116–126. JSTOR.

Frey, R. J. 2009. *On the Ω (Omega) ratio*. Applied Mathematics and Statistics Department, Stony Brook University.

Frost, P. A.; Savarino, J. E. 1988. For better performance: constrain portfolio weights, *The Journal of Portfolio Management* 15(1): 29–34. Institutional Investor Journals. http://dx.doi.org/10.3905/jpm.1988.409181

Gilli, M.; Schumann, E.; Di Tollo, G.; Cabej, G. 2011. Constructing 130/30-portfolios with the omega ratio, *Journal of Asset Management* 12(2): 94–108. Nature Publishing Group. http://dx.doi.org/10.1057/jam.2010.25

Gilli, M.; Schumann, E. 2010. Distributed optimisation of a portfolio's Omega, *Parallel Computing* 36(7): 381–389. http://dx.doi.org/10.1016/j.parco.2009.10.001

Gilli, M.; Schumann, E. 2011. Risk–reward optimisation for long-run investors: an empirical analysis, *European Actuarial Journal* 1(2): 303–327. http://dx.doi.org/10.1007/s13385-011-0024-2

Grossman, S. J.; Zhou, Z. 1993. Optimal investment strategies for controlling drawdowns, *Mathematical Finance* 3(3): 241–276. Blackwell Publishing Ltd. http://dx.doi.org/10.1111/j.1467-9965.1993.tb00044.x

Hentati-Kaffel, R.; Prigent, J.-L. 2012. *Structured portfolio analysis under SharpeOmega ratio*. Universite Pantheon-Sorbonne (Paris 1), Centre d'Economie de la Sorbonne.

Jagannathan, R.; Ma, T. 2003. Risk reduction in large portfolios: why imposing the wrong constraints helps, *The Journal of Finance* 58(4): 1651–1684. Wiley Online Library. http://dx.doi.org/10.1111/1540-6261.00580

Kahneman, D.; Tversky, A. 1979. Prospect theory: an analysis of decision under risk, *Econometrica: Journal of the Econometric Society* 47(2): 263–291. http://dx.doi.org/10.2307/1914185

Kane, S. J.; Bartholomew-Biggs, M. C.; Cross, M.; Dewar, M. 2009. Optimizing Omega, *Journal of Global Optimization* 45(1): 153–167. Springer US. http://dx.doi.org/10.1007/s10898-008-9396-5

Kaplan, P. D.; Knowles, J. 2004. A. Kappa: a generalized downside risk-adjusted performance measure, *Journal of Performance Measurement* 8: 42–54. TGS Publishing.

Kaplan, P. D.; Siegel, L. B. 1994. Portfolio theory is alive and well, *The Journal of Investing* 3(3): 18–23. http://dx.doi.org/10.3905/joi.3.3.18

Kasten, G. W. 2007. High transaction costs from portfolio turnover negatively affect 401 (K) participants and increase plan sponsor fiduciary liability, *Journal of Pension Benefits* 14(3): 50–64. Aspen Publishers, Inc.

Kazemi, H.; Schneeweis T.; Gupta, B. 2004. Omega as a performance measure, *Journal of Performance Measurement* 8: 16–25.

Keating, C.; Shadwick, W. F. 2002a. A universal performance measure, *Journal of Performance Measurement* 6(3): 59–84. EDHEC.

Keating, C.; Shadwick, W. F. 2002b. An introduction to omega, *AIMA Newsletter*.

Ledoit, O.; Wolf, M. 2004. A well-conditioned estimator for large-dimensional covariance matrices, *Journal of Multivariate Analysis* 88(2): 365–411. Elsevier. http://dx.doi.org/10.1016/S0047-259X(03)00096-4

Ledoit, O.; Wolf, M. 2003. Improved estimation of the covariance matrix of stock returns with an application to portfolio selection, *Journal of Empirical Finance* 10(5): 603–621. Elsevier.

Maillard, S.; Roncalli, T.; Teïletche, J. 2010. The properties of equally weighted risk contribution portfolios, *The Journal of Portfolio Management* 36(4): 60–70.

Mandelbrot, B. 1967. The variation of some other speculative prices, *Journal of Business* 40(4): 393–413. http://dx.doi.org/10.1086/295006

Markowitz, H. 1952. Portfolio selection, *The Journal of Finance* 7(1): 77–91. Wiley Online Library.

Markowitz, H. M. 1959. Portfolio selection: efficient diversification of investments, *Cowles Foundation Monograph* 16.

Mausser, H.; Saunders, D.; Seco, L. 2006. Optimizing omega, *Risk Magazine,* November, 88–92.

Merton, R. C. 1980. On estimating the expected return on the market: an exploratory investigation, *Journal of Financial Economics* 8(4): 323–361. Elsevier. http://dx.doi.org/10.1016/0304-405X(80)90007-0

Michaud, R. O. 1989. The Markowitz optimization enigma: is 'optimized' optimal?, *Financial Analysts Journal* 45(1): 31–42. JSTOR. http://dx.doi.org/10.2469/faj.v45.n1.31

Pflug, G. Ch. 2000. Some remarks on the value-at-risk and the conditional value-at-risk, *Probabilistic Constrained Optimization* 49: 272–281. http://dx.doi.org/10.1007/978-1-4757-3150-7_15

Rockafellar, R. T.; Uryasev, S. 2000. Optimization of conditional value-at-risk, *Journal of Risk* 2: 21–42.

Rockafellar, R. T.; Uryasev, S. 2002. Conditional value-at-risk for general loss distributions, *Journal of Banking & Finance* 26(7): 1443–1471. http://dx.doi.org/10.1016/S0378-4266(02)00271-6

Roy, A. D. 1952. Safety first and the holding of assets, *Econometrica* 20: 431–449. http://dx.doi.org/10.2307/1907413

Shaw, W. T. 2011. *Portfolio optimization for VAR, CVaR, Omega and utility with general return distributions: a Monte Carlo approach for long-only and bounded short portfolios with optional robustness and a simplified approach to covariance matching*, Working paper [online], [cited 6 October 2014]. Available from Internet: http://ssrn.com/abstract=1856476 http://dx.doi.org/10.2139/ssrn.1856476.

Sortino, F. A.; Van Der Meer, R.; Plantinga, A. 1999. The Dutch triangle, *The Journal of Portfolio Management* 26: 50–57. Institutional Investor Journals. http://dx.doi.org/10.3905/jpm.1999.319775

Sortino, F. A.; Van Der Meer, R. 1991. Downside risk, *The Journal of Portfolio Management*, 17: 27–31. Institutional Investor Journals. http://dx.doi.org/10.3905/jpm.1991.409343

Vilkancas, R. 2014. Omega atžvilgiu optimizuoto akcijų portfelio empiriniai tyrimai, *Verslas: teorija ir praktika* 15(1): 58–70. http://dx.doi.org/10.3846/btp.2014.01

Renaldas VILKANCAS is currently working as an assistant at the Department of Finance Engineering of Vilnius Gediminas Technical University, Lithuania.